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Theory and Methodology

Characterizing an equitable allocation of shared costs: A DEA approach [1]

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Abstract

In many applications to which DEA could be applied, there is often a fixed or common cost which is imposed on all decision making units. This would be the case, for example, for branches of a bank which can be accessed via the numerous automatic teller machines scattered throughout the country. A problem arises as to how this cost can be assigned in an equitable way to the various DMUs. In this paper we propose a DEA approach to obtain this cost allocation which is based on two principles: invariance and pareto-minimality. It is shown that the proposed method is a natural extension of the simple one-dimensional problem to the general multiple-input multiple-output case. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

An issue of considerable importance, both from a practical organizational standpoint and from a costs research perspective, involves the allocation of *fixed* resources or costs across a set of competing entities in an *equitable* manner. The problem, for example, of how to allocate ongoing overhead expenditures among a set of departments or divisions within an organization, across multiple branches of a bank, among a set of schools in a district, and so on, is one with which we are all familiar. In this paper we investigate the particular problem of allocating a fixed cost across a set of *comparable* decision making units (DMUs). By 'comparable' we will mean that each DMU has access to, and consumes an amount of each of a set of *inputs*; similarly, each DMU produces some amount of each of a set of defined *outputs*. So, the DMUs are all doing basically the same kinds of things. For each DMU the amounts of inputs and outputs used individually can be clearly distinguished or measured. At the same time, we assume that the set of DMUs may *share* or incur a com-

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mon cost such as a fixed overhead. Consider the example of a set of automobile dealerships wherein two types of advertising expenditures are incurred:

(1) Direct advertisements (TV, radio, newspaper) pertaining to a particular dealership and

(2) General or blanket advertisements issued by the corporation for particular models of vehicles sold by all dealerships.

While both (1) and (2) affect the sales output of a dealership, only (1) is taken into account in the performance measurement, because there is no way to associate a part of (2) with a certain DMU, and as well there is no direct consumption of this input by the DMU.

Suppose now that the corporation decides to pass on (allocate) the cost of the TV campaign X_{J+1} among the dealerships. That is, it wishes to assign "General TV-advertising costs" $X_{1J+1}, \ldots, X_{nJ+1}$ to the n dealerships. This cost becomes a new, non-discretionary input. Specifically, while all dealerships benefit from the blanket advertisement, it is not under their control to utilize more or less of that resource. In particular, *no dealership* is in a position to *substitute* an amount of any other discretionary input for more or less of the blanket advertising input. The issue is how to split that blanket cost among the DMUs in the best or most equitable way.

To provide a practical setting within which to investigate this issue, we refer to the recent paper by Cook et al. [5]. There, the authors present a model for evaluating the relative efficiencies of a set of highway maintenance crews or patrols in the province of Ontario, Canada. The model is based on the data envelopment analysis (DEA) procedure of Charnes et al. [3]. Each maintenance patrol is responsible for some designated number of lane kilometers of highway along with all of the activities associated with that portion of the network. The more than 100 different categories of maintenance activities can be grouped under five general headings: 'surface', 'shoulder', 'median', 'right of way', and 'winter operations'. In the specific example discussed, each patrol is examined in terms of two inputs and two outputs:

Outputs:

size of system: this is a measure comprising for each patrol, a combination of the number of lane kilometers of highway served together with the number of hectares of road side environment;

traffic: this output is a measure of the average daily traffic.

Inputs:

maintenance budget: this is the aggregate of all direct maintenance expenditures attributable to a patrol's activities, but does not include those fixed costs at the district level that cannot be immediately attached to specific crews;

annual capital budget: expenditure on major resurfacing.

The 246 maintenance patrols in Ontario are organized into 18 geographical districts, which are further grouped into 5 regions. In the initial stages of the study of maintenance activities carried out in Ontario, a pilot study of fourteen patrols in one district was conducted. It is this single district study that is reported on by Cook et al. [5]. The hierarchical arrangement of patrols (into districts, then regions) gives rise to the need to look at the issue of distributing fixed cost. A good example in this particular setting is the fixed administrative expenditures consumed at the district level, as opposed to those expenditures pertaining to the individual patrol. One component of this fixed expenditure, for example, is the salary and benefits of district staff, in particular the District Engineer, whose task it is to coordinate activities of all patrols in his/her jurisdiction.

