Conservation Scheme for sigma Ocean Models

(Finite Volume Consideration for Pressure Gradient Force)

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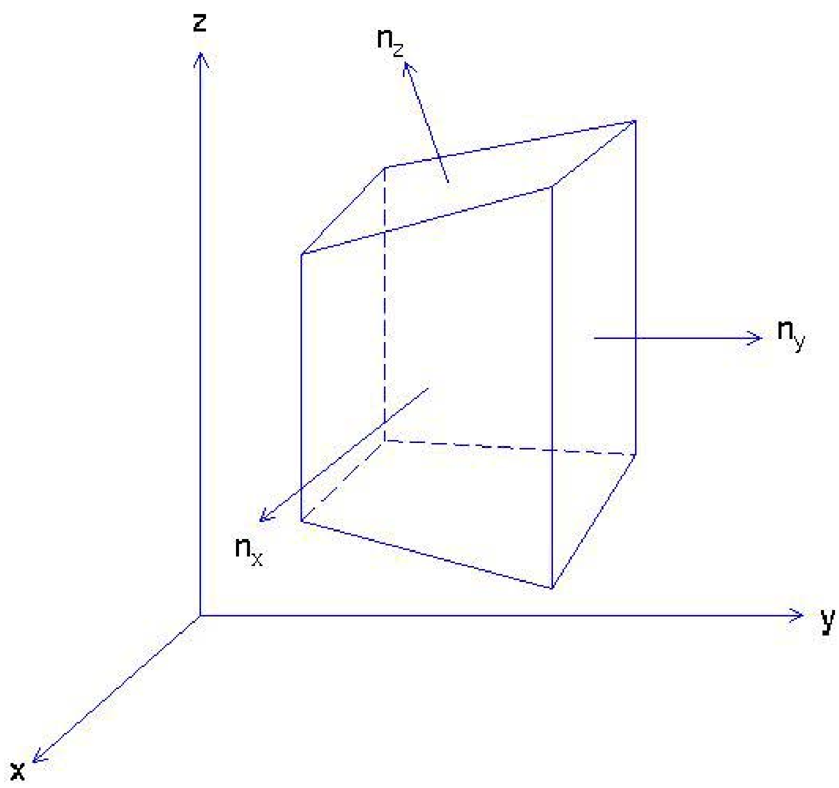
Four Types of Numerical Models

- Spectral Model (not suitable for oceans due to irregular lateral boundaries)
- Finite Difference (z-coordinate, sigma-coordinate, …)
- Finite Element
- Finite Volume

Finite Volume



Finite Volume Model

- Transform of PDE to Integral Equations
- Solving the Integral Equation for the Finite Volume
- Flux Conservation

Dynamic and Thermodynamic Equations

- Continuity $$\nabla \cdot (\rho \mathbf{V}) = \mathbf{0}$$
- Momentum

$$\frac{\partial(\rho \mathbf{V})}{\partial t} + \nabla \cdot (\rho \mathbf{V}\mathbf{V}) = -\nabla p + \nabla \cdot (\mu \nabla \mathbf{V}) + \mathbf{F}$$

- Thermodynamic

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\mathbf{V}\phi) = \nabla \cdot (\kappa_\phi \nabla \phi) + F_\phi$$

Integral Equations for Finite Volume

- Continuity

$$\int_\Omega \nabla \cdot (\rho \mathbf{V})\, d\Omega = \oint_\Gamma \rho \mathbf{V} \cdot \mathbf{n} d\Gamma = \mathbf{0}$$

- Momentum

$$\int_\Omega \frac{\partial(\rho \mathbf{V})}{\partial t} d\Omega + \oint_\Gamma \rho \mathbf{V}\mathbf{V} \cdot \mathbf{n} d\Gamma = - \oint_\Gamma p d\Gamma + \oint_\Gamma \mu \nabla \mathbf{V} \cdot \mathbf{n} d\Gamma + \int_\Omega \mathbf{F} d\Omega \quad l\Omega$$

