Model Valid Predictability Period (VPP)

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References

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- Chu, P.C., L.M. Ivanov, and C.W. Fan, 2002: Backward Fokke-Planck equation for determining model valid prediction period. Journal of Geophysical Research, in press.
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Question

- How long is an ocean (or atmospheric) model valid once being integrated from its initial state?

- Or what is the model valid prediction period (VPP)?

Atmospheric & Oceanic Model (Dynamic System with Stochastic Forcing)

-
  - $d\,\boldsymbol{X}/dt = \boldsymbol{f}(\boldsymbol{X}, t) + q(t)\,\boldsymbol{X}$
  - Initial Condition: $\boldsymbol{X}(t_0) = \boldsymbol{X}_0$
  - Stochastic Forcing:
  - $\langle q(t) \rangle = 0$
  - $\langle q(t)q(t') \rangle = q^2 \delta(t-t')$

Prediction Model

- $\boldsymbol{Y}$ --- Prediction of $\boldsymbol{X}$
- Model: $d\boldsymbol{Y}/dt = \boldsymbol{h}(\boldsymbol{y}, t)$
- Initial Condition: $\boldsymbol{Y}(t_0) = \boldsymbol{Y}_0$

Model Error

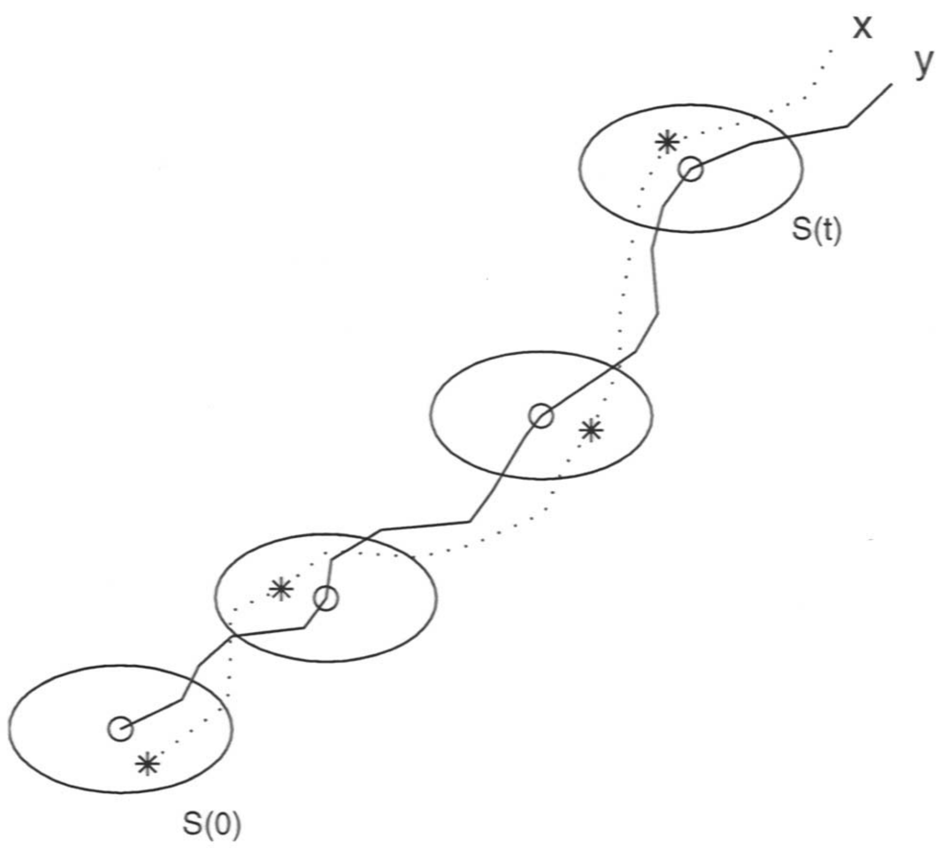
-
- $$Z = X - Y$$

- Initial: $Z_0 = X_0 - Y_0$

Definition of VPP

- VPP is defined as the time period when the prediction error first exceeds a pre-determined criterion (i.e., the tolerance level ε).

VPP



Predictability

- Conventional
  - Error   Growth
  - $\boldsymbol{Z}(t) = ?$
  - For operational model, the vector $\boldsymbol{Z}$ may have many components
- Using VPP
  - One Scalar

Uncertain Initial Error

- The prediction is meaningful only if

$$Var(\mathbf{Z}) \le \varepsilon^2 \quad \Leftarrow \quad ellipsoid \;\; S_\varepsilon(t)$$

-
- VPP time period $(t - t_0)$

- Such that $\mathbf{Z} \in S_\varepsilon(t)$

Conditional Probability Density Function

- Initial Error: $\boldsymbol{z}_0$
- $(t - t_0)$ Random Variable
- Conditional PDF of $(t - t_0)$ with given $\boldsymbol{z}_0$
  - $P[(t - t_0) \mid \boldsymbol{z}_0]$
  - Backward Fokker-Planck Equation

Backward Fokker-Planck Equation

$$\frac{\partial P}{\partial t} - \left[\mathbf{f}(\mathbf{z}_0, t)\right]\frac{\partial P}{\partial \mathbf{z}_0} - \frac{1}{2} q^2 \mathbf{z}_0{}^2 \frac{\partial^2 P}{\partial \mathbf{z}_0 \partial \mathbf{z}_0} = 0$$

Moments

$$\tau_1(\mathbf{z}_0) = \int_{t_0}^{\infty} P(t_0, \mathbf{z}_0, t - t_0)(t - t_0)\,dt$$

$$\tau_2(\mathbf{z}_0) = \int_{t_0}^{\infty} P(t_0, \mathbf{z}_0, t - t_0)(t - t_0)^2\,dt$$

Mean & Variance of VPP

- Mean VPP: tau1
- Variance of VPP:
- $tau2 - tau1^2$

Linear Equations for Mean and Variance of VPP

- For an autonomous dynamical system
- $d\,\boldsymbol{X}/\,dt = \boldsymbol{f}(\boldsymbol{X}) + q(t)\,\boldsymbol{X}$
- Integration of [Backward F-P Eq. *
- $(t - t_0)$, $(t - t_0)^2$] from t_0 to infinity.

