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First-passage time for stability analysis of the Kaldor model

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Abstract

Every economic model should include an estimate of its stability and predictability. A new measure, the first passage time (FPT) which is defined as the time period when the model error first exceeds a pre-determined criterion (i.e., the tolerance level), is proposed here to estimate the model predictability. A theoretical framework is developed to determine the mean and variance of FPT. The classical Kaldor model is taken as an example to show the robustness of using FPT as a quantitative measure for identifying the model stability.
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1. Introduction

One of the most interesting theories of business cycles in the Keynesian vein is that expounded in a pioneering article by Kaldor [1]. It is distinguishable from most other contemporary treatments since it utilizes non-linear functions, which produce endogenous cycles, rather than the linear multiplier-accelerator kind which rely largely on exogenous factors to maintain regular cycles.

The savings and investment functions in the income-expenditure theory of Keynes are linear. Kaldor [1] proposed that the treatment of savings and investment as linear curves simply does not correspond to empirical reality. Let (Y, K) denote gross product and capital stock. The investment function $I(Y, K)$ and savings function $S(Y, K)$ are increasing functions with respect to Y. Over the savings and investment functions, Kaldor superimposed Keynes's multiplier theory, namely, that gross product changes to clear the goods market.

Chang and Smyth [2] and Varian [3] translated Kaldor's trade cycle model into more rigorous context: the former into a limit cycle and the latter into catastrophe theory. Output, as we saw via the theory of the multiplier, responds to the difference between savings and investment. If there is excess goods demand (which translates to saying that investment exceeds savings, $I > S$), then gross product rises (d Y/d$t > 0$), whereas if there is excess goods supply (which translates to savings exceeding investment, $I < S$), then Y falls. The Kaldor system is represented by

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = \mu(I - S), \tag{1a}$$

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$$\frac{dK}{dt} = I,\tag{1b}$$

where μ is the adjustment coefficient in the goods market.

It is widely recognized that the uncertainty can be traced back to three factors: (a) measurement errors, (b) model errors such as uncertain model parameters, and (c) chaotic dynamics. Measurement errors cause uncertainty in initial conditions. Discretization causes truncation errors. The chaotic dynamics may trigger a subsequent amplification of small errors through a complex response.

Traditionally, the small amplitude stability analysis (linear error dynamics) is used to study the model stability. This method is divided into two steps. First step is to find equilibrium states of the dynamic system. The second step is to investigate temporal evolution of perturbations relative to the equilibrium states. It is well known that comparative static analysis is only valid if there is a tendency of the variables towards the new equilibrium. If such movement does not take place, the comparative static exercises do not really contribute to the knowledge of the evolution of the economic systems [9]. In modeling the endogenous business cycles, the macroeconomic equilibrium is never reached, but oscillatory motion is usually found around it. Therefore, it is more appropriate to examine the comparative dynamics of the systems (1a) and (1b), i.e., to investigate the whole change over time caused by a change in initial conditions, exogenous parameters or reaction coefficients [4]. In the context of business cycle theory, the change of amplitude of business cycle is caused by the changes in the parameters and initial conditions.

A question arises: how long is the Kaldor model (1) valid since being integrated from its initial state? This has great practical significance. For example, if the model validity time shorter than the business cycle, the model does not have any capability to predict the business cycle. In this paper, probabilistic stability analysis is proposed to investigate the model valid period. This method is on the base of the first-passage time (FPT) for model prediction.

2. FPT for prediction

Let an N-dimensional vector, $\mathbf{x}(t) = [x^{(1)}(t), x^{(2)}(t), \ldots, x^{(N)}(t)]$ represent a set of economic variables governed by

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t),\tag{2}$$

which is the extension of (1a) and (1b). Here $\mathbf{f}$ is a functional. Individual economic prediction is to find the solution of (2) with an initial condition

$$\mathbf{x}(t_0) = \mathbf{x}_0.\tag{3}$$

Uncertainty in economic models leads to the addition of stochastic forcing. For simplicity, a stochastic forcing ($\mathbf{f}'$) is assumed to be white multiplicative or additive noise, and (2) becomes

$$\frac{d\hat{\mathbf{x}}}{dt} = \mathbf{f}(\hat{\mathbf{x}}, t) + \mathbf{f}'(\hat{\mathbf{x}}, t), \quad \mathbf{f}'(\hat{\mathbf{x}}, t) = \mathbf{k}(\hat{\mathbf{x}}, t)\,\mathbf{g}(t),\tag{4}$$

where $\mathbf{k}(\mathbf{x}, t)$ and $\mathbf{g}(t)$ are the forcing covariance matrix $\{\mathbf{k}_{ij}\}$ (dimension of $N \times N$) and the vector delta-correlated process (dimension of N), respectively.

Let $\hat{\mathbf{x}}(t)$ be the reference solution which satisfies (3) with the initial condition, $\hat{\mathbf{x}}(t_0) = \mathbf{x}_0$. The model error $\mathbf{z}$ is determined as

$$\mathbf{z}(t) = \hat{\mathbf{x}}(t) - \mathbf{x}'(t),\tag{5}$$

where $\mathbf{x}'(t)$ is one of individual prediction. Two vectors $\hat{\mathbf{x}}(t)$ and $\mathbf{x}'(t)$ are considered as reference and prediction points in the N-dimensional phase space.

To quantify FPT, we first define two model error limits. First, the forecast error cannot be less than a minimum scale δ, which depends on the intrinsic noises existing in the model. Second, the forecast error cannot be more than a maximum scale (tolerance level)ε. The prediction is valid if the reference point $\hat{\mathbf{x}}(t)$ is situated inside the ellipsoid (S_ε, called tolerance ellipsoid) with center at $\mathbf{x}'(t)$ and size ε. When $\hat{\mathbf{x}}(t)$ coincides with $\mathbf{x}'(t)$, the model has perfect prediction. The prediction is invalid if the reference point $\hat{\mathbf{x}}(t)$ touches the boundary of the tolerance ellipsoid at the first time from the
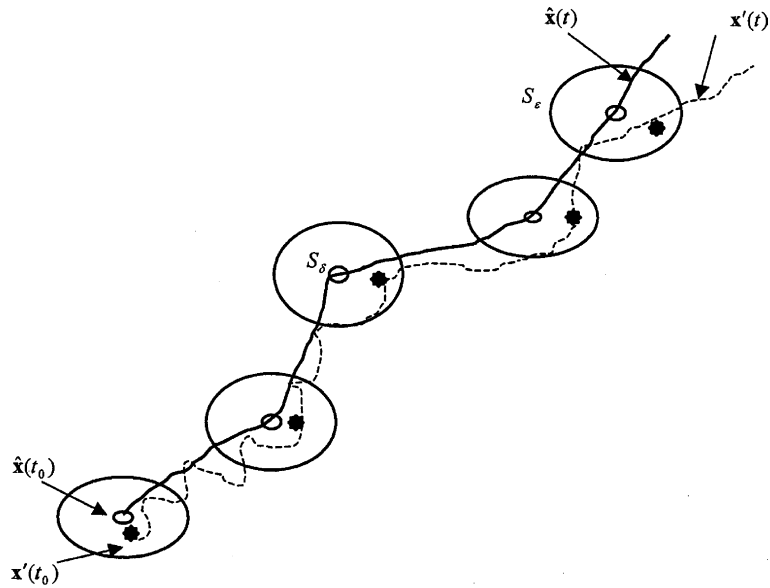
Fig. 1. Phase space trajectories of model prediction **y** (solid curve) and reality **x** (dashed curve) and error ellipsoid $S_\varepsilon(t)$ centered at **y**. The positions of reality and prediction trajectories at time instance are denoted by "∗" and "○" respectively. A valid prediction is represented by a time period $(t - t_0)$ at which the error first goes out of the ellipsoid $S_\varepsilon(t)$.

initial state that is FPT for prediction (Fig. 1). FPT is a random variable when the model has stochastic forcing or initial condition has random error. Its statistics such as the probability density function, mean and variance can represent how long the model can predict. The FPT, $\tau = t - t_0$, depends on the initial model error, $\mathbf{z}_0 \equiv \mathbf{z}(t_0)$, tolerance level ε, and model parameters. The longer the FPT, the more stable of the economic model is.