The analysis carried out in Cook et al. [5] utilizes only those factors, i.e., outputs and inputs, for which measurable patrol-specific data exists. What is not utilized in the analysis on a patrol by patrol basis is the fixed district (overhead) costs, nor is it clear how this cost should be split. There are a number of reasons, however, to be discussed below, for wanting to obtain an allocation of such an overhead across the patrols in the district in the most equitable way possible. Clearly, the cost that is imposed on a DMU constitutes an additional input which may alter the *absolute efficiency* rating of the DMU. The objective of management is to allocate these costs in such a way that the *relative* (*radial*) *efficiency* is not changed. In the DEA setting, we require that no DMU will appear relatively "better" just because its allocated cost was too small.

It should be emphasized that the DMU has no control on this cost. Its performance relies entirely on its existing inputs and outputs. We will argue in the subsequent sections that any allocation of costs that does not alter the value of the radial efficiency measure is equitable. We, therefore, take this as a necessary condition for any such allocation.

In Section 2 we examine the basic concept of equity in the one dimensional case which motivates the analysis and provides a backdrop for the subsequent DEA model. In Section 3 we look at the concept of fair allocation in the DEA setting and examine what that should mean on a problem setting such as that discussed above. In Section 4, we go back to the one-dimensional case and examine it vis-à-vis the DEA framework that was laid out in Section 3. Section 5 examines a special case involving only inputs. Here it is shown that the intuitively desirable result occurs, namely, that the optimal amount of the fixed cost to be allocated to a DMU is proportional to its consumption of the variable inputs. In Section 6 we examine the general multiple-inputs multiple-outputs case. We characterize the set of equitable allocations and present a reasonable model for arriving at a unique such allocation. A numerical example is presented. Concluding remarks follow in Section 7.

2. The one-dimensional case

We start off with the one-dimensional case where each DMU has one input and one output. For $j = 1, \ldots, n$ let x_j and y_j be the input and output respectively of DMU_j. One measure of efficiency of each DMU is given by

$$E_j = y_j/x_j. \tag{1}$$

Suppose that a cost R is to be distributed among the n DMUs. That is, each DMU is to be allocated a cost r_j such that

$$\sum_{j=1}^{n} r_j = R. \tag{2}$$

A reasonable and "fair" allocation is such that the relative efficiencies of the DMUs remain unchanged after the allocated costs are added as inputs to the various DMUs. The rationale for this is as follows: the existing efficiency rating E_{j*} for any DMU $j*$ is a reflection of that DMUs consumption of the specific amounts of inputs that it has at its disposal. Moreover, that rating is also a reflection of any other noncontrollable factors present at the time, whether they are explicitly included in the analysis or not (e.g., blanket advertisement for all DMUs). Thus, the allocation of the fixed cost (or fixed resource) should be made in a way that is consistent with the computed influence that the fixed cost is presently having on performance. In other words, if the efficiency of DMU j, after adding the cost r_j is E'_j, then we would require that

$$\frac{E'_s}{E'_t} = \frac{E_s}{E_t}, \quad s, t = 1, \ldots, n. \tag{3}$$

Lemma 1. *The cost allocation* $r_1, \ldots, r_n$, *with* $\sum_{j=1}^{n} r_j = R$ *that satisfies Eq. (3) is unique and is given by*

$$r_j = \frac{Rx_j}{\sum_{s=1}^{n} x_s}, \quad j = 1, \ldots, n. \tag{4}$$

Proof.

$$\frac{E_s}{E_t} = \frac{E'_s}{E'_t} \text{ if and only if } \frac{y_s/x_s}{y_t/x_t} = \frac{y_s/(x_s + r_s)}{y_t/(x_t + r_t)} \tag{5}$$

or

$$\frac{x_s}{x_t} = \frac{x_s + r_s}{x_t + r_t}. \tag{6}$$

From Eq. (6) we get that

$$\frac{r_s}{r_t} = \frac{x_s}{x_t} \tag{7}$$

or

$$r_s = \frac{x_s}{x_t} r_t \tag{8}$$

and the result follows. □

From this elementary exercise we may conclude that: (a) the equitable allocation is *unique*, (b) it is a function of the total cost R and the *inputs* that are used, and (c) it is *independent* of the *output* levels.