$$\mathbf{f}(\mathbf{z}_0)\frac{\partial \tau_1}{\partial \mathbf{z}_0} + \frac{q^2 \mathbf{z}_0^2}{2}\frac{\partial^2 \tau_1}{\partial \mathbf{z}_0 \partial \mathbf{z}_0} = -1$$

$$\mathbf{f}(\mathbf{z}_0)\frac{\partial \tau_2}{\partial \mathbf{z}_0} + \frac{q^2 \mathbf{z}_0^2}{2}\frac{\partial^2 \tau_2}{\partial \mathbf{z}_0 \partial \mathbf{z}_0} = -2\tau_1 \; .$$

Example 1: One Dimensional Model (Nicolis 1992)

- 1D Dynamical System

$$\frac{d\xi}{dt} = (\sigma - g\xi^2) + v(t)\xi\,, \qquad 0 \le \xi < \infty$$

$$\langle v(t) \rangle = 0, \qquad \langle v(\mathrm{t}) v(\mathrm{t}') \rangle = q^2 \delta(t - t')\,.$$

$$\sigma = 0.64, \qquad g = 0.3\,, \qquad q^2 = 0.2\,.$$

Mean and Variance of VPP

$$(\sigma\xi_0 - g\xi_0^2)\frac{d\tau_1}{d\xi_0} + \frac{q^2\xi_0^2}{2}\frac{d^2\tau_1}{d\xi_0^2} = -1$$

$$(\sigma\xi_0 - g\xi_0^2)\frac{d\tau_2}{d\xi_0} + \frac{q^2\xi_0^2}{2}\frac{d^2\tau_2}{d\xi_0^2} = -2\tau_1$$

$\tau_1 = 0, \quad \tau_2 = 0$ for $\xi_0 = \varepsilon$.

$\frac{d\tau_1}{d\xi_0} = 0, \quad \frac{d\tau_2}{d\xi_0} = 0$ for $\xi_0 = \xi_{noise}$.

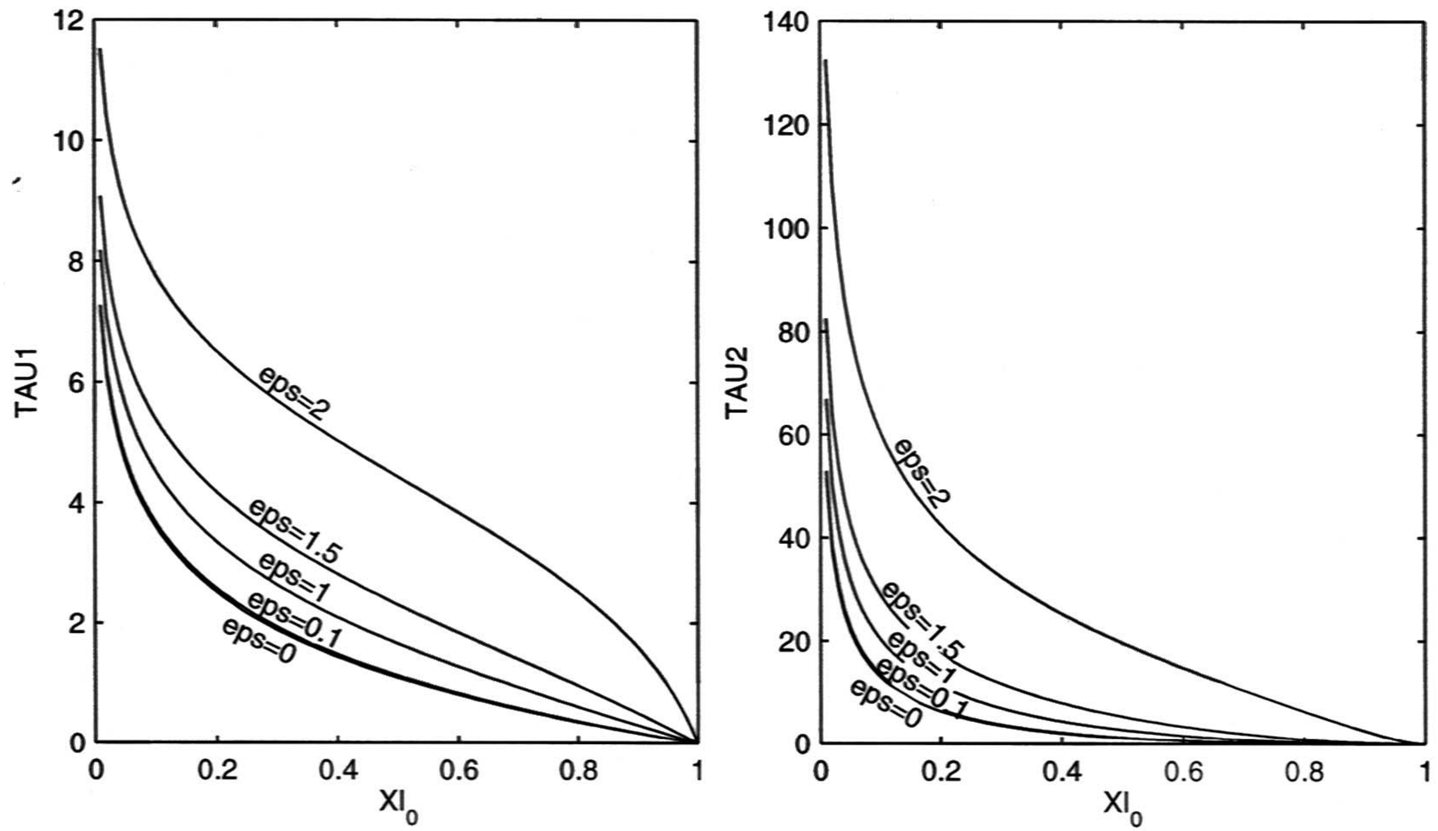
Analytical Solutions

$$\tau_1(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon) = \frac{2}{q^2}\int_{\bar{\xi}_0}^{1} y^{-\frac{2\sigma}{q^2}} \exp\left(\frac{2\varepsilon g}{q^2} y\right)\left[\int_{\bar{\xi}_{noise}}^{y} x^{\frac{2\sigma}{q^2}-2} \exp\left(-\frac{2\varepsilon g}{q^2} x\right) dx\right] dy$$

$$\tau_2(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon) = \frac{4}{q^2}\int_{\bar{\xi}_0}^{1} y^{-\frac{2\sigma}{q^2}} \exp\left(\frac{2\varepsilon g}{q^2} y\right)\left[\int_{\bar{\xi}_{noise}}^{y} \tau_1(x) x^{\frac{2\sigma}{q^2}-2} \exp\left(-\frac{2\varepsilon g}{q^2} x\right) dx\right] dy$$