- Thermodynamic

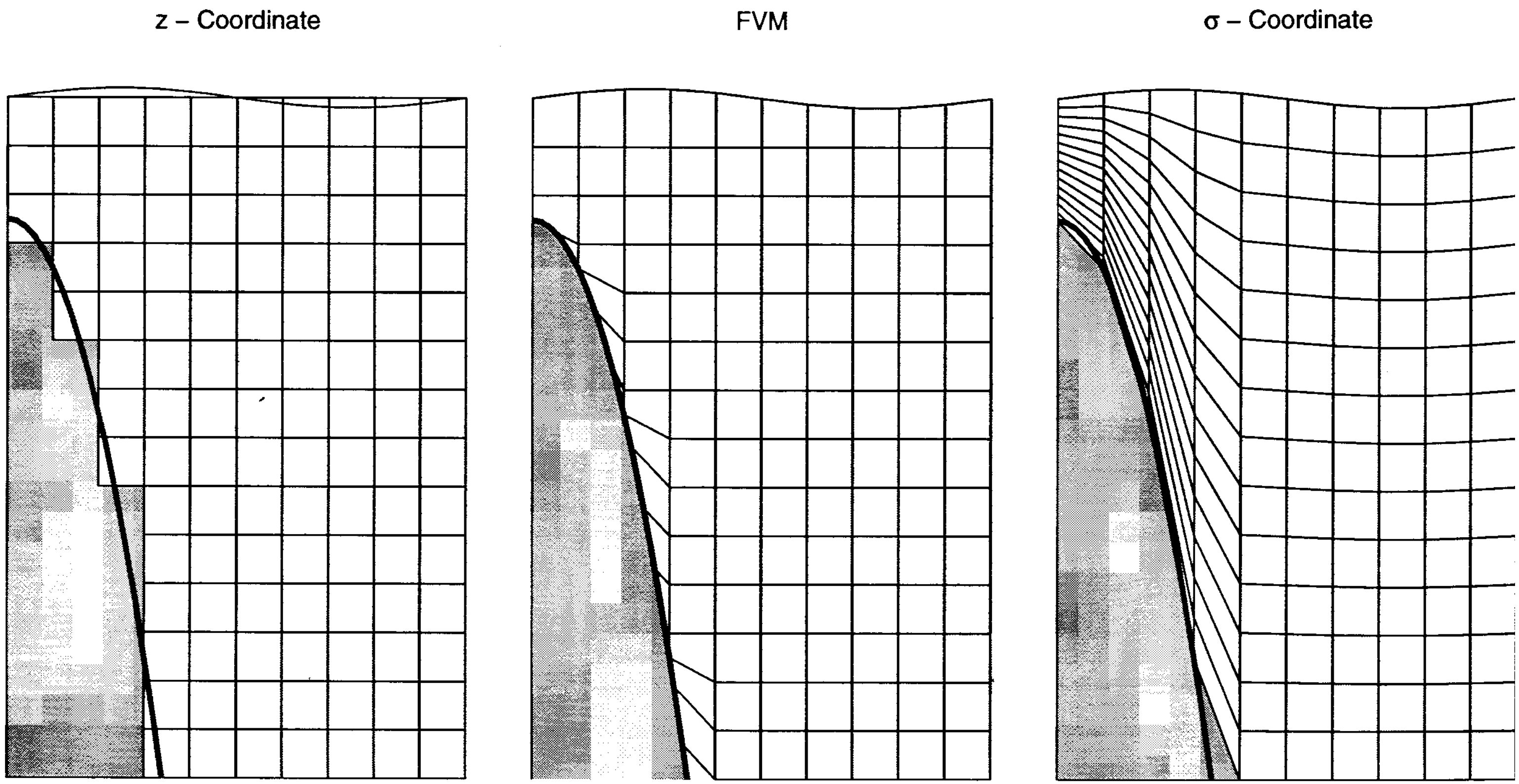
$$\int_\Omega \frac{\partial \phi}{\partial t} d\Omega + \oint_\Gamma \phi \mathbf{V} \cdot \mathbf{n} d\Gamma = \oint_\Gamma \kappa_\phi \nabla \phi \cdot \mathbf{n} d\Gamma + \int_\Omega F_\phi d\Omega$$

Time Integration of Phi-Equation

$$\int_{\Omega} \phi(t_2) d\Omega - \int_{\Omega} \phi(t_1) d\Omega = - \Delta t \oint_{\Gamma} \phi(t^*) \mathbf{V} \cdot \mathbf{n} d\Gamma$$

$$+ \Delta t \oint_{\Gamma} \kappa_{\phi} \nabla \phi(t^*) \cdot \mathbf{n} d\Gamma + \Delta t \int_{\Omega} F_{\phi}(t^*) d\Omega \text{ -}$$

Comparison Between Finite Difference (z- and sigma-coordinates) and Finite Volume Schemes



CFL Condition

$U \, \Delta x / \Delta t < 1$

Easily to identify in z-coordinate and finite volume models.