3. Backward Fokker–Planck equation

The conditional probability density function (PDF) of FPT with a given initial error, $P[(t - t_0)|\mathbf{z}_0]$, satisfies the backward Fokker–Planck equation [5,6]

$$\frac{\partial P}{\partial t} - \sum_{i=1}^{N} \left(f_i' - \frac{\mathrm{d}\hat{x}_i}{\mathrm{d}t} \right) \frac{\partial P}{\partial z_i^0} - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{l=1}^{N} k_{il} k_{lj} \frac{\partial^2 P}{\partial z_i^0 \partial z_j^0} = 0, \tag{6}$$

where the coefficients k_{ij} are the components of the forcing covariance matrix $\kappa(\mathbf{x}, t)$ and $(z_1^0, z_2^0, \ldots z_N^0)$ are the components of the initial error $\mathbf{z}_0$. Integration of PDF over t leads to

$$\int_{t_0}^{\infty} P[(t - t_0)|\mathbf{z}_0]\mathrm{d}t = 1. \tag{7}$$

The kth FPT moment ($k = 1, 2, \ldots$) is calculated by

$$\tau_k(\mathbf{z}_0) = k \int_{t_0}^{\infty} P[(t - t_0)|\mathbf{z}_0](t - t_0)^{k-1}\mathrm{d}t, \quad k = 1, \ldots, \infty. \tag{8}$$

If the initial error $\mathbf{z}_0$ reaches the tolerance level, the model loses prediction capability initially (i.e., FPT is zero)

$$P[(t - t_0)|\mathbf{z}_0] = 0 \quad \text{at } J(\mathbf{z}_0) = \varepsilon^2, \tag{9a}$$

which is the absorbing type boundary condition. Here $J(\mathbf{z}_0)$ denotes the norm of $\mathbf{z}_0$. If the initial error reaches the noise level the boundary condition becomes [6]

$$\frac{\partial P[(t - t_0)|\mathbf{z}_0]}{\partial z_0^{(j)}} = 0 \quad \text{at } J(\mathbf{z}_0) = \delta^2, \tag{9b}$$

which is the reflecting boundary conditions. Here, ξ is the noise level. Usually,

$$\delta \ll \varepsilon.$$

Mean, variance, skewness, and kurtosis of the FPT are calculated from the first four moments

$$\langle \tau \rangle = \tau_1, \tag{10a}$$

$$\langle \delta\tau^2 \rangle = \tau_2 - \tau_1^2, \tag{10b}$$

$$\langle \delta\tau^3 \rangle = \tau_3 - 3\tau_2\tau_1 + 2\tau_1^3, \tag{10c}$$

$$\langle \delta\tau^4 \rangle = \tau_4 - 4\tau_3\tau_1 + 6\tau_2\tau_1^2 - 3\tau_1^4, \tag{10d}$$

where the bracket denotes the ensemble average over realizations generated by stochastic forcing.

4. Kaldor model

4.1. Average method

The Kaldor model (1) can be written by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} Y \\ K \end{bmatrix} = \mathbf{A}\begin{bmatrix} Y \\ K \end{bmatrix} + \begin{bmatrix} f_1(Y, K) \\ f_2(Y, K) \end{bmatrix}, \tag{11}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \mu(I_Y - S_Y) & \mu(I_K - S_K) \\ I_Y & I_K \end{bmatrix}_{(Y^*, K^*)} \tag{12}$$

is the Jacobian matrix evaluated at the equilibrium $(Y^*, K^*) = (0, 0)$, and f_1 and f_2 are nonlinear terms. Here, the subscripts denote the partial differentiation; $-I_K$ is the depreciation coefficient; I_Y is the coefficient for reinvested profits. The Jacobian matrix $\mathbf{A}$ has a determinant:

$$|\mathbf{A}| = \mu(I_Y - S_Y)I_K - \mu(I_K - S_K)I_Y = \mu(S_K I_Y - I_K S_Y), \tag{13}$$

where, since $I_K < 0$ and $S_K, S_Y, I_Y > 0$ then $|\mathbf{A}| > 0$, thus we have regular (non-saddle point) dynamics.

Following Chiarella's [7] approach of using the polar coordinates, $Y = r\cos\theta$, $K = r\sin\theta$, the Kaldor model is transformed into

$$\frac{\mathrm{d}r}{\mathrm{d}t} = r[a_{11}\cos^2\theta + a_{22}\sin^2\theta + (a_{12} + a_{21})\cos\theta\sin\theta] + g_1(r, \theta)\cos\theta + g_2(r, \theta)\sin\theta, \tag{14a}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = a_{21}\cos^2\theta - a_{12}\sin^2\theta + (a_{22} - a_{11})\sin\theta\cos\theta + \frac{[g_2(r, \theta)\cos\theta - g_1(r, \theta)\sin\theta]}{r}, \tag{14b}$$

where

$$g_i(r, \theta) = f_i(r\cos\theta, r\sin\theta), \quad i = 1, 2.$$

The right-hand sides of Eqs. (14a,b) are periodic in θ and can be expanded into Fourier series with desired degree of accuracy. The first term of this expansion is obtained by averaging the right-hand sides of Eqs. (14a) and (14b) with respect to θ on the interval $[0, 2\pi]$. In this way, the equation for averaged amplitude of the business cycle [7]

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\mathrm{Tr}(\mathbf{A})}{2}\rho + G(\rho), \quad \mathrm{Tr}(\mathbf{A}) = \mu(I_Y - S_Y) + I_K > 0, \tag{15}$$

where ρ is the first-order approximation to r and

$$G(\rho) = \frac{1}{2\pi} \int_0^{2\pi} [g_1(\rho, \theta) \cos \theta + g_2(\rho, \theta) \sin \theta] d\theta. \tag{16}$$

The nonlinear term $G(\rho)$ may have several different forms. Here, we use the profit-capital-accumulation dynamic model as an example for illustration. Semmler [8] developed an endogenous cycle model of profit and capital accumulation dynamics on the base of the Kaldor dynamics. The Kaldor nonlinear term $G(\rho)$ has the following form [9]:

$$G(\rho) = -\frac{\mu}{8} \rho^3. \tag{17}$$

The coefficients (μ, I_Y, I_K, S_Y) in Eq. (15) are usually difficult to measure. This may lead to the addition of stochastic forcing to the business cycle model. For simplicity, the stochastic forcing is assumed to be white multiplicative or additive noise, and Eq. (15) becomes

$$\frac{d\rho}{dt} = \sigma\rho + G(\rho) + m(t)\rho, \quad \sigma \equiv \frac{\text{Tr}(\mathbf{A})}{2}, \tag{18a}$$

where $m(t)$ is a random parameter with zero mean and pulse-type variance,

$$\langle m(t) \rangle = 0, \quad \langle m(t)m(t') \rangle = q^2 \Delta(t - t'), \tag{18b}$$

where the bracket indicates the ensemble average; q represents the strength of the uncertainty in coefficients in σ (such as in the depreciation coefficient and the coefficient for reinvested profits); and $\Delta(t)$ is a delta function.