$\bar{\xi}_0 = \xi_0 / \varepsilon$, $\bar{\xi}_{noise} = \xi_{noise} / \varepsilon$

Dependence of tau1 & tau2 on Initial Condition Error (ξ_0/ε)

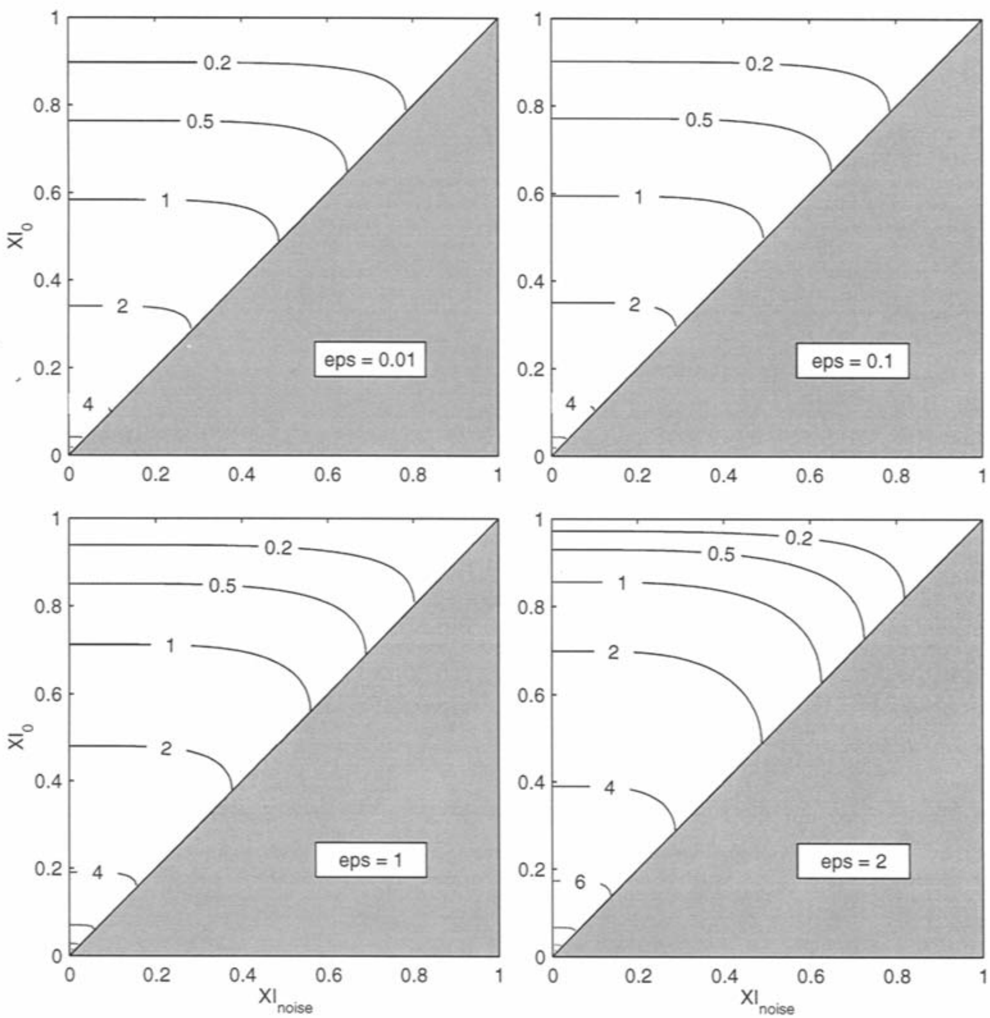


Small Tolerance Error ($\varepsilon \to 0$)