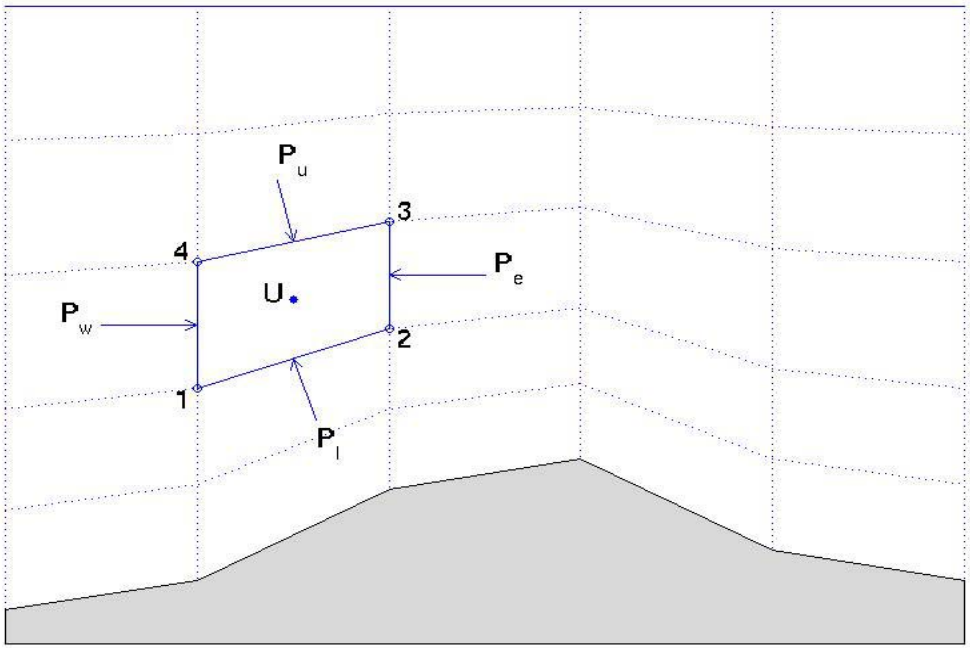
Hard to identify in sigma-coordinate model

How to reduce errors with low computer cost?

Hydrostatic Correction

- Chu, P.C., and C.W. Fan, 2003: Hydrostatic correction for reducing horizontal pressure gradient errors in sigma coordinate models. Journal of Geophysical Research, J. Geophys. Res. Vol. 108, No. C6, 3206, 10.1029/2002JC001668.

Finite Volume Consideration



$$\mathbf{F} = L_y \oint_C p\mathbf{n}ds$$

Horizontal Pressure Gradient

$$\frac{\partial p}{\partial x} \equiv \frac{F_x}{L_y \Delta S} = -\frac{1}{\Delta S}\left(\int_1^2 p_l dz + \int_2^3 p_e dz + \int_3^4 p_u dz + \int_4^1 p_w dz \right)$$

Second-Order Scheme

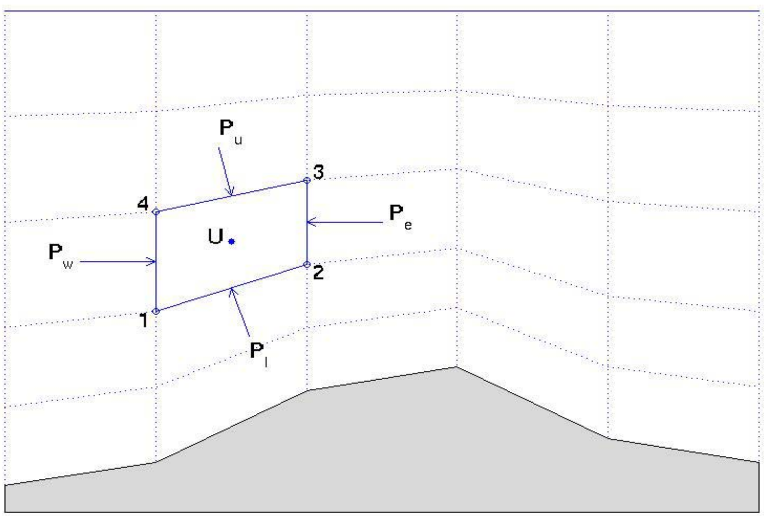
$$\frac{\Delta p}{\Delta x} = \frac{1}{\Delta S}\left[\begin{array}{l} \overline{p}_l(z_{i+1,k+1} - z_{i,k+1}) + \overline{p}_e(z_{i+1,k} - z_{i+1,k+1}) \\ +\overline{p}_u(z_{i,k} - z_{i+1,k}) + \overline{p}_w(z_{i,k+1} - z_{i,k}) \end{array}\right]$$