4.2. FPT moments

Let $\hat{\rho}(t)$ be the reference solution which satisfies (18a) with the initial condition, $\rho(t_0) = \rho_0$. The forecast error Z is determined as

$$Z(t) = \hat{\rho}(t) - \rho'(t),$$

where $\rho'(t)$ is one of individual prediction corresponded to perturbing initial condition and/or stochastic forcing. The backward Fokker–Planck Eq. (6) for this case is simplified to

$$\frac{\partial P}{\partial t} - \left[\sigma Z_0 - \frac{\mu}{8} Z_0^3\right] \frac{\partial P}{\partial Z_0} - \frac{1}{2} q^2 \frac{\partial^2 P}{\partial Z_0^2} = 0. \tag{19}$$

We multiply Eq. (19) by $(t - t_0)^n$, integrate with respect to t from t_0 to ∞, use the condition (7), and obtain the equations of the nth moment of FPT

$$\frac{q^2 \rho_0^2}{2} \frac{d^2 \tau_n}{dZ_0^2} + \left[\sigma Z_0 - \frac{\mu}{8} Z_0^3\right] \frac{d\tau_n}{dZ_0} = -n\tau_{n-1}, \quad \tau_0 = 1. \tag{20}$$

Eq. (20) is linear, time-independent, and second-order differential equations with the initial error Z_0 as the only independent variable. Two boundary conditions for τ_1 and τ_2 can be derived from (9a) and (9b)

$$\tau_k = 0, \quad \text{for } |Z_0| = \varepsilon; \tag{21a}$$

$$\frac{\partial \tau_k}{\partial Z_0} = 0, \quad \text{for } |Z_0| = \delta. \tag{21b}$$

Analytical solution of (20) with the boundary conditions (21a) and (21b) is

$$\tau_n(z_0, \xi, \varepsilon, \sigma, q^2) = \frac{2}{q^2} \int_{z_0}^1 y^{-\frac{2\sigma}{q^2}} \exp\left(\frac{\varepsilon^2 \mu}{8q^2} y^2\right) \left[\int_\xi^y n\tau_{n-1}(x) x^{\frac{2\sigma}{q^2}-2} \exp\left(-\frac{\varepsilon^2 \mu}{8q^2} x^2\right) dx\right] dy, \tag{22}$$

where $\tau_0 \equiv 1$, and

$$z_0 \equiv \frac{Z_0}{\varepsilon}, \quad \xi \equiv \frac{\delta}{\varepsilon} \tag{23}$$

are non-dimensional initial error and noise level scaled by the tolerance level ε. The moments of FPT depend on two types of parameters: (a) prediction parameters (z_0, ξ, ε), and (b) model parameters (σ, q^2, μ), It is convenient to use the two lowest order statistics, mean FTP (τ_1) and the standard deviation of FTP $\left[s = \sqrt{\langle \delta \tau^2 \rangle}\right]$, to analyze the stability of the Kaldor model.

5. Model predictability

The first two moments τ_1 and τ_2 can be taken as the stability measure of the dynamic system. The longer the mean FPT, the more the stable of the system is. The dependence of τ_1 and τ_2 on the two types of parameters is investigated separately. The three model parameters are taken as $q^2 = 0.2$, $\sigma = 1.0$, $\mu = 1$.
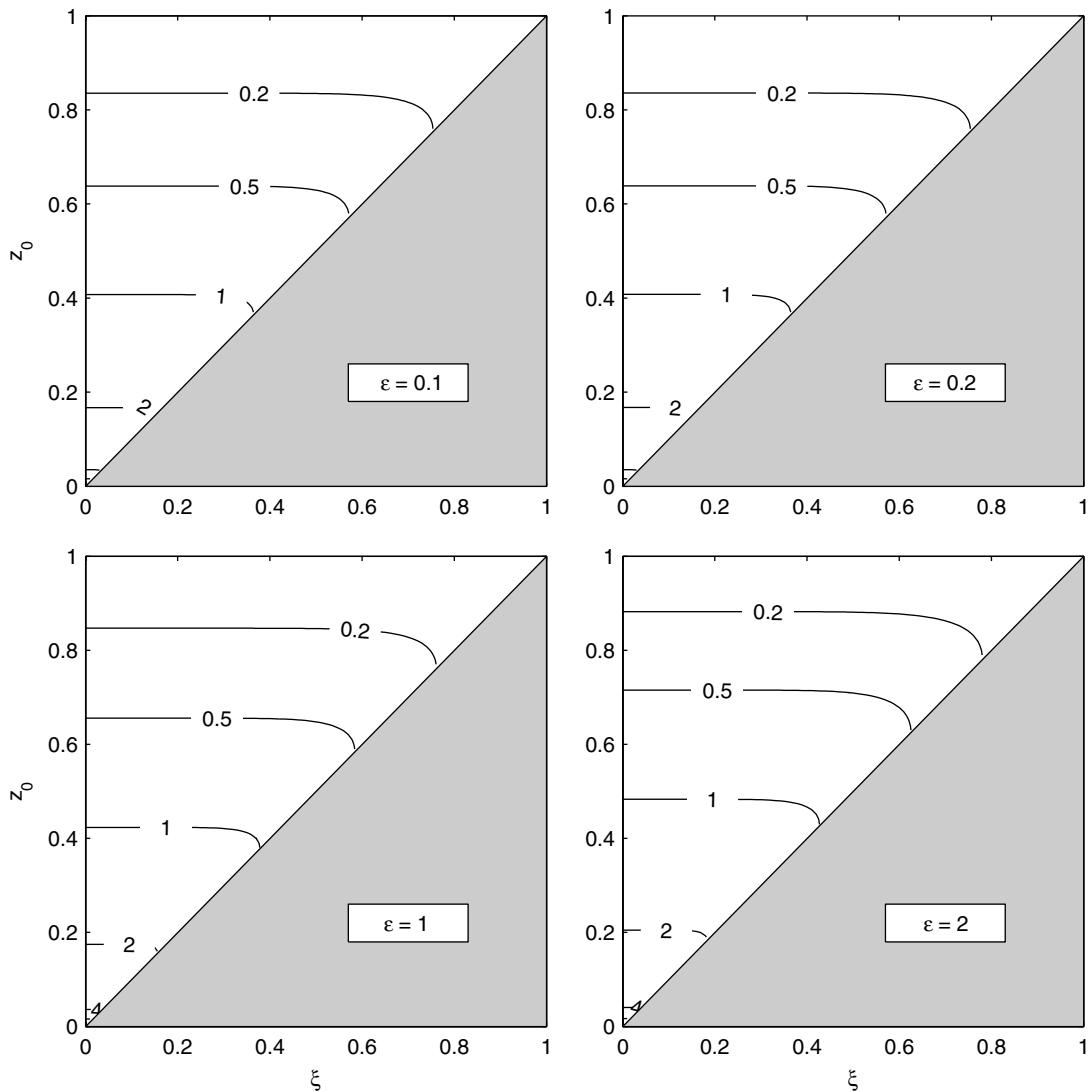