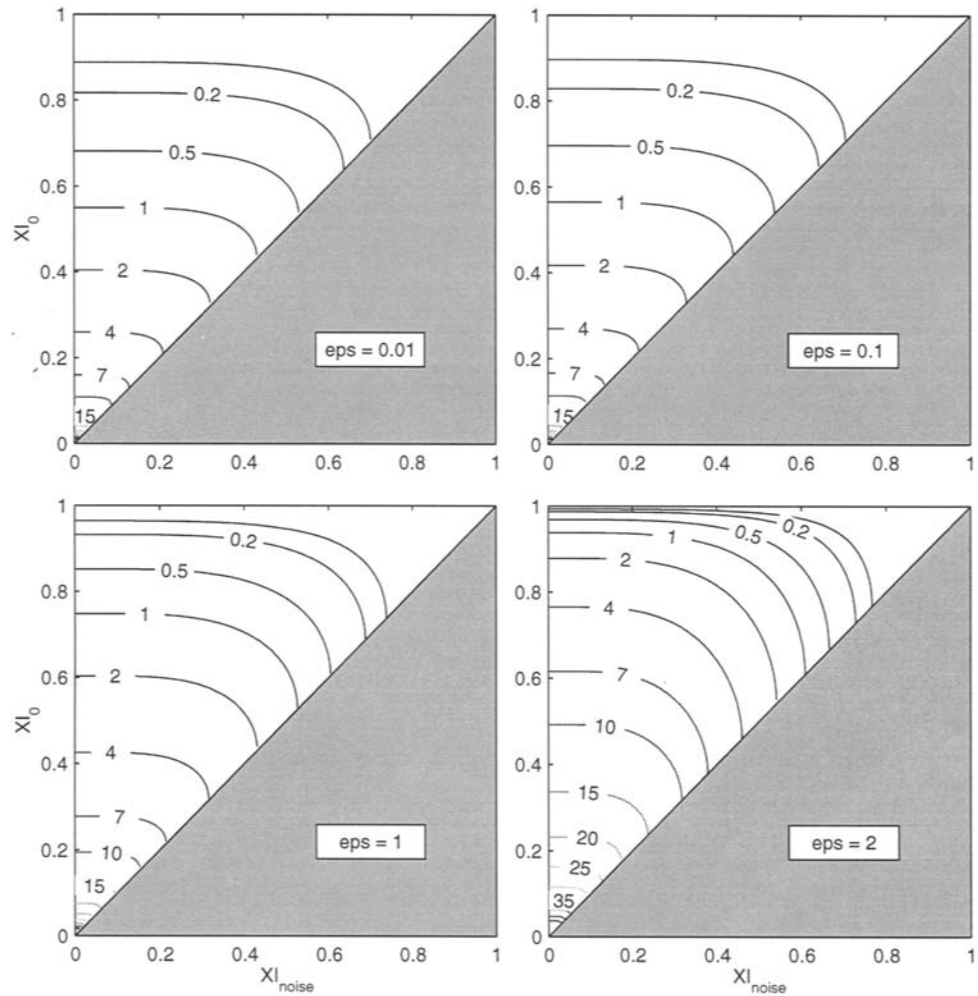
$$\lim_{\varepsilon \to 0} \tau_1(\bar{\xi}_0, \bar{\xi}_{noise}, \varepsilon) = \frac{1}{\sigma - q^2/2}\left[\ln\left(\frac{1}{\bar{\xi}_0}\right) - \frac{q^2}{2\sigma - q^2}\left(\frac{\bar{\xi}_{noise}}{\bar{\xi}_0}\right)^{\frac{2\sigma}{q^2}-1} + \frac{q^2}{2\sigma - q^2}\bar{\xi}_{noise}^{\frac{2\sigma}{q^2}-1}\right]$$

- (1) Lyapunov Exponent: ($\sigma - q^2/2$)
- (2) Stochastic Forcing (q 0):
  - Multiplicative White Noise
  - Reducing the Lyapunov exponent (Stabilizing the dynamical system)

Dependence of Mean VPP on initial error and tolerance level



Dependence of Variance of VPP on initial error and tolerance level



Example 2: Multi-Dimensional Models: Power Decay Law in VPP

Model Error

-
- $$Z = X - Y$$
- Initial: $Z_0 = X_0 - Y_0$

Error Mean and Variance

Error Mean $$\mathbf{L}_1 = \langle \mathbf{z} \rangle$$

Error Variance $$\mathbf{L}_2 = \left\langle \left(\mathbf{z} - \langle \mathbf{z} \rangle \right)^t \left(\mathbf{z} - \langle \mathbf{z} \rangle \right) \right\rangle$$

Exponential Error Growth

$$L_1 \propto e^{\sigma t}, \qquad L_2 \propto e^{\omega t},$$

Classical Linear Theory

No Long-Term Predictability

Power Law

$$L_1 \propto t^{\alpha}, \qquad L_2 \propto t^{\beta},$$

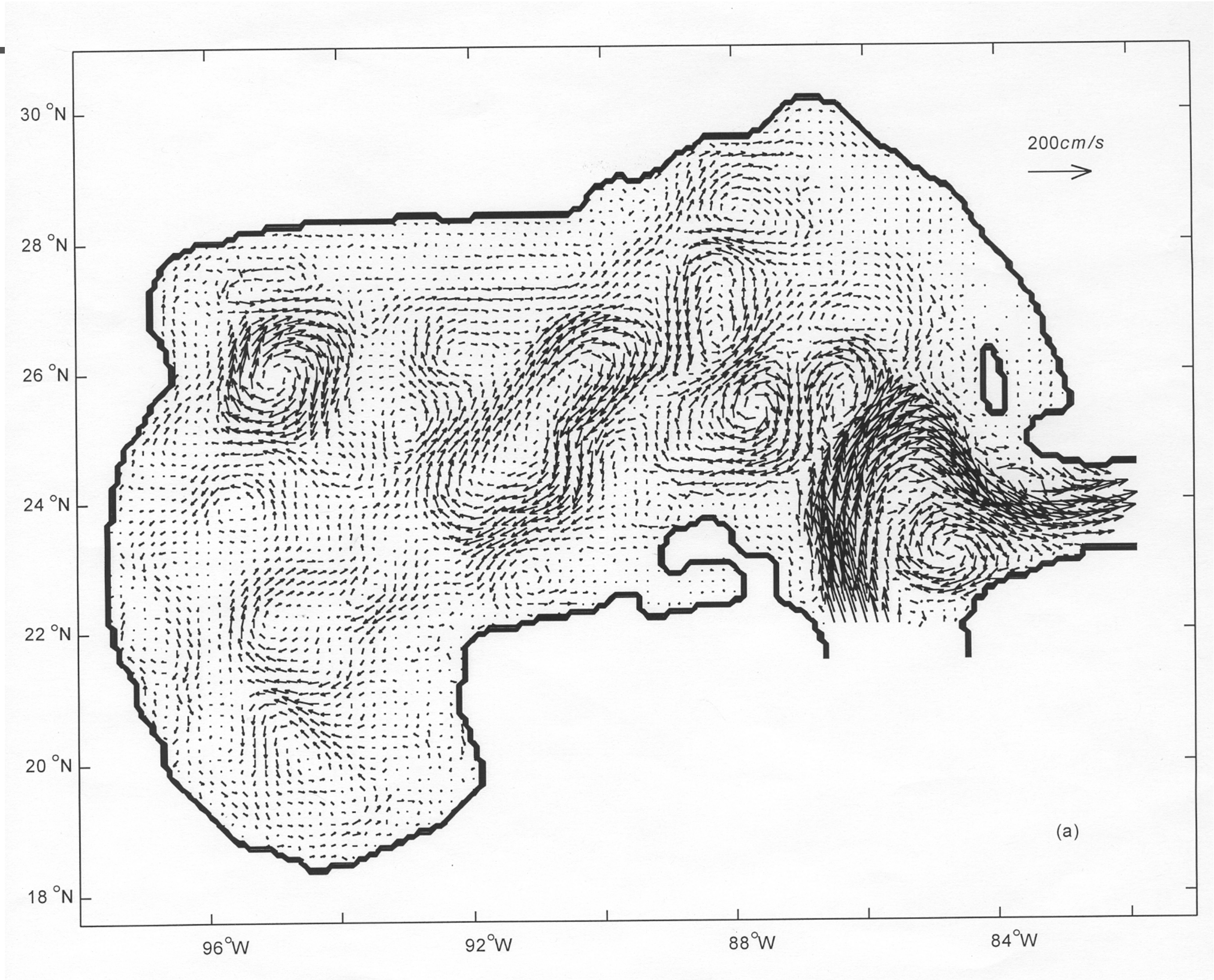
$$P(t_0, \mathbf{z}_0, \varepsilon, t - t_0) \sim t^{-\gamma} \quad \text{for large } t.$$