$$\int_1^2 p_l dz = \overline{p}_l(z_{i+1,k+1} - z_{i,k+1}), \quad \int_2^3 p_e dz = \overline{p}_e(z_{i+1,k} - z_{i+1,k+1}),$$

$$\int_3^4 p_l dz = \overline{p}_u(z_{i,k} - z_{i+1,k}), \quad \int_4^1 p_w dz = \overline{p}_w(z_{i,k+1} - z_{i,k}),$$

$$\left(\frac{\Delta p}{\Delta x}\right)_{i,k} = \frac{\left(p_{i+1,k+1} - p_{i,k}\right)\left(H_{i+1}\sigma_k - H_i\sigma_{k+1}\right) + \left(p_{i+1,k} - p_{i,k+1}\right)\left(H_i\sigma_k - H_{i+1}\sigma_{k+1}\right)}{\Delta x_i \Delta\sigma_k\left(H_i + H_{i+1}\right)}$$

Steep Topography – Hermit Polynomial Integration



$$\xi = \frac{l - l_1}{\Delta l}, \quad \Delta l \equiv l_2 - l_1$$

$$\int_{l_1}^{l_2} p\left(l\right) dl = \Delta l \int_0^1 \Psi\left(\xi\right) d\xi$$

Hermit Polynomial Integration

$$\int_{l_1}^{l_2} p\,dl = \int_0^1 \Psi(\xi)\,d\xi = \frac{\Delta l}{2}\left(p_1 + p_2\right) + \frac{\Delta l^2}{12}\left(\left(\frac{\partial p}{\partial l}\right)_1 - \left(\frac{\partial p}{\partial l}\right)_2\right)$$

$$\Psi\left(\xi\right) = p_{l_1}\Phi_1 + p_{l_2}\Phi_2 + \Delta l\left(\frac{\partial p}{\partial l}\right)_{l_1}\Phi_3 + \Delta l\left(\frac{\partial p}{\partial l}\right)_{l_2}\Phi_4$$

