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A new look at the 3:1 rule of combat through Markov Stochastic Lanchester models

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The 3:1 rule of combat states that in order that for the attacker to win the battle, his forces should be at least three times the force of the defender. This somewhat vague statement has resulted in numerous interpretations and discussions from historical and military science points of view. In this paper we attempt to examine this rule by utilising a number of Markov Stochastic Lanchester models that correspond to various basic combat situations and to draw some conclusions from their implementations. We identify general combat situations where the 3:1 rule is reasonable as well as situations where the force ratio should be either smaller or larger. Since the analysis is performed in the formal and somewhat 'sterile' setting of (pure) mathematical modeling, the results should be appropriately interpreted as reasoning of a certain abstraction of the battlefield.

Keywords: battle; attrition; force ratio; Markov Stochastic Lanchester; breakpoint; exchange ratio

Introduction

The question of how wars are fought and what factors constitute victory in them has drawn much attention throughout history. In particular, the issue of force ratio and its impact on the outcome of battles was addressed as early as the 5th century BC by Sun Tzu and later by Clausewitz, Lanchester, Liddell Hart and many others. The question of 'necessary' or 'optimal' force ratio that is required to achieve a victory has been analysed extensively.[1–6] Mearsheimer[7] echoes the rule of thumb shared by many that 'An attack requires more than a 3:1 advantage on each main axis to succeed'. In more recent papers this general statement is challenged by Epstein.[8] The arguments that are used to support or to oppose this rule range from quotations of combat commanders, and reference to historical data, to the utilisation of formal macroscopic models.

In this paper we propose a new look at this issue by applying the theory of Stochastic Lanchester Modeling. A major feature of combat is the inherent randomness that dominates its events and the way they unfold.[9] One way to account for the randomness in analysis, and in particular in the attrition process that constitutes a central factor in combat, is by utilising Stochastic Lanchester models. These stochastic models provide a simple representation of the attrition process in combat and offer a convenient tool for obtaining probability measures of combat events. These probabilities facilitate a reasonable way for evaluating the 3:1 rule.

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The 3:1 rule is examined in this paper with respect to the models that correspond to four combat situations. The first two models—the Ancient battle and the Modern battle—correspond to battles in which the attrition follows Lanchester's linear law and Lanchester's square law, respectively. Although these two types of battles are used commonly in combat analysis, they are not very realistic and serve here as a point of reference for the last two models that are more realistic. The other two models— the Fixed Front battle and the Variable Front battle— represent a more realistic setting in which the attacker commits to the attack only part of his force while the rest is kept as a reserve. The two models correspond to different tactical rules according to which the reserve force is employed.

It is shown that, with respect to these models, the 3:1 rule may be appropriate only in extreme cases which are identified quantitatively in terms of the *individual exchange ratio.* This ratio, which is defined in Section 3, represents the relative effectiveness and vulnerability of the weapon systems on the two sides. It is also shown that the endurance of the two forces—represented by the acceptable attrition levels on each side—plays a major role in determining the validity of this rule. Specifically, it is shown to what extent high motivation and determination can be traded off for an inferior individual exchange ratio in the 'validity region' of the 3:1 rule.

Section 2 presents a brief, and basically qualitative, review of stochastic battle models. The technical details regarding these models are given in Appendix A. Section 3 introduces notation and preliminary results that are used throughout this paper. The Ancient battle, Modern battle, Fixed Front battle and Variable Front battle are described in Sections 4, 5, 6 and 7 respectively, and results pertaining to the 3:1 rule are presented there. The corresponding formal mathematical models are presented in Appendices B, C, D, E. Section 8 summarises the analysis and provides general conclusions regarding the four combat situations that have been examined.