Fig. 2. Contour plots of $\tau_1(z_0, \xi, \varepsilon)$ versus (z_0, ξ) for four different values of ε (0.1, 0.2, 1.0, 2.0) using the Kaldor model with given model parameters $\sigma = 1.0$, $q^2 = 0.2$, $\mu = 1$. The contour plot covers the half domain due to $z_0 \geqslant \xi$.
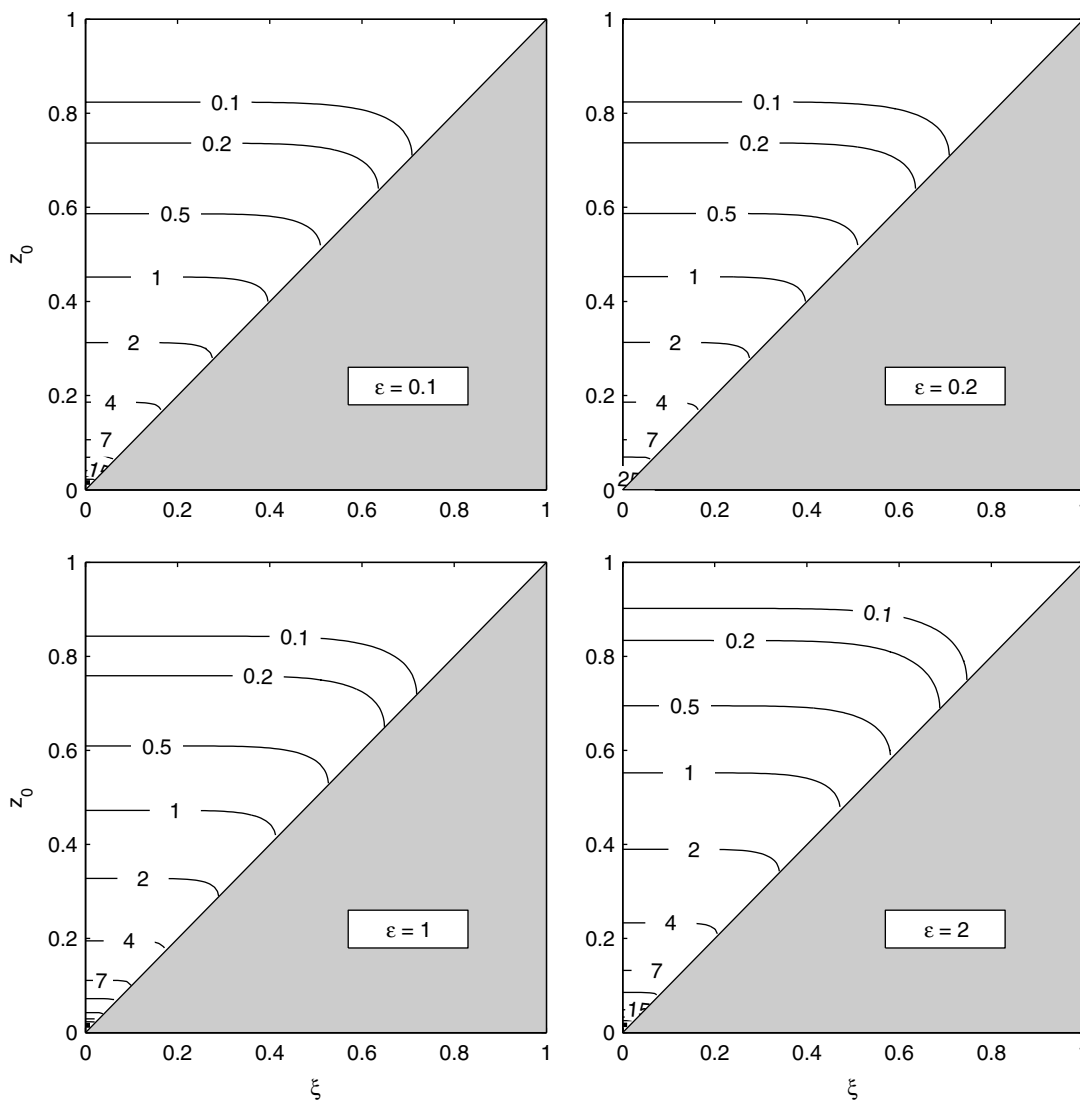
Fig. 3. Contour plots of $\tau_2(z_0, \xi, \varepsilon)$ versus (z_0, ξ) for four different values of ε (0.1, 0.2, 1.0, 2.0) using the Kaldor model with given model parameters $\sigma = 1.0$, $q^2 = 0.2$, $\mu = 1$. The contour plot covers the half domain due to $z_0 \geqslant \xi$.

Figs. 2 and 3 show the contour plots of $\tau_1(z_0, \xi, \varepsilon)$ and $\tau_2(z_0, \xi, \varepsilon)$ versus (z_0, ξ) for four different values of ε (0.1, 0.2, 1.0, and 2.0). Following features can be obtained: (a) for given values of (z_0, ξ) [i.e., the same location in the contour plots], both τ_1 and τ_2 increase with the tolerance-level ε. (b) For a given value of tolerance-level ε, both τ_1 and τ_2 are almost independent on the noise level ξ (contours are almost paralleling to the horizontal axis) when the initial error (z_0) is much larger than the noise level (ξ). This indicates that the effect of the noise level (ξ) on τ_1 and τ_2 becomes evident only when the initial error (z_0) is close to the noise level (ξ). (c) For given values of (ε, ξ), both τ_1 and τ_2 decrease with increasing initial error z_0.

Figs. 4 and 5 show the curve plots of $\tau_1(z_0, \xi, \varepsilon)$ and $s(z_0, \xi, \varepsilon)$ versus z_0 for four different values of tolerance level, ε (0.1, 1, 2, and 3) and four different values of random noise ξ (0.1, 0.2, 0.4, and 0.6). Following features are obtained: (a) τ_1 and s decrease with increasing z_0, which implies that the higher the initial error, the shorter the mean FPT (or lower model predictability) and the smaller the s (or lower variability of the model predictability) are; (b) τ_1 and s decrease with increasing noise level ξ, which implies that the higher the noise level, the lower the FPT and its variability are; and (c) τ_1 and s increase with increasing ε, which implies that the higher the tolerance level, the longer the FPT (or higher
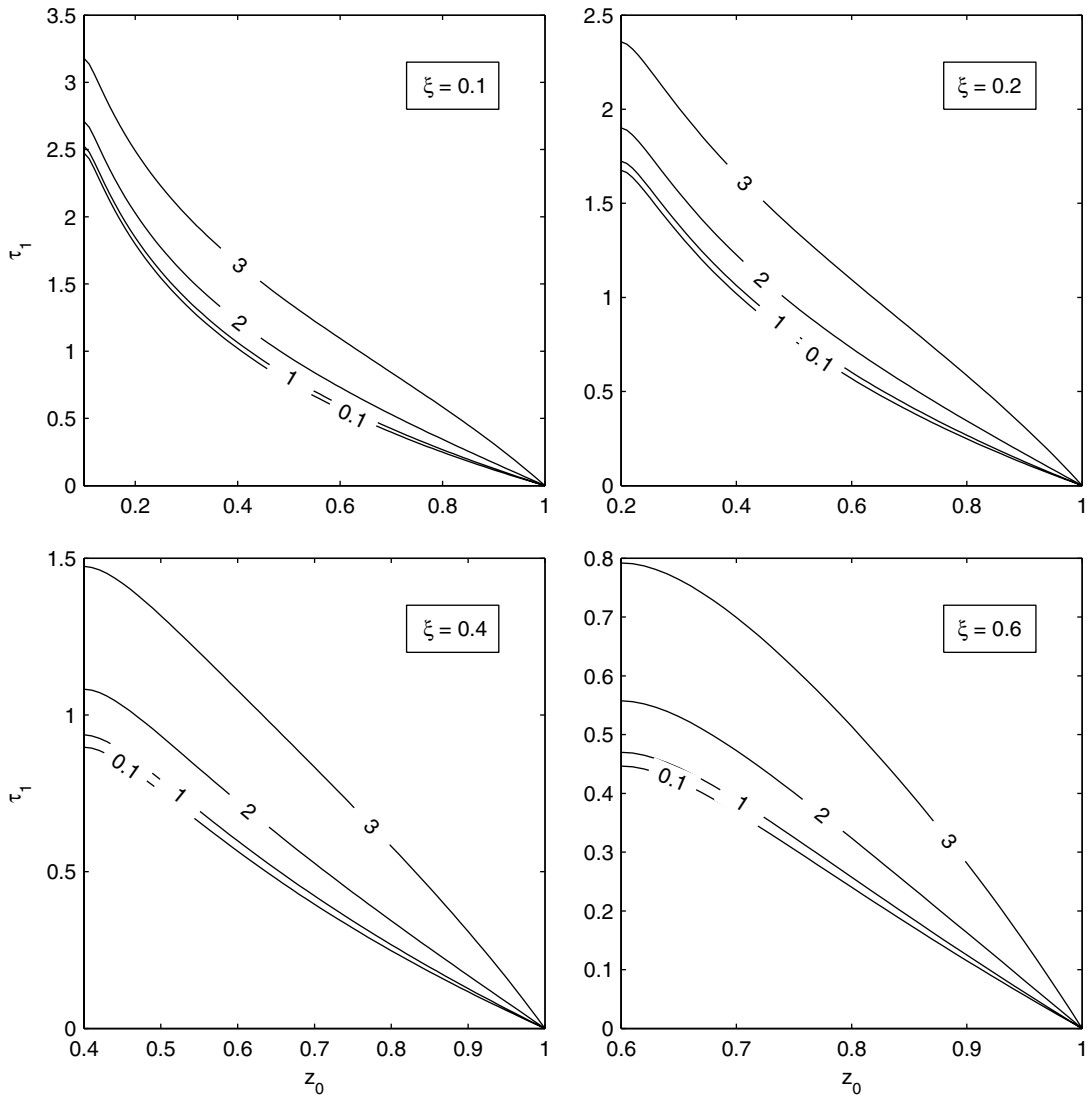
Fig. 4. Dependence of $\tau_1(z_0, \xi, \varepsilon)$ on the initial condition error z_0 for four different values of ε (0.1, 1.0, 2.0, 3.0) and four different values of random noise ξ (0.1, 0.2, 0.4, 0.6) using the Kaldor model with given model parameters $\sigma = 1.0$, $q^2 = 0.2$, $\mu = 1$.