3. Cost allocation equity utilizing DEA

For purposes herein we will utilize the original CCR-model [3] for relative efficiency measurement. [2] Specifically, we concentrate on the *constant returns-to-scale* case. Furthermore, it is instructive to apply the output-oriented version of the CCR model, given by

$$(P_{j_o}) \qquad f_{j_o} = \min \quad \sum_{i=1}^{I} v_{ij_o} x_{ij_o} \tag{9}$$

s.t.

$$\sum_{k=1}^{K} \mu_{kj_o} y_{kj_o} = 1, \tag{10}$$

$$-\sum_{k=1}^{K} \mu_{kj_o} y_{kj} + \sum_{i=1}^{I} v_{ij_o} x_{ij} \geqslant 0, \quad j = 1, \ldots, n, \tag{11}$$

$$\mu_{kj_o}, v_{ij_o} \geqslant 0, \qquad \forall i, k.$$

Here, it is assumed that each decision making unit (DMU) j consumes a known amount x_{ij} of each of I inputs $i = 1, \ldots, I$ in the production of K outputs in the amounts $y_{kj}, k = 1, \ldots, K$. The model (P_{j_o}) finds the best set of multipliers μ_{kj_o}, v_{ij_o} for each DMU j_o, in the sense of minimizing the *inefficiency* score f_{j_o}. Further, it is assumed that the production function is adequately explained by the existing input–output bundle (x, y).

Recall that $f_{j_o} (\geq 1)$ yields the *output expansion factor* in the sense that the outputs would need to be increased by $(f_{j_o} - 1) \times 100\%$ in order to render DMU j_o efficient. It is noted also that for the CCR model, the *measure* e_{j_o} that would come about from the input-oriented version (max outputs rather than min inputs), and which is traditionally interpreted as the measure of efficiency, is such that $e_{j_o} = 1/f_{j_o}$. Due to this connection, we will from this point on refer to the f_{j_o} as the efficiency scores.

Given the resulting efficiency scores f_j from model (P_{j_o}), we wish to allocate, in an equitable manner, a given amount R of a fixed resource or cost among the n DMUs. In a pure accounting sense, one would arguably allocate a fixed cost or resource to a DMU in a manner consistent with the way other inputs are consumed by that DMU. If, for example, one DMU utilizes twice as much labor and capital as another DMU, then it is reasonable to allocate twice as much of the overhead expenditures to the former DMU as compared to the latter. In the typical DEA setting, however, such an approach is a problem in that multiple factors are involved, and are generally in non-commensurate units.

Consistent with the assumption that the given inputs and outputs adequately explain the production function, we may require that the allocated cost in question should have no effect on this function. We call this requirement *invariance* of the relative efficiency scores to the allocated costs. Thus, following the discussion in Section 2, a reasonable principal for the partitioning of R into n pieces $r_1, r_2, \ldots, r_n$, is to do so in such a manner as to preserve the relative efficiency ratings for the n DMUs. Specifically, the r_j should be chosen so that if they were to be included after the fact as an $(I{+}1)$th input, the re-evaluated efficiencies would remain unchanged. Otherwise a DMU is either penalized (if the efficiency rating is decreased) or benefits (if the efficiency rating increases) because of a decision it does not make.

Unfortunately, allocation according to this principle is not unique. One can, for example, readily see that if R were distributed in its entirety among only the inefficient DMUs in any proportion whatever, the ratings would not change, and the principle would be satisfied. This is the case since the optimal multipliers (which are unique to each DMU) would be such that $v_{I+1j_o} = \varepsilon$ for all j_o. Such an allocation renders the new input *redundant* in terms of its impact on the evaluation process. Clearly, however, any allocation which “penalizes” only the inefficient DMUs, would generally be unacceptable to the organization. Thus, while the invariance requirement discussed

[2] We use the non-archimedian version of the CCR-model in this paper. Our development, therefore, does not take into account any consideration or importance that one may wish to accord to slacks. See Thrall [6].

above is necessary for an equitable allocation of cost, it is not sufficient and, therefore, another condition is needed. This condition is called *Input Pareto-Minimality*. Formally, we define a cost allocation to be *input pareto-minimal* if no cost can be transferred from one DMU to another without violating the invariance principle. Clearly, the allocation mentioned above where only inefficient DMUs are assigned costs is not input pareto-minimal since some costs may be transferred to efficient DMUs without violating invariance.