Stochastic Lanchester Models

Consider a battle where two opposing homogeneous forces, that is all weapons on any one side are the same, are engaged in a continuous-time combat. The combat terminates when one of the forces reaches a specified level of attrition called *terminating attrition level.* The defending force is called *Blue* and the attacking force is called *Red.*

The *state* of the engagement at time t is defined by the (cumulative) attrition of each side at that time. A particular engagement state can give rise to a number of other engagement states, however the probability for that transition does not depend on the past history of the engagement but only on its present state. This is the Markovian property. See References 10–12 for reviews of these types of models.

The *intrinsic individual attrition intensity* is the kill intensity of a combatant in a duel. The *individual exchange ratio* between Blue and Red is the ratio between the two respective individual attrition intensities. The *break point* of Blue (Red) is a value between 0 and 1 that indicates the attrition ratio at which the Blue (Red) force disengages the battle. That is, the break point is the ratio between the terminating attrition level and the initial force size.

Notations

$M(N)$ = Initial force size of Blue (Red);

$m_0(n_0)$ = Terminating attrition level of Blue (Red);

(i, j) = A state in the engagement which represents the cumulative attrition of Blue(i) and Red(j). Clearly, the initial state of the engagement is (0, 0) and the absorbing states are either of the form (i, n_0), where $i = 0, \ldots, m_0 - 1$, in which case Blue wins, or of the form (m_0, j), where $j = 0, \ldots, n_0 - 1$, in which case Red wins. As is shown in Appendix A, the state (m_0, n_0), which implies a draw, cannot be reached under the assumptions that are made.

$h_b(i, j)$, $(h_r(i, j))$ = *Attrition function* of Blue (Red), that reflects the particular engagement pattern of the battle. These functions depend on the number of live combatants on both sides and are defined separately for each combat situation, as it is shown in the subsequent sections.

$\lambda(\mu)$ = Intrinsic individual attrition intensity of a Blue (Red) combatant.

$b(i, j)\,(r(i, j))$ = *Attrition rate* of Blue (Red). The attrition rate is the product of the intrinsic individual attrition intensity and the appropriate attrition function. That is,

$$b(i, j) = \lambda h_b(i, j) \quad \text{and} \quad r(i, j) = \mu h_r(i, j).$$

α = Individual exchange ratio; that is, $\alpha = \lambda/\mu$.

f_b, (f_r) = Break point of Blue (Red). $0 < f_b, f_r \leqslant 1$,

$P[R](P[B])$ = Win probabilities of Red (Blue). Clearly: $P[B] = 1 - P[R]$

$E[R](E[B])$ = Expected number of casualties of Red (Blue) at the end of the battle.

C = Expected casualties ratio between Red and Blue. That is: $C = E[R]/E[B]$.

Assumptions

A1—Each time step in the engagement is small enough such that at most one casualty on both sides can be recorded in each time step.

A2—The probability of a casualty is stationary and is proportional to the length of the time step.

A3—The probability of the transition from state (i, j) to state $(i, j+1)$ is obtained by dividing the attrition rate $b(i, j)$ of Blue by the sum of the attrition rates $b(i, j) + r(i, j)$. The probability of the transition from state (i, j) to state $(i+1, j)$ is obtained similarly.

In Appendix A we present a detailed description of the Markov Stochastic Lanchester Model that is implemented in our analysis.

A deterministic counterpart to a stochastic model is defined next. The results that are obtained from this deterministic representation of the battle are used as reference points in the stochastic analysis that is performed. This family of well known models, called Deterministic Lanchester models[13] describe the transitions in the state of a battle with respect to time in terms of fixed average values called *attrition coefficients.* Probably the best-known deterministic model is Lanchester's Square Law given by

$$\begin{aligned} B'(t) &= -\rho \cdot R(t) \\ R'(t) &= -\beta \cdot B(t) \end{aligned} \qquad (1)$$

where $B(t)$ and $R(t)$ are the number of Blue survivors and Red survivors at time t respectively, β and ρ are the corresponding attrition coefficients, and the initial force sizes are $B(0) = M$ and $R(0) = N$.