$$\Phi_1 = 1 - 3\xi^2 + 2\xi^3, \quad \Phi_2 = 3\xi^2 - 2\xi^3,$$

$$\Phi_3 = \xi - 2\xi^2 + \xi^3, \quad \Phi_4 = \xi^3 - \xi^2$$

Hydrostatic Correction

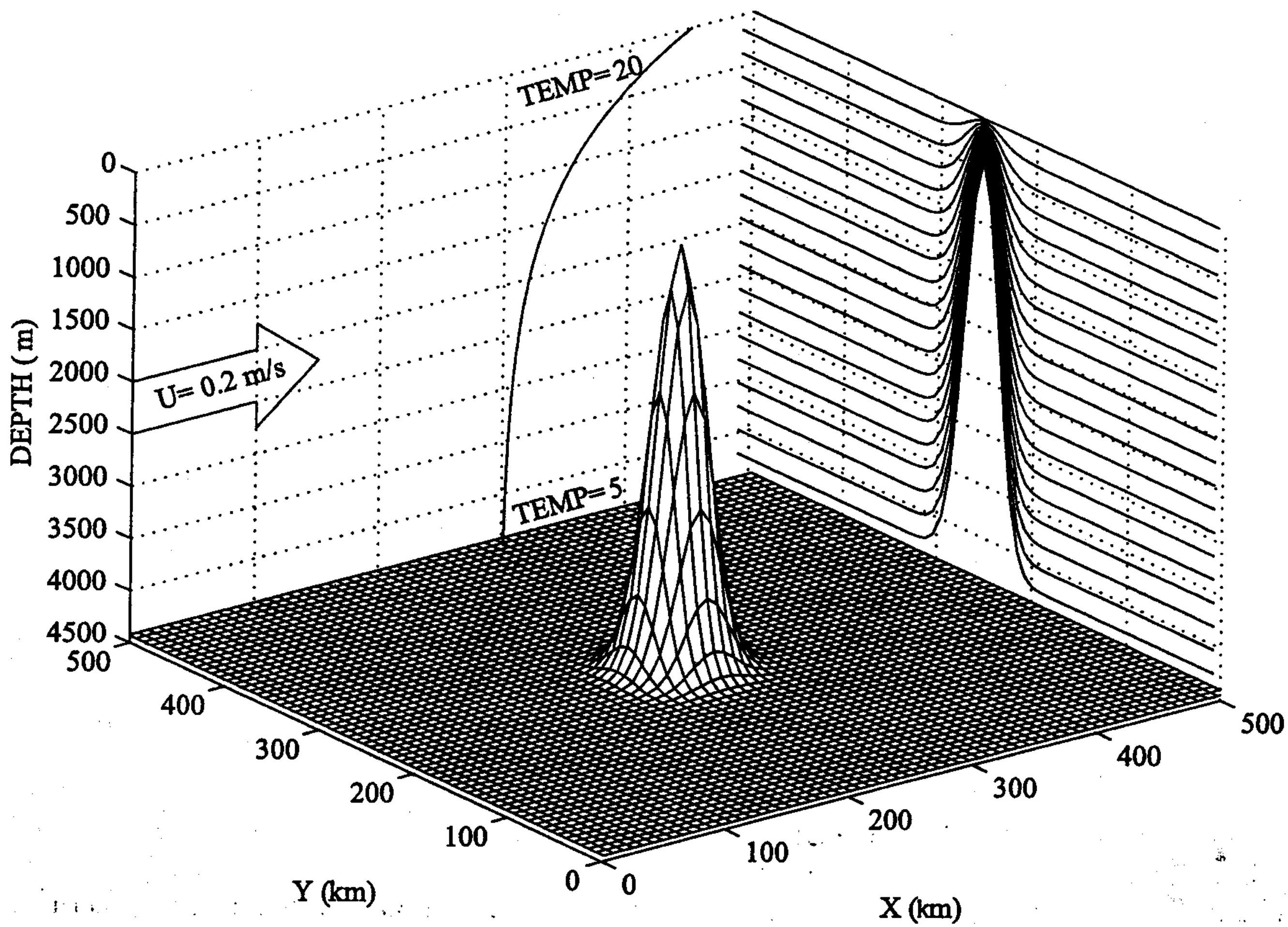
$$\left(\frac{\Delta p}{\Delta x}\right)_{i,k} \simeq \frac{\left[\left(p_{i+1,k+1}-p_{i,k}\right)\left(z_{i+1,k}-z_{i,k+1}\right)+\left(p_{i+1,k}-p_{i,k+1}\right)\left(z_{i,k}-z_{i+1,k+1}\right)\right]}{\left(x_{i+1}-x_{i}\right)\left(z_{i,k}+z_{i+1,k}-z_{i,k+1}-z_{i+1,k+1}\right)}+\Omega_{ik}$$

$$\Omega_{ik}=\frac{g\Gamma_{ik}}{6\left(x_{i+1}-x_{i}\right)\left(z_{i,k}+z_{i+1,k}-z_{i,k+1}-z_{i+1,k+1}\right)},$$