model predictability) and the larger the s (or higher variability of the model predictability). For the same values of model parameters, the standard deviation of FTP is much smaller than the mean FTP. Note that the results presented in this subsection is for a given values of growth rate ($\sigma = 1.0$), stochastic forcing ($q^2 = 0.2$), and the Kaldor model parameter ($\mu = 1$) only.

6. Dependence of τ_1 and s on the Kaldor model parameters

6.1. Dependence on the growth rate σ

To investigate the sensitivity of τ_1 and s to the model parameter σ, the other two model parameters are kept unchanged ($q^2 = 0.2$, $\mu = 1$). The model parameter σ takes values of 0.25, 0.5, and 1.0. Figs. 6 and 7 show the curve plots of $\tau_1(z_0, \xi, \sigma)$ and $s(z_0, \xi, \sigma)$ versus z_0 for two tolerance levels ($\varepsilon = 0.05, 0.25$), two noise levels
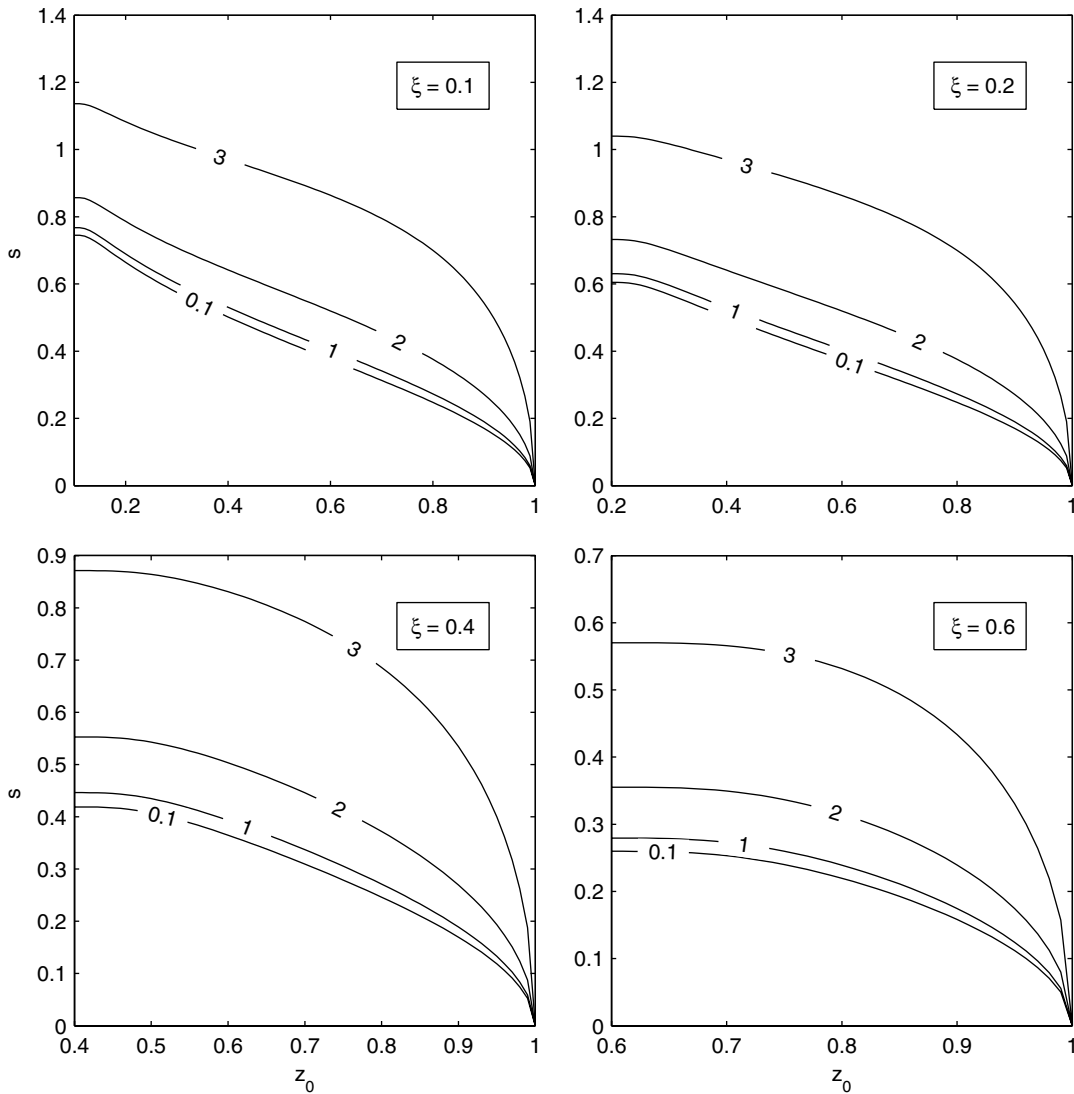
Fig. 5. Dependence of $s(z_0, \xi, \varepsilon)$ on the initial condition error z_0 for four different values of ε (0.1, 1.0, 2.0, 3.0) and four different values of random noise ξ (0.1, 0.2, 0.4, 0.6) using the Kaldor model with model parameters $\sigma = 1.0$, $q^2 = 0.2$, $\mu = 1$.