Before we apply the ideas presented above to the multiple-inputs multiple-outputs case, we look at the one-dimensional case again, but from a DEA point of view.

4. The one-dimensional case and the DEA formulation

The DEA (CCR) formulation for the (trivial) one-dimensional case is

$$\text{(P1)} \qquad \min \quad vx_{jo} \tag{12}$$

s.t.

$$vx_j \geqslant y_j/y_{j_o}, \qquad j = 1, \ldots n, \tag{13}$$

$$v \geqslant 0.$$

By adding the new cost r_j, (P1) becomes

$$\text{(P2)} \qquad \min \quad vx_{jo} + wr_{jo} \tag{14}$$

s.t.

$$vx_j + wr_j \geq y_j/y_{j_o}, \qquad j = 1, \ldots, n, \tag{15}$$

$$v, w \geq 0.$$

Going back to Eq. (3), one can argue now that a necessary condition for an allocation to be equitable is that no DMU can utilize this new input to improve its relative efficiency. In LP terminology, this requirement amounts to keeping the w variable in (P2) out of the basis.

For each DMU j_o, w remains out of the basis if and only if the reduced costs are non-negative. That is:

$$r_{j_o} \geq \sum_{j=1}^{n} u_j^{j_o} r_j, \tag{16}$$

where $u_j^{j_o}$ are the dual optimal variables of (P1).

Evidently, as discussed in Section 3 above, this invariance condition is not sufficient to determine an equitable allocation. The Input Pareto-Minimality condition is needed as well, and therefore we require that

$$r_{j_o} = \sum_{j=1}^{n} u_j^{j_o} r_j \tag{17}$$

for all inefficient DMUs j_o.

The dual of (P1) is:

$$\text{(D1)} \qquad \max \quad \sum_{j=1}^{n} u_j y_j / y_{j_o} \tag{18}$$

s.t.

$$\sum_{j=1}^{n} u_j x_j \leq x_{j_o}, \tag{19}$$

$$u_j \geq 0.$$

The extreme points of the feasible set defined by Eq. (19) have all components but one equal zero. Thus, an optimal solution for (D1) is of the form

$$u^* = (0, \ldots, x_{j_o}/x_j^*, 0, \ldots, 0). \tag{20}$$

Clearly, this solution may not be unique when the maximum of $\{y_i/x_j\}$ is obtained by more than one j. Let $j_1, \ldots, j_l$ be the efficient DMUs; then from Eq. (17) it follows that

$$r_{j_o} = \frac{x_{j_o}}{x_{j_1}} r_{j_1} = \cdots = \frac{x_{j_o}}{x_{j_l}} r_{j_l}. \tag{21}$$

Hence,

$$\frac{v_s}{v_t} = \frac{x_s}{x_t} \tag{22}$$

and therefore

$$r_j = \frac{x_j}{\sum_{s=1}^{n} x_s} R, \tag{23}$$

as was obtained in Eq. (4).

We conclude that input pareto-minimality may indeed be a reasonably sufficient criterion for equity.

Thus, we have established the applicability of the proposed DEA cost-allocation approach for the single-input single-output case. It is instructive to point out here that if one assessed DMUs on a periodic basis (e.g. annually), the relative positioning of those DMUs may change. This means, of course, that a DMUs share of a fixed cost burden can fluctuate. Arguably, this may be an undesirable property in the case of *one-time* fixed costs that are amortized over future periods and where a DMUs percentage of the burden would be best left at a fixed value. One-time plant construction might be an example. The proposed DEA approach may be more suitable to ongoing, fixed expenses such as those arising from annual blanket advertising. In this case, each years allocation (and total amount to be shared) may reasonably be expected to change, depending upon performance.

Next, we examine the pure multiple-input case.

5. The pure input case

Consider the case where the n DMUs use a number of inputs to produce the same unique output. For example, local television stations utilize inputs such as reporters, technicians, telecommunication systems, video cameras, etc. to produce the 6 o'clock news which, we assume here, is of a uniform format. We can, therefore, discard the uniform output and look at a pure input version of (P_{j_o}) (Eqs. (9)–(11) in Section 3) where we wish to evaluate the DMUs in terms of efficiency with which the inputs are consumed.