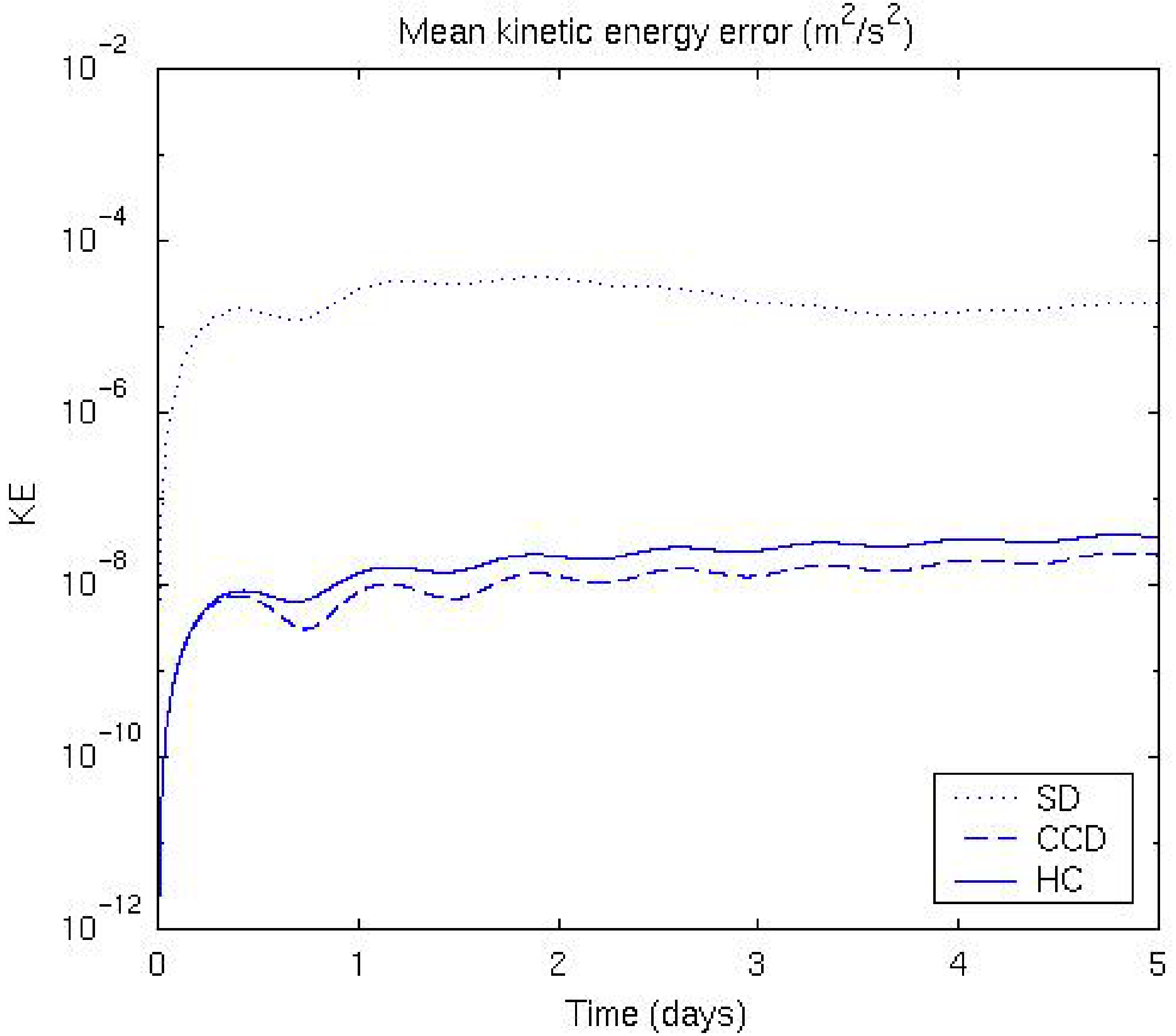
$$\Gamma_{ik}\equiv\left[\left(H_{i+1}\sigma_{k}-H_{i+1}\sigma_{k+1}\right)^{2}\left(\rho_{i+1,k}-\rho_{i+1,k+1}\right)-\left(H_{i}\sigma_{k}-H_{i}\sigma_{k+1}\right)^{2}\left(\rho_{i,k}-\rho_{i,k+1}\right)\right.$$