The connection between a stochastic model and its corresponding deterministic model is stated in the following assumption:

A4—The individual exchange ratios in both a stochastic model and its corresponding deterministic model are equal. That is: $\lambda/\mu = \beta/\rho$.

The outcome of a deterministic model is uniquely determined by $B(0)$, $R(0)$, β and ρ. An interesting question, regarding combat models is: when are the two sides of equal strength?

Definition The two sides, Blue and Red, are said to be at *deterministic-parity* if the battle between them, as depicted by an appropriate deterministic model, results in a draw. The individual exchange ratio α for which this parity is obtained is called *deterministic-parity point* and is denoted by α^*.

It can be shown, for example, that if the battle is described by Lanchester's Square Law, then the two forces are at deterministic-parity if and only if

$$\rho(1-(1-f_r)^2)R(0)^2 = \beta(1-(1-f_b)^2)B(0)^2 \quad (2)$$

where f_b and f_r are the corresponding break points. Therefore the deterministic-parity point is,

$$\alpha^* = \frac{1-(1-f_r)^2}{1-(1-f_b)^2}\left(\frac{N}{M}\right)^2 \quad (3)$$

Unless otherwise indicated, henceforth we will assume that the initial force sizes are $M = 10, 20, 30, 40, 50$, and $N = 3M$, for Blue and Red respectively; these numbers represent typical tactical level battles.

The calculation of $P[R]$ and C are made by repeatedly solving a set of difference equations, until the sum $P[R] + P[B]$ coverges to 1. At the beginning of the process, this sum equals to 0, and it increases up to 1 as the process advances; for more details, see Appendix A.

The analysis that is presented in the subsequent sections is concentrated in the following question: *What does the 3:1 rule mean, in terms of* $P[R]$ *and* C, *as a function of the exchange ratio* α *and the size of the battle* M?

In other words, we wish to identify combat situations where the 3:1 rule seems to be applicable and other situations where this rule is either optimistic or pessimistic.

In the following sections we characterise four types of battles. For each battle we examine the 3:1 rule with respect to three situations of terminating attrition levels:

(a) $m_0 = M$ and $n_0 = N$: Both sides are ready to fight to annihilation.

(b) $m_0 = 0.3M$ and $n_0 = 0.3N$: Both sides are equally determined, but ready to endure only 30% attrition.

(c) $m_0 = 0.5M$ and $n_0 = 0.3N$: Red (the attacker) is less determined than Blue (the defender). Red is ready to endure 30% attrition while Blue may endure 50% attrition.

The ancient battle

The ancient battle is characterised by a set of duels that are fought individually. These duels may be executed concurrently, as is the case when two opposing phalanxes of swordsmen are engaged in combat, for example the Three Musketeers taking on a larger size enemy but in separate duels, or executed sequentially, as is the case in contests of gladiators in an arena. In Lanchester Theory this type of combat is represented by the Linear Law.[14] In more recent combat scenarios, this model may represent, for example, a situation where two columns of tanks meet on a narrow mountain road. Only the front tank in each one of two opposing columns may engage in combat. A variant of this mountain battle is analysed in Reference 15.

At any point of time in the Ancient battle, the number of duels that are in progress is determined by the number of survivors on both sides. If the battle is of the 'concurrent' type (for example, an engagement of two phalanxes) then the number of duels in progress is the minimum between the number of Blue survivors and Red survivors. If the battle is of the 'sequential' type (for example, two columns of tanks on a narrow mountain road) then this number is equal to one.

Analysis

It is easily seen (see Appendix B) that the win probability $P[R]$ is determined by the individual exchange ratio α and the terminating attrition levels m_0 and n_0. The initial sizes of the forces, M and N, may influence the level at which these attrition levels are set, but otherwise they have no direct effect on the outcome of the battle. It follows that any Ancient battle with parameters M, N, f_b, f_r, and α is equivalent to an Ancient battle with initial force sizes $m_0 = f_b M$ and $n_0 = f_r N$, for Blue and Red respectively, that continues until annihilation. Therefore, the interpretation of the 3:1 rule here is in terms of the attrition thresholds which may be considered as the initial force sizes in an equivalent combat situation.