Thus, the problem that we look at is

$$(P'_{j_o}) \qquad f_{j_o} = \min \quad \sum_{i=1}^{I} v_{ij_o} x_{ij_o} \tag{24}$$

s.t.

$$\sum_{i=1}^{I} v_{ij_o} x_{ij} \geq 1, \qquad j = 1, \ldots, n, \tag{25}$$

$$v_{ij_o} \geq 0.$$

We now show that if efficiency is viewed only in terms of inputs, then the appropriate allocation $\{r_j\}$ of a fixed resource is one whereby r_j is proportional to the virtual or aggregated input. Hence, the amount of fixed cost to be assigned to a DMU is proportional to that DMU's consumption of *variable* resources.

If a new (fixed) input is introduced, we may consider an augmented version of (P'_{j_o}):

$$(Q'_{j_o}) \qquad \min \quad \sum_{i=1}^{I} v_{ij_o} x_{ij_o} + v_{I+1j_o} r_{j_o} \tag{26}$$

s.t.

$$\sum_{i=1}^{I} v_{ij_o} x_{ij} + v_{I+1j_o} r_j \geq 1, \qquad j = 1, \ldots, n, \tag{27}$$

$$v_{ij_o} \geqslant 0, \qquad \forall i.$$

As was shown in Section 3 above, the condition for invariance and pareto-minimality is that the reduced cost of the new cost variable vanishes.

$$0 = -r_{j_o} + \sum_{j=1}^{n} u_j^{j_o} r_j, \tag{28}$$

where the $u_j^{j_o}$ are the optimal dual variables of (P'_{j_o}).

The dual of problem of (P'_{j_o}) is:

$$(D'_{j_o}) \qquad \max \quad \sum_{j=1}^{n} u_j \tag{29}$$

s.t.

$$\sum_{j=1}^{n} u_j x_{ij} \leq x_{ij_o}, \qquad i = 1, \ldots, I, \tag{30}$$

$$u_j \geq 0, \quad \forall j.$$

In the case that DMUj_o is not efficient, then

$$\sum_{i=1}^{I} v^*_{ij_o} x_{ij_o} = f_{j_o} > 1,$$

where the $v^*_{ij_o}$ are the optimal solutions for (P'_{j_o}). Letting J_{j_o} denote the binding constraints in (P'_{j_o}) corresponding to the efficient reference set for j_o, it follows that

$$-r_{j_o} + \sum_{j \in J_{j_o}} u_j^{j_o} r_j = 0, \tag{31}$$

since the other dual variables $u_j^{j_o}$ are all zeros, due to complementary slackness.

Denote J_e as the set of all efficient DMUs. Clearly

$$J_e = U^n_{j_o=1} J_{j_o}.$$

5.1. Allocation among efficient DMUs

For an efficient DMU, that is a DMU_{j_o} for which

$$\sum_{i=1}^{I} v_{i_{j_o}} x_{i_{j_o}} = f_{j_o} = 1,$$

we can assign any value r_{j_o} in (P'_{j_o}) without altering its optimal objective value (Eq. (24)) since we can always choose $v_{I+1jO} = 0$ in (Q'_{j_o}). Therefore, we may make the assumption in the pure input case that a fair allocation of the fixed resource to the efficient DMUs is the uniform allocation. Since no outputs are involved, no normalizing conditions such as Eq. (10) are imposed. Any two members of J_e here are judged to be the same from an aggregate input standpoint, whereas in the general case, two efficient DMUs are the same only from an aggregate input/aggregate output perspective. One could, for example, in the general case have two DMUs j_1 and j_2 where one is twice the size of the other (in each of the inputs and outputs), and yet both could be efficient. In such a case an equal allocation $r_{j_1} = r_{j_2}$ might seem unreasonable where $x_{ij_1} = 1/2 x_{ij_2}$ for all i. Such a situation could, of course, not happen in the pure input case, since if j_1 is efficient (i.e., $f_{j_1} = 1$), then $f_{j_2} = 2$; that is the larger $\mathrm{DMU}j_2$ will not be efficient.