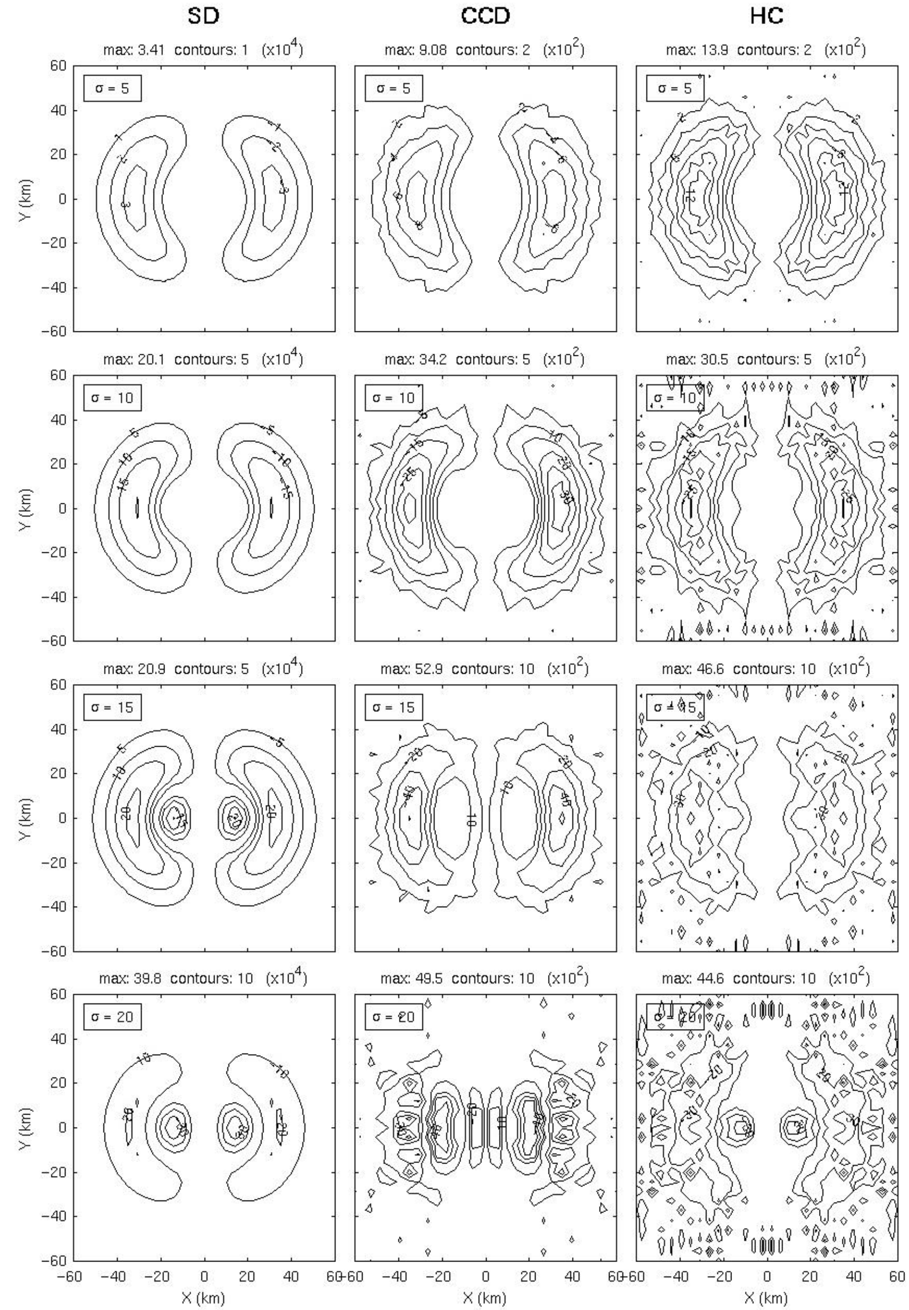
$$\left.+\left(H_{i+1}\sigma_{k+1}-H_{i}\sigma_{k+1}\right)^{2}\left(\rho_{i+1,k+1}-\rho_{i,k+1}\right)-\left(H_{i+1}\sigma_{k}-H_{i}\sigma_{k}\right)^{2}\left(\rho_{i+1,k}-\rho_{i,k}\right)\right]$$

Verification of Current Schemes by Ezer, Arango, Shchepetkin (OCEMOD, 2002)