Long-Term Predictability May Occur

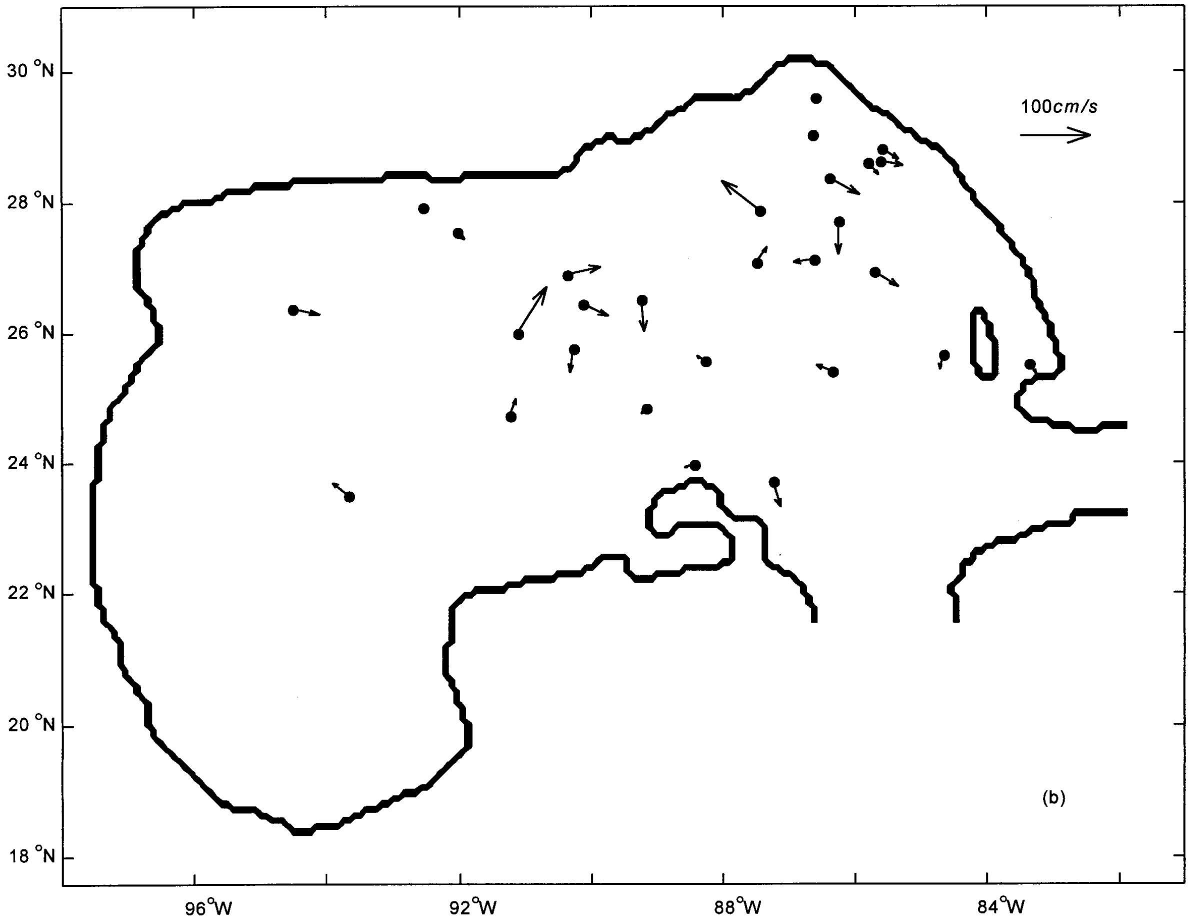
Gulf of Mexico Nowcast/Forecast System

- University of Colorado Version of POM
- 1/12^{o} Resolution
- Real-Time SSH Data (TOPEX, ESA ERS-1/2) Assimilated
- Real Time SST Data (MCSST, NOAA AVHRR) Assimilated
- Six Months Four-Times Daily Data From July 9, 1998 for Verification