If we then make the assumption that $r_j = r_o$ for $j \in J_{j_o}$, then from Eq. (31)

$$r_{j_o} = r_o \sum_{j \in I_{j_o}} u_j^{j_o}. \tag{32}$$

From the dual theorem of linear programming however, the objective functions of (P'_{j_o}) and (D'_{j_o}) are equal, hence $f_{j_o} = \sum_{j \in J_{j_o}} u_j^{j_o}$, and

$$r_{j_o} = r_o \cdot f_{j_o}. \tag{33}$$

Thus, in the pure input case a fair allocation of a fixed resource to a set of n DMUs is one which assigns $\mathrm{DMU}j_o$ an amount proportional to its aggregated or virtual input, as obtained from the DEA exercise. This result complies with the allocation rule of the one-dimensional case.

We now wish to apply the *invariance* and *input pareto-minimality* principles to the allocation of shared costs in the general multiple-inputs multiple-outputs case.

6. The general case

Consider an augmented version of model (P_{j_o}), namely:

$$(Q_{j_o}) \qquad \hat{f}_{j_o} = \min \quad \sum_{i=1}^{I} v_{ij_o} x_{ij_o} + v_{I+1_{j_o}} r_{j_o} \tag{34}$$

s.t.

$$\sum_{k=1}^{K} \mu_{kj_o} y_{kj_o} = 1, \tag{35}$$

$$-\sum_{k=1}^{K} \mu_{kj_o} y_{k_j} + \sum_{i=1}^{I} v_{ij_o} x_{ij} + v_{I+1_{j_o}} r_j \geq 0, \quad j = 1, \ldots, n, \tag{36}$$

$$v_{ij_o}, \mu_{kj_o} \geq 0, \qquad \forall i, k.$$

As before, the condition that satisfies the two principles is

$$\bar{z}_{I+1} = -r_{j_o} + \sum_{j=1}^{n} u_j^{j_o} r_j = 0, \tag{37}$$

where $u_j^{j_o}, j = 1, \ldots, n$, are the optimal dual variables of (P_{j_o}) corresponding to constraints Eq. (11) in (P_{j_o}). As before, letting J_e denote the set of indices of all efficient DMUs, it follows from the complementary slackness property of linear programming that

$$-r_{j_o} + \sum_{j \in J_e} u_j^{j_o} r_j = 0, \quad j_o \in \bar{J}_e, \tag{38}$$

must hold for any inefficient DMU j_o. We, therefore, conclude that any cost allocation $r = (r_1, \ldots, r_n)$ must satisfy the set of equations

$$r_\ell = \sum_{j \in J_e} u_j^\ell r_j \text{ for all } \ell \in \bar{J}_e \tag{39}$$

and

$$\sum_{\ell=1}^{n} r_\ell = R. \tag{40}$$

The following two properties hold by virtue of Eqs. (39) and (40):

Property 1. The allocation r_j of the cost to the efficient DMUs $j \in J_e$ is such that

$$\sum_{j \in J_e} \left(1 + \sum_{\ell \in \bar{J}_e} u_j^\ell\right) r_j = R. \tag{41}$$

Property 2. For a given relative distribution of the cost across the efficient DMUs, the allocation $\{r_j\}_{j \in \bar{J}_e}$ to the inefficient DMUs is uniquely determined.

Thus, we have obtained a characterization for an equitable allocation of shared costs in a multiple-input multiple-output case. Specifically, any allocation that belongs to the set

$$\mathscr{A} = \left\{ r | r_l = \sum_{j \in J_e} u_j^\ell r_j, \ell \in \bar{J}_e \right\}$$

is an equitable allocation. It satisfies both the invariance and the pareto-minimality principles. Evidently, this allocation is not unique. It has degrees of freedom the number for which is equal to the number of efficient DMUs minus one. Therefore, $\mathscr{A}$ cannot be used to determine a cost allocation among the DMUs but rather to examine existing costing rules for equity.