($\xi = 0.1, 0.6$), and three different values of σ (0.25, 0.5, and 1.0). τ_1 and s decrease with increasing σ for all combinations of noise level ($\xi = 0.1, 0.6$) and tolerance level ($\varepsilon = 0.05, 0.25$). This indicates that increase of Tr(**A**) [i.e., σ] destabilizes the Kaldor model (decreasing τ_1) and decreases the variability of the model predictability (decreasing s); and decrease of Tr(**A**) stabilizes the Kaldor model (increasing τ_1) and increases the variability of the model predictability (increasing s).

6.2. Dependence on the stochastic forcing q^2

To investigate the sensitivity of τ_1 and s to stochastic forcing q^2 and the Kaldor model parameter μ, the growth rate is kept unchanged ($\sigma = 1$). Figs. 8 and 9 show the curve plots of τ_1 (z_0, ξ, q^2) and $s(z_0, \xi, q^2)$ versus z_0 for two adjustment coefficients ($\mu = 0.01, 10$), two noise levels ($\xi = 0.1, 0.6$), and three different values of q^2 (0.1, 0.25, and 0.5) representing weak, normal, and strong stochastic forcing. Two regimes are found: (a) τ_1 and s decrease with increasing q^2 for large
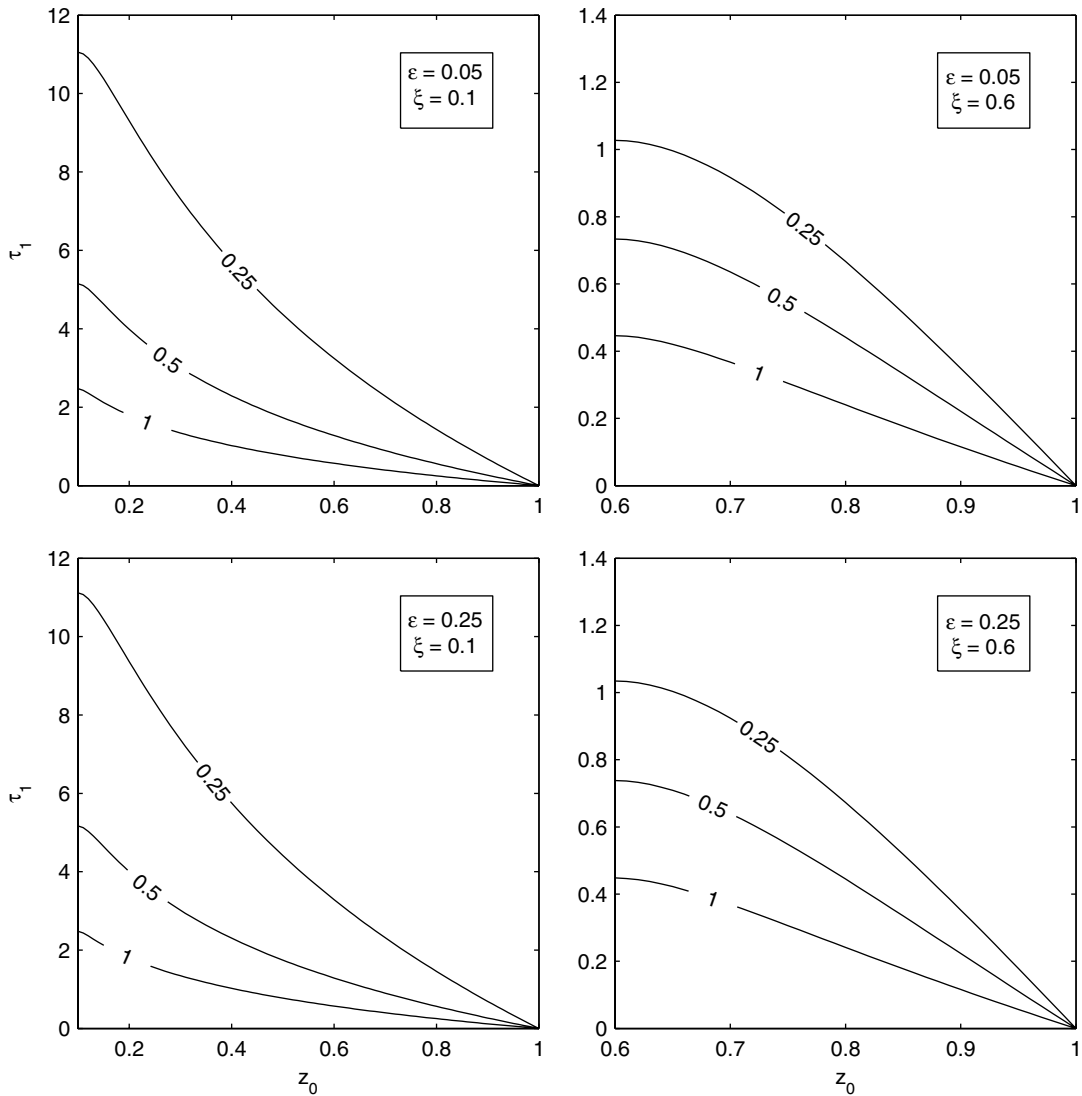
Fig. 6. Dependence of $\tau_1(z_0, \xi, q^2)$ on the initial condition error z_0 for three different values of σ (0.25, 0.5, 1.0) and given values of q^2 (=0.2) and μ(=1) using the Kaldor model with two different values of ε (0.05, 0.25) and two different values of noise level ξ (0.1, 0.6). Note that the mean FTP is not sensitive to σ.

noise level ($\xi = 0.6$), (b) τ_1 and s increase with increasing q^2 for small noise level ($\xi = 0.1$) and (c) both relationships (increase and decrease of τ_1 and s with increasing q^2) are independent of μ.