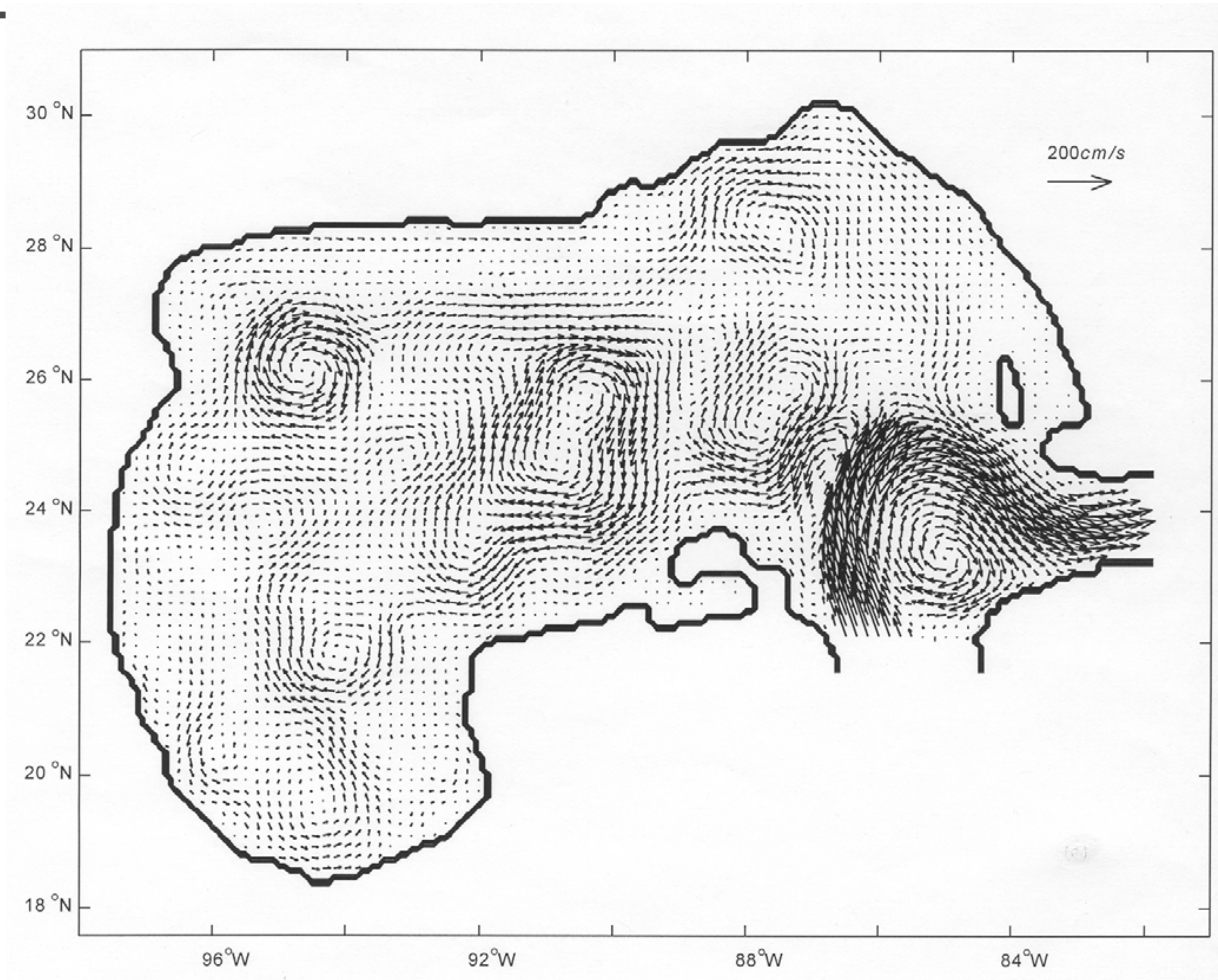
Model Generated Velocity Vectors at 50 m on 00:00 July 9, 1998



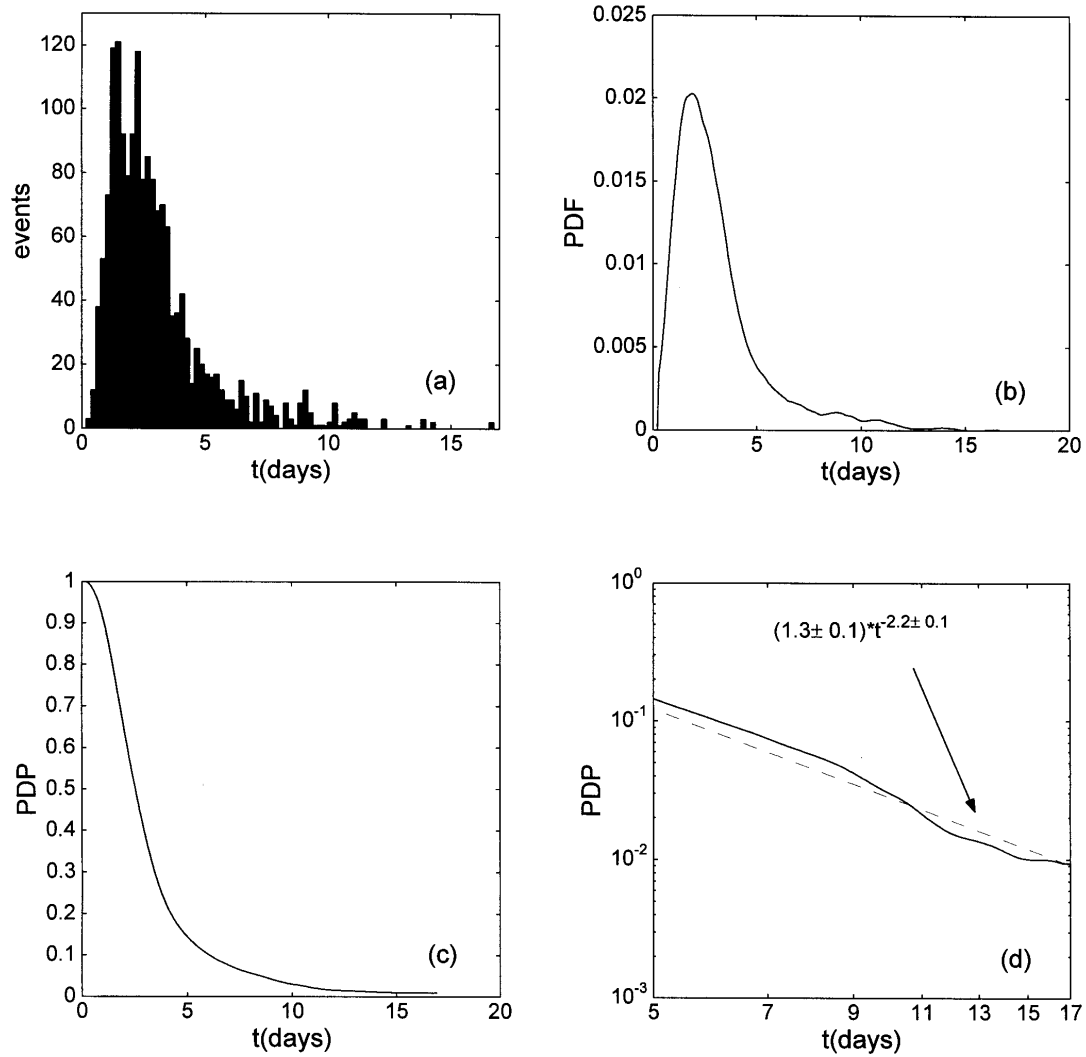
(Observational) Drifter Data at 50 m on 00:00 July 9, 1998