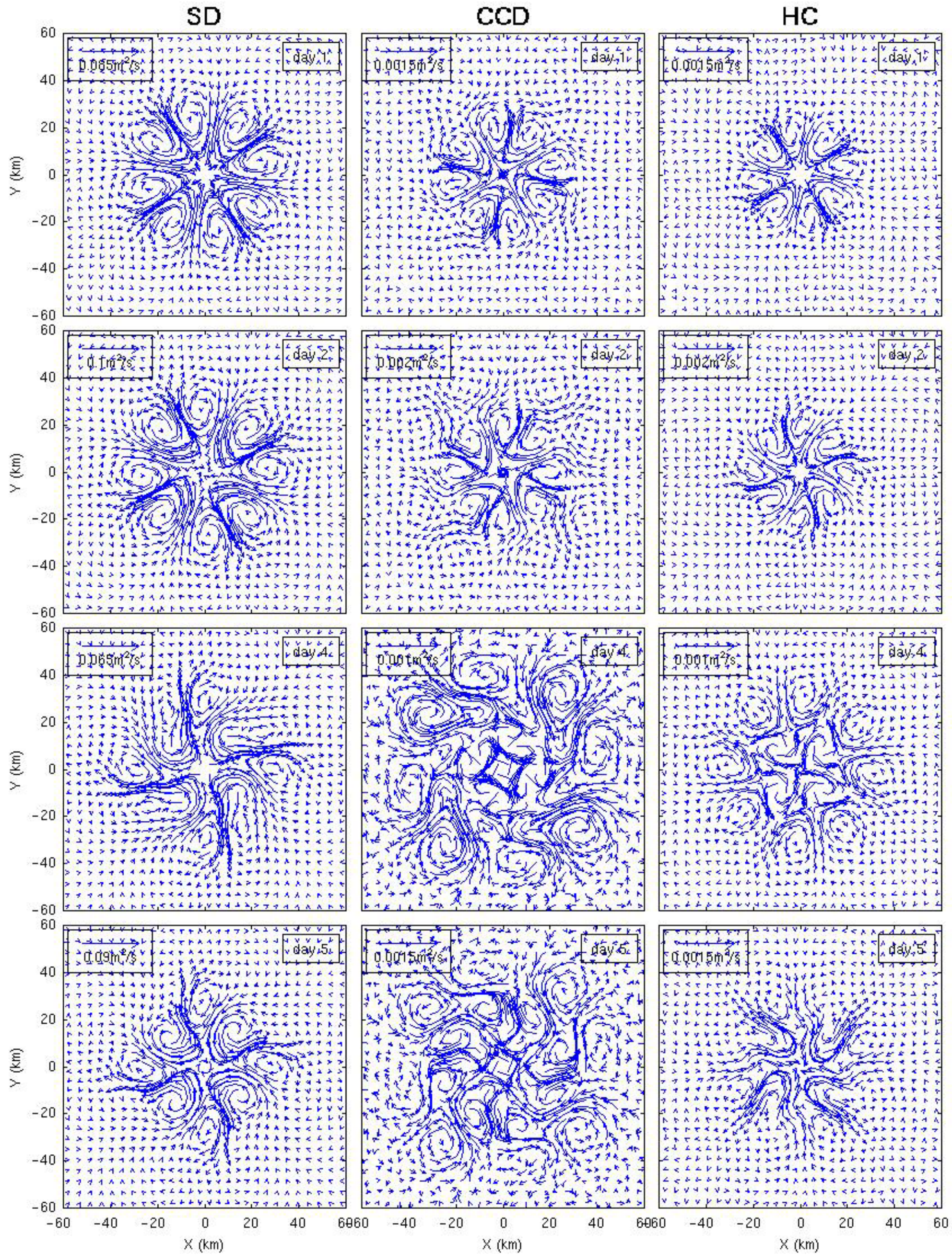
HC has the same accuracy as CCD Scheme





SD
CCD
HC
max: 3.41 contours: 1 (x10^4)
max: 9.08 contours: 2 (x10^2)
max: 13.9 contours: 2 (x10^2)
σ = 5
max: 20.1 contours: 5 (x10^4)
max: 34.2 contours: 5 (x10^2)
max: 30.5 contours: 5 (x10^2)
σ = 10
max: 20.9 contours: 5 (x10^4)
max: 52.9 contours: 10 (x10^2)
max: 46.6 contours: 10 (x10^2)
σ = 15
max: 39.8 contours: 10 (x10^4)
max: 49.5 contours: 10 (x10^2)
max: 44.6 contours: 10 (x10^2)
σ = 20
Y (km)
X (km)

- HC has the same accuracy as the sixth-order CCD scheme



CPU time (minute) at the end of 5 day runs of the POM seamount test case using the SD and HC schemes

Scheme	Second-order	HC
CPU Time	171.51	176.92
Ratio	1	1.03

Conclusions

- Finite volume model combines the strength of z- and sigma-coordinate models.
- Hydrostatic correction scheme is a simple scheme with high accuracy and efficiency.
- The accuracy of the HC scheme is similar to the sixth-order CCD scheme.
- The CPU cost is similar to the ordinary second-order scheme.