7. Stabilizing and destabilizing regimes

This indicates the existence of stabilizing and destabilizing regimes of the dynamical system depending on stochastic forcing. For a small noise level, the stochastic forcing stabilizes the dynamical system and increase the mean FPT. For a large noise level, the stochastic forcing destabilizes the dynamical system and decreases the mean FPT. The two regimes
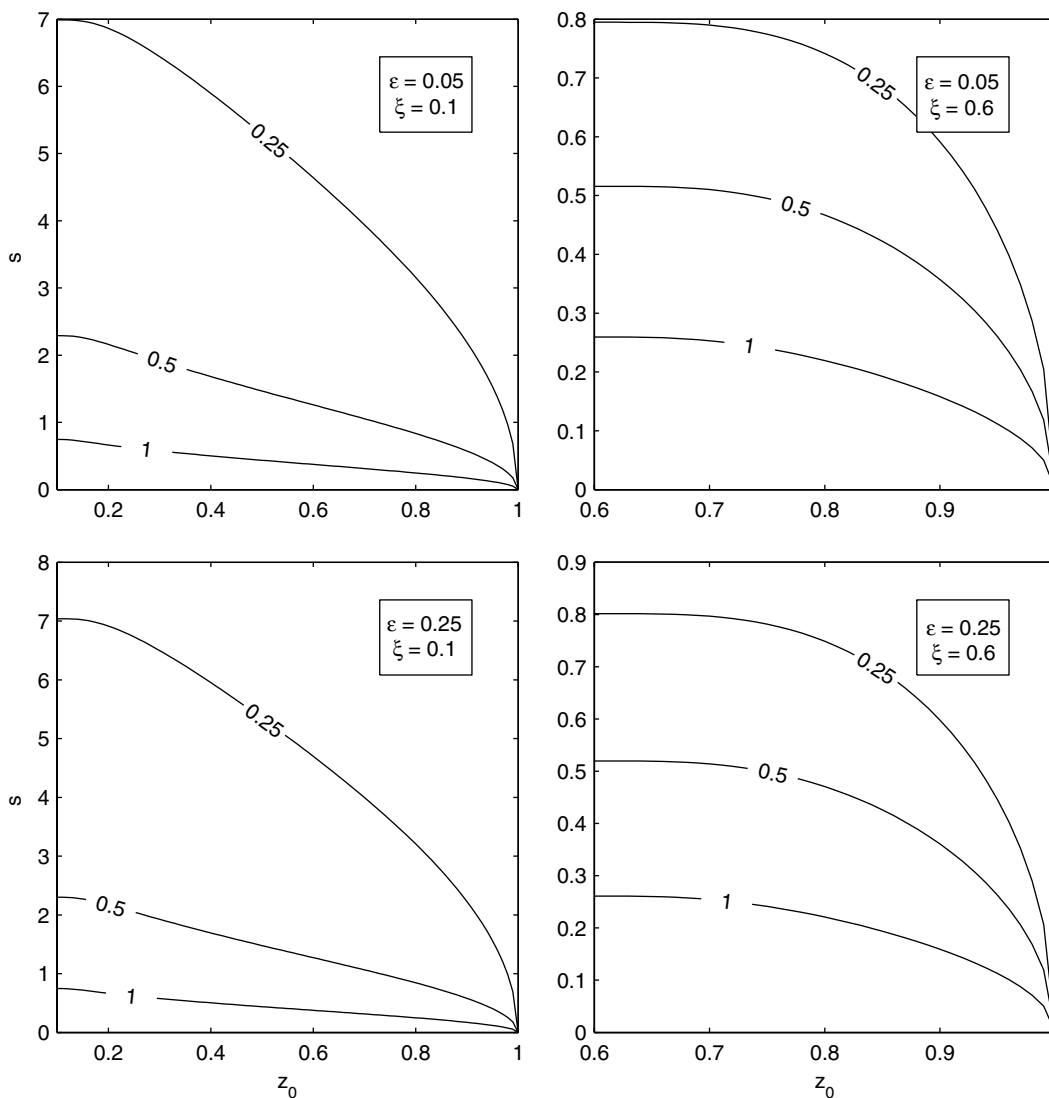
Fig. 7. Dependence of $s(z_0, \xi, q^2)$ on the initial condition error z_0 for three different values of σ $(0.25, 0.5, 1.0)$ and given values of q^2 $(=0.2)$ and $\mu(=1)$ using the Kaldor model with two different values of ε $(0.05, 0.25)$ and two different values of noise level ξ $(0.1, 0.6)$. Note that the standard deviation of FTP is not sensitive to σ.

can be identified analytically for small tolerance level ($\varepsilon \to 0$). The initial error z_0 should also be small ($z_0 \sim \varepsilon$). The solutions (22) becomes

$$\lim_{\varepsilon \to 0} \tau_1(z_0, \xi, \varepsilon, \sigma, q^2) = \frac{1}{\sigma - q^2/2} \left\{ \ln\left(\frac{1}{z_0}\right) - \frac{q^2}{2\sigma - q^2} \xi^{\frac{2\sigma}{q^2}-1} \left[\left(\frac{1}{z_0}\right)^{\frac{2\sigma}{q^2}-1} - 1 \right] \right\}. \tag{24}$$

The Lyapunov exponent is identified as $(\sigma - q^2/2)$ for dynamical system (16) [4]. For a small noise level ($\xi \ll 1$), the second term in the bracket of the right-hand of (24)

$$R = -\frac{q^2}{2\sigma - q^2} \xi^{\frac{2\sigma}{q^2}-1} \left[\left(\frac{1}{z_0}\right)^{\frac{2\sigma}{q^2}-1} - 1 \right], \tag{25}$$
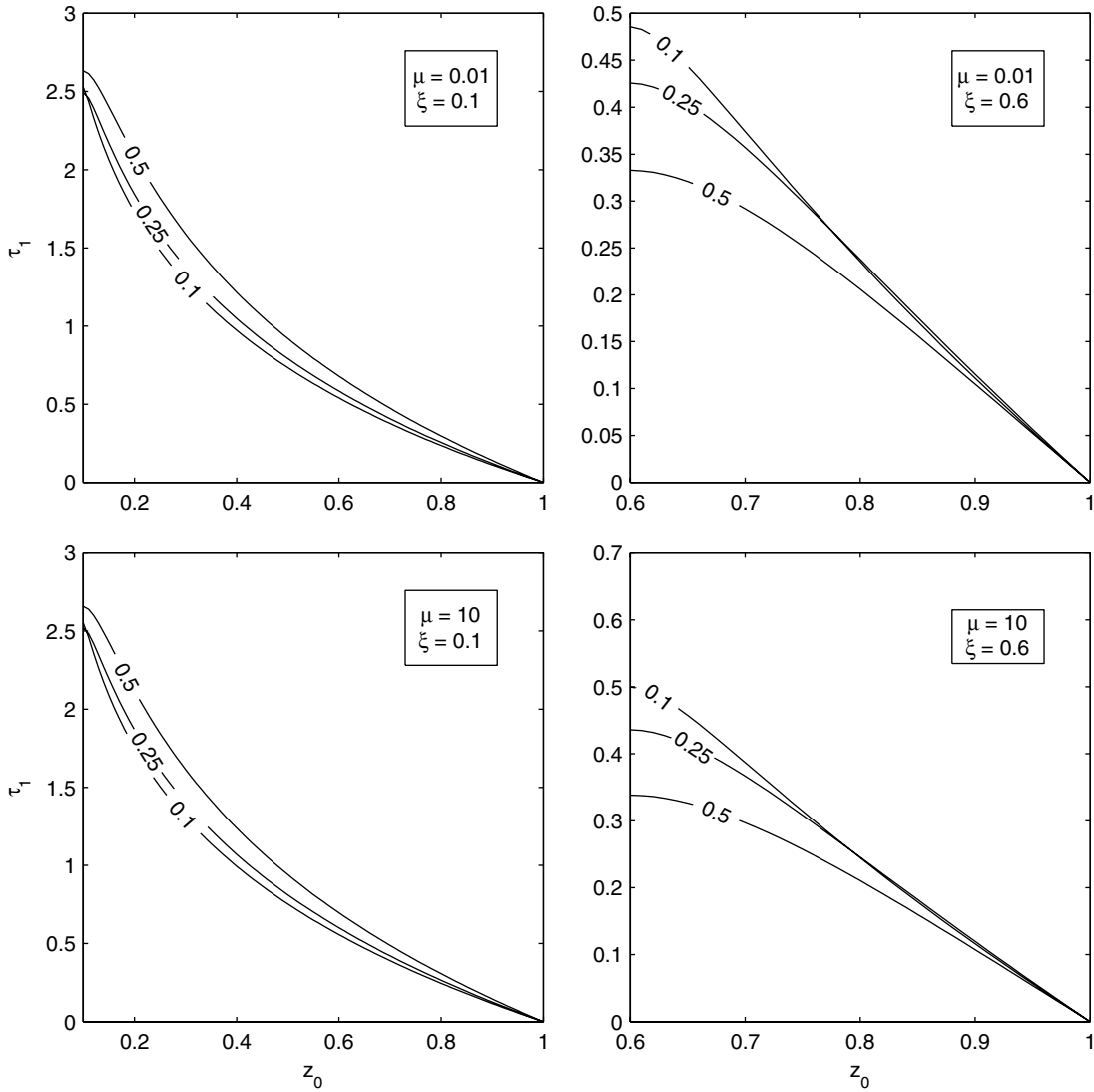
Fig. 8. Dependence of $\tau_1(z_0, \xi, q^2)$ on the initial condition error z_0 for three different values of q^2 (0.1, 0.25, 0.5) and given parameters ($\sigma = 1.0, \varepsilon = 0.2$) using the Kaldor model with two different values of the adjustment coefficient μ (0.01, 10) and two different values of noise level ξ (0.1, 0.6). Note that the mean FTP is not sensitive to varying μ.

is negligible. The solution (24) becomes

$$\lim_{\varepsilon \to 0} \tau_1(z_0, \xi, \varepsilon, \sigma, q^2) = \frac{1}{\sigma - q^2/2} \ln \left(\frac{1}{z_0} \right), \tag{26}$$

which shows that the stochastic forcing ($q \neq 0$), reduces the Lyapunov exponent ($\sigma - q^2/2$), stabilizes the dynamical system (15), and in turn increases the mean FPT. On the other hand, the initial error z_0 reduces the mean FPT.