Reconstructed Drift Data at 50 m on 00:00 July 9, 1998 (Chu et al. 2002 a, b, JTECH)



Statistical Characteristics of VPP for zero initial error and 55 km tolerance level (Non-Gaussion)



Scaling behavior of the

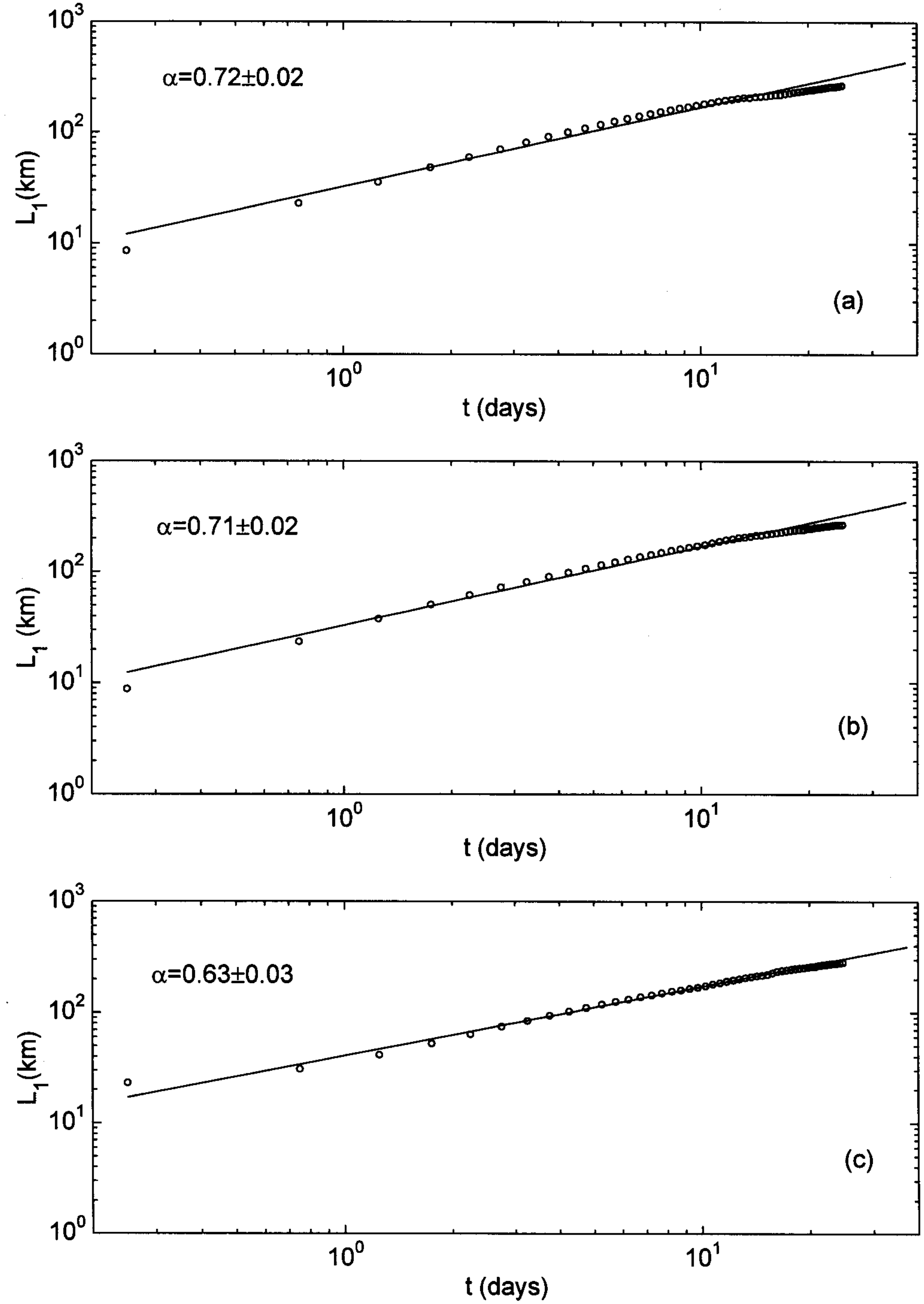
mean error (L_1) growth

for initial error levels:

(a) 0

(b) 2.2 km

(c) 22 km



Scaling behavior of the

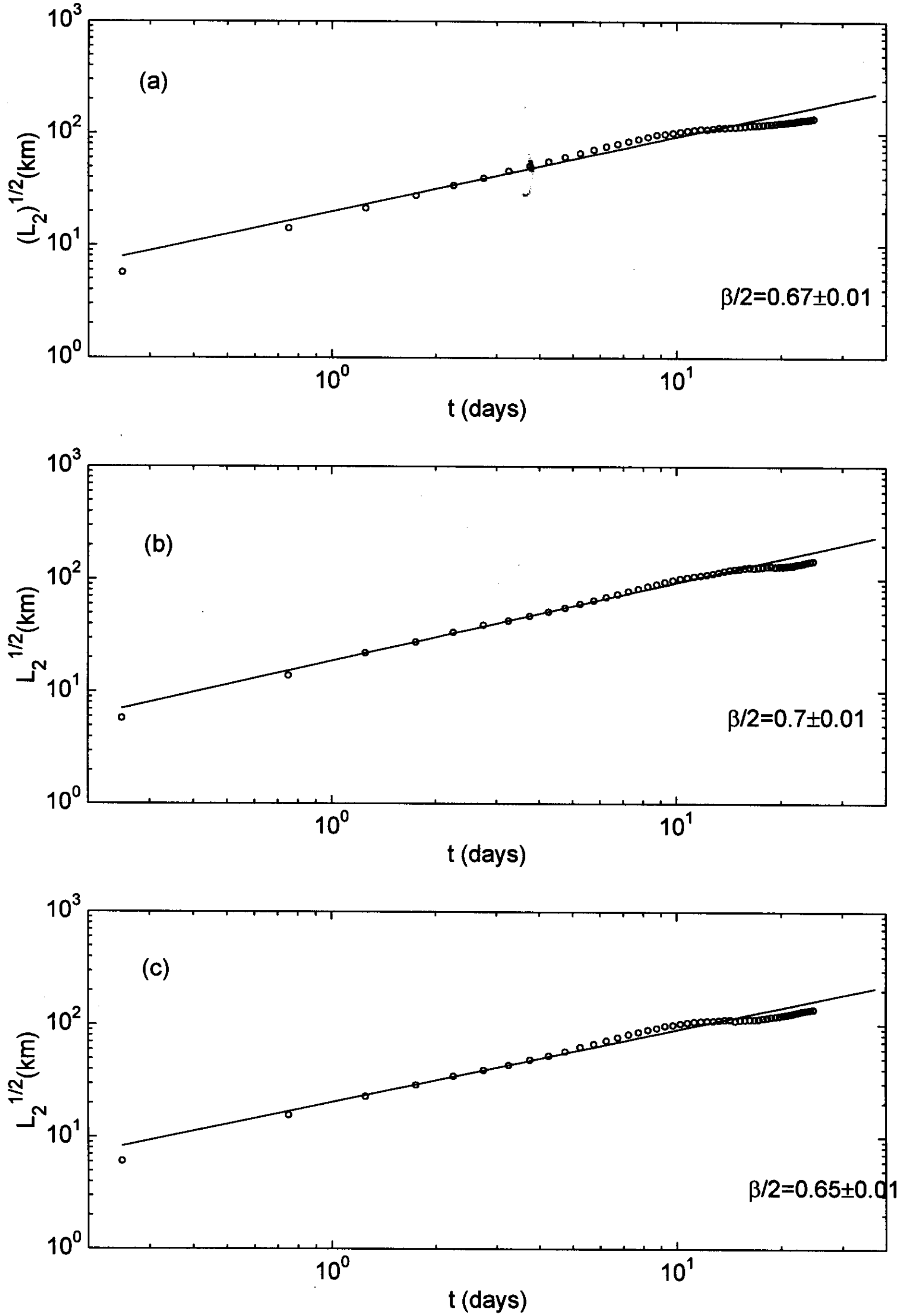
Error variance (L_2) growth

for initial error levels:

(a) 0

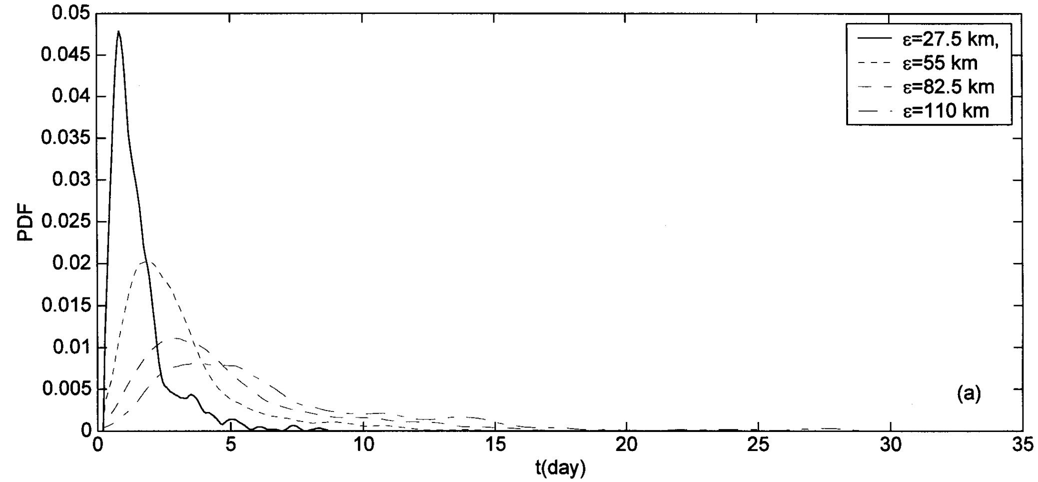
(b) 2.2 km

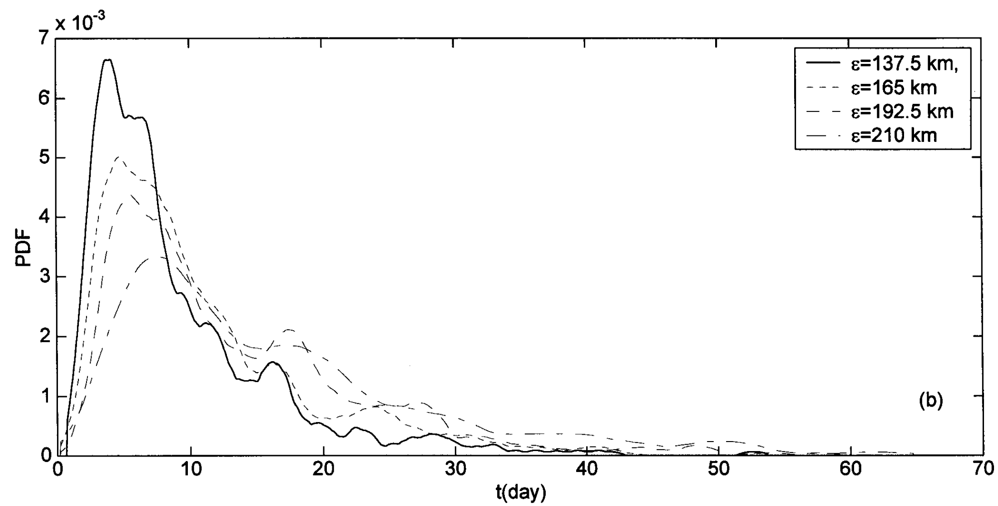
(c) 22 km



Probability Density Function of VPP calculated with different tolerance levels

Non-Gaussian distribution with long tail toward large values of VPP (Long-term Predictability)





Conclusions

- (1) VPP is an effective prediction skill measure (scalar).
- (2) Backward Fokker-Planck equation is a useful tool for predictability study.
- (3) Stochastic-Dynamic Modeling