If the preliminary DEA analysis produced only one efficient DMU, then the allocation is unique. This situation, however, is very unlikely to occur in real world problems. One can reach such situations by prioritizing the efficient DMUs. Several methods for prioritizing efficient units are reported in the literature – see, for instance, [1,4]. One way to obtain a single allocation in this case is to impose cone-ratio type constraints (see e.g., Charnes et al. [2]) on the weights. Specifically, we add the following constraints to Eqs. (34)–(36):

$$\frac{1}{c} \le \mu_k/\mu_s \le c, \quad s, k = 1, \ldots, K, \tag{42}$$

$$\frac{1}{c} \le v_i/v_t \le c, \quad t, i = 1, \ldots, I. \tag{43}$$

These constraints are used to identify the most robust efficient DMU, that is, the DMU that maintains efficiency as the weights get more and more "spread out" among the various inputs and outputs. As $c \to 1$, the most robust efficient DMU emerges, where robustness is measured here in terms of efficiency invariance to a wide range of non-zero multiplier values. In that case, a unique set of relative costs is obtained, as it is readily seen from Eq. (38).

To demonstrate this method for prioritizing the efficient units, consider the data in Table 1.

Table 1
Input–output data

DMU	Input1	Input2	Input3	Output 1	Output2
1	350	39	9	67	751
2	298	26	8	73	611
3	422	31	7	75	584
4	281	16	9	70	665
5	301	16	6	75	445
6	360	29	17	83	1070
7	540	18	10	72	457
8	276	33	5	78	590
9	323	25	5	75	1074
10	444	64	6	74	1072
11	323	25	5	25	350
12	444	64	6	104	1199

Running a (CCR) DEA model on these data results in four efficient DMUs: 4, 5, 8, and 9. Therefore, the allocation is not unique and therefore the set $\mathscr{A}$ can be used only to examine any given cost allocation for equity. If we impose ratio restrictions on the weights as in Eq. (43) above, then for $c = 12.4$, DMU 9 emerges as the single efficient one. The efficiency ratings θ and the optimal u_9 value – which is to be used in Eq. (38) – are shown in Table 2.

The u_9 values for the various DMUs represent – as per Eq. (38) above – the relative cost allocation for the corresponding DMUs. For example, the cost that is to be allocated to DMU 3 is 27.5% higher than that cost to DMU 9 (the efficient one), and the cost allocation to DMU 4 is only 76.6% of that of DMU 9.

Note that these relative cost allocations reflect the activity of a DMU, as represented by the *inputs*. For example, DMU 4, with input vector (281,16,9), represents a general lower activity rate than DMU 9 with an input vector of (323, 25, 5). Moreover, the outputs are used *only* to determine the reference (efficient) DMU. As was the case in the single-input single-output case, once the efficient DMU is found, the *allocation is determined entirely by the input side*. To demonstrate this property, consider DMUs 11 and 12 which have identical input vectors to that of DMUs 9 and 10, respectively. Their output vectors are, however, quite different – DMU 11 has a lower output vector than DMU 9 while DMU 12 has a higher output vector than DMU 10. This latter situation is well reflected in their corresponding efficiency ratings θ. However, their shares of the fixed cost are identical to those of DMUs 9 and 10, respectively. Thus, only the activity level of a DMU indeed affects its corresponding cost allocation – as one will naturally expect.

Table 2
Efficiency ratings and dual variable values

DMU	θ	u_9
1	1.377	1.987
2	1.220	0.923
3	1.688	1.275
4	1.001	0.766
5	1.155	0.792
6	1.480	1.115
7	1.799	1.212
8	1.103	0.857
9	1.000	1.000
10	1.382	1.379
11	3.036	1.000
12	1.111	1.379

7. Discussion

The problem of allocating an ongoing fixed cost such as annual overhead, is important in many managerial decision problems. When similar units share a common resource pool, such as head office management expenses, centralized technology, or annual advertising expenses, cost center considerations point to a need to allocate the cost fairly across the various units. For the simple and straightforward case of one input and one output – through the pure input case – to the general multiple-inputs multiple-outputs case, we have shown that DEA can be used to obtain a characterization of an equitable cost allocation. This DEA-oriented cost allocation approach reflects the activity level represented by the input consumption of the DMU.

As emphasized earlier, the method is a generalization of the simpler idea of fixed allocations being proportional to variable resource consumption. To facilitate *buy-in* by management it is recommended that the review of a DMU's share of fixed costs be undertaken only at convenient points in time. This might occur annually, when new and up-to-date cost figures are available; such would be the case for, say, annual advertising budgets.

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