For a large noise level ξ, the second term in the bracket of the right-hand of (24) is not negligible. For a positive Lyapunov exponent, $2\sigma - q^2 > 0$, this term is always negative [see (25)]. The absolute value of R increases with increasing q^2 (remember that $\xi < z_0 < 1$). Thus, the term (R) destabilizes the stochastic Kaldor system (15), and reduces the mean FPT.
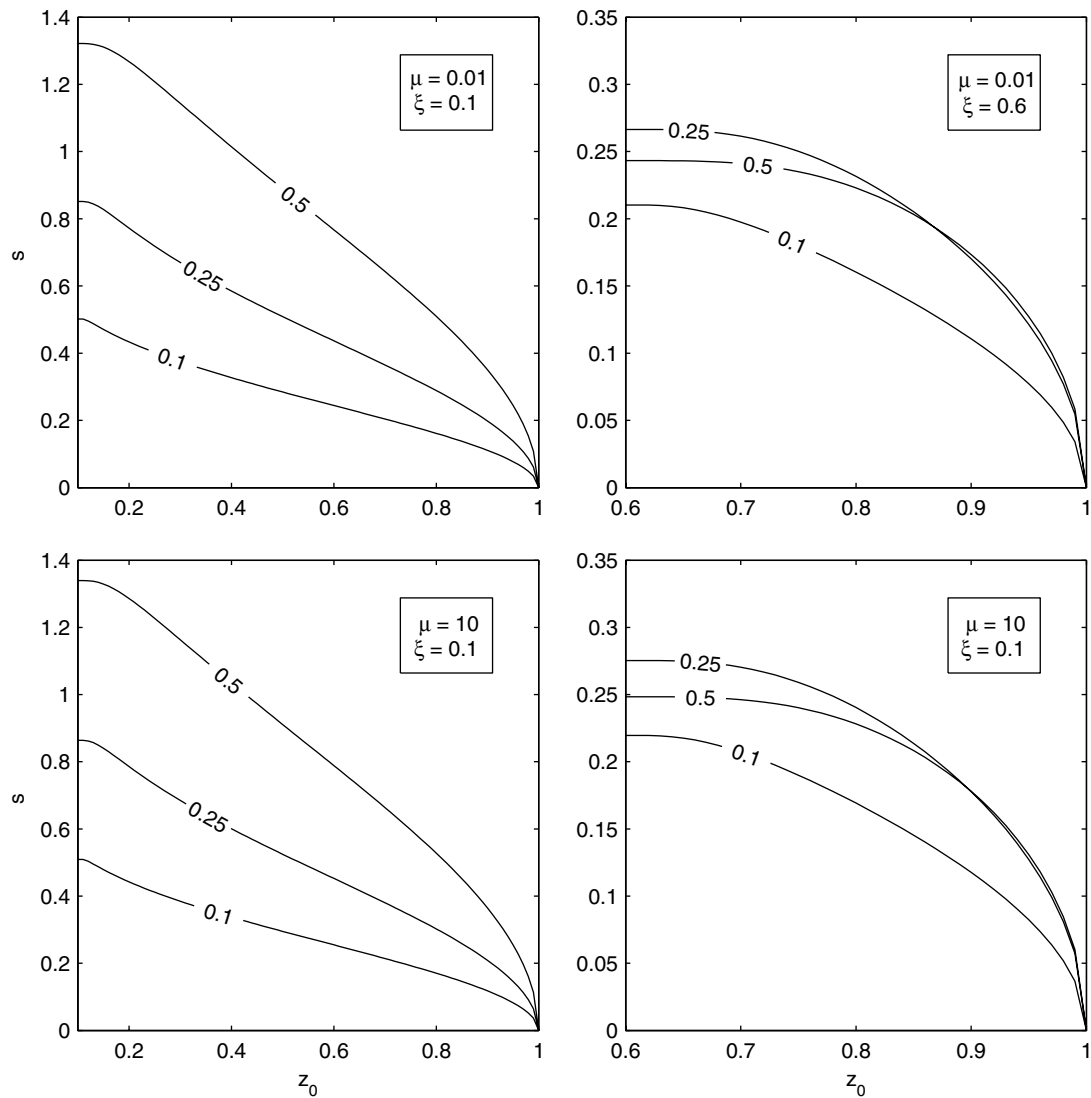
Fig. 9. Dependence of $s(z_0, \xi, q^2)$ on the initial condition error z_0 for three different values of q^2 (0.1, 0.25, 0.5) and given parameters ($\sigma = 1.0$, $\varepsilon = 0.2$) using the Kaldor model with two different values of the adjustment coefficient μ (0.01, 10) and two different values of noise level ξ (0.1, 0.6). Note that the mean FTP is not sensitive to varying μ.

8. Conclusions

(1) The Kaldor model stability and predictability are not only affected by the model parameters such as the depreciation coefficient, the coefficient for reinvested profits [i.e., Tr(**A**)], the adjustment, but also by the prediction parameters (such as initial error, tolerance level, and noise level. The capability of the FPT approach in evaluating model stability and predictability is demonstrated using the nonlinear Kaldor model.

(2) Uncertainty in economic models is caused by uncertain measurements, computational accuracy, and uncertain model parameters. This motivates to the inclusion of stochastic forcing in economic models such as the Kaldor model. The backward Fokker–Planck equation can be used for evaluation of economic model stability and predictability through the FPT calculation.

(3) A theoretical framework was developed in this study to determine various FPT moments (τ_k), which satisfy time-independent second-order linear differential equations with given boundary conditions. This is a well-posed problem and the solutions are easily obtained.

(4) For the Kaldor model, the following features are detected from the FPT calculation: (a) decrease of τ_1 and s with increasing initial condition error (or with increasing random noise), (b) slow increase of τ_1 and s with increasing tolerance level ε, (c) For the same values of model parameters, the standard deviation of FTP is much smaller than the mean FTP, which indicates that the first moment is a reliable indicator of the model stability.

(5) Both stabilizing and destabilizing regimes are found in the Kaldor model depending on stochastic forcing. For a small noise level, the stochastic forcing stabilizes the Kaldor system and increases the mean FPT. For a large noise level, the stochastic forcing destabilizes the Kaldor system and decreases the mean FPT.

(6) Model stability depends on Tr($\mathbf{A}$). Increase of Tr($\mathbf{A}$) destabilizes the Kaldor model (decreasing τ_1) and decrease of Tr($\mathbf{A}$) stabilizes the Kaldor model (increasing τ_1). The model stability does not depend on the Kaldor model parameter μ.

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