Table 1 presents the 3:1 win probabilities $P[R]$ for various values of α and $M(= m_0)$. The deterministic-parity point in this combat situation is equal to the initial force ratio (See (19) in Appendix B), hence $\alpha^* = 3$.

It can be seen that at deterministic-parity Red has a slight advantage for all battle sizes M. That is, $P[R] > P[B]$ for all M. This advantage however decreases, and $P[R]$ approaches 0.5, as the size of the battle M increases. Moreover, as M increases, $P[R]$ approaches faster to 0 or 1 for individual exchange ratios α larger or smaller than α^*, respectively. This means that the deterministic-parity point α^* becomes more indicative in terms of separating between the win and the loss situations. For example, if $\alpha = 4$ Red has an approximately even chance to defeat Blue if $M = 1$ and $N = 3$. If however $M = 50$ ($N = 150$) this win probability is reduced to 5%.

These characteristics are depicted in Figure 1 below, where the horizontal axis corresponds to the individual exchange ratio α, and the vertical axis represents the win

Table 1 Win probabilities for $R - P[R]$

α	$M = 1$	$M = 3$	$M = 5$	$M = 10$	$M = 20$	$M = 30$	$M = 40$	$M = 50$
7	0.33	0.15	0.08	0.02	0.00	0.00	0.00	0.00
4	0.49	0.38	0.33	0.24	0.15	0.10	0.07	0.05
3.50	0.53	0.46	0.42	0.36	0.29	0.25	0.21	0.18
3.25	0.55	0.50	0.47	0.44	0.40	0.37	0.34	0.32
3.10	0.57	0.53	0.51	0.49	0.47	0.45	0.44	0.43
3	**0.58**	**0.54**	**0.53**	**0.52**	**0.52**	**0.51**	**0.51**	**0.51**
2.90	0.59	0.56	0.56	0.56	0.57	0.58	0.59	0.59
2.75	0.61	0.60	0.60	0.62	0.65	0.67	0.69	0.71
2.50	0.64	0.65	0.67	0.71	0.78	0.82	0.85	0.88
2	0.70	0.77	0.81	0.88	0.95	0.98	0.99	1.00
1	0.88	0.97	0.99	1.00	1.00	1.00	1.00	1.00

probability $P[R]$. Computing the expected number of casualties, on both sides, and taking their ratio C, results in values that are constant for all battle sizes M. A generalisation of this property is shown to be true for the Ancient battle in the following lemma. Its proof is given in Appendix B.

Lemma 1 *For the Ancient battle the expected casualties ratio between Red and Blue is independent of the size of the battle M. Moreover, this ratio is independent of both the initial force ratio and the attrition levels and is always equal to the individual exchange ratio between Blue and R. That is,* $C = \alpha$.

The 3:1 rule

The 3:1 ratio is clearly insufficient for values of α that are significantly larger than the deterministic-parity point 3. However, this ratio may be also overstated if Red is willing to accept win probabilities that are, say, less than 95%. For example, if $\alpha = 2$ (Blue is twice as effective as Red on an individual basis) and $m_0 = 30$ then a 3:1 ratio seems to be too excessive since Red could reduce its force to, say, 2.5:1 ratio and still maintain a win probability of over 80%. On the other hand it is clear that at deterministic-parity ($\alpha = 3$) the 3:1 rule may not be sufficient to an attacker that requires more than a slight advantage over Blue in the odds of winning the battle.

Notice that as the battle size (expressed in terms of m_0 and n_0) increases the information regarding the enemy (Blue) becomes more critical. A small error in estimating α and m_0 may result in either committing too much force to the battle, which could have been won with less force, or a sure defeat.

The modern battle

This battle proceeds as follows: Each shooter in Blue (Red) selects randomly a target to engage from Red (Blue). A shooter keeps on engaging its target until either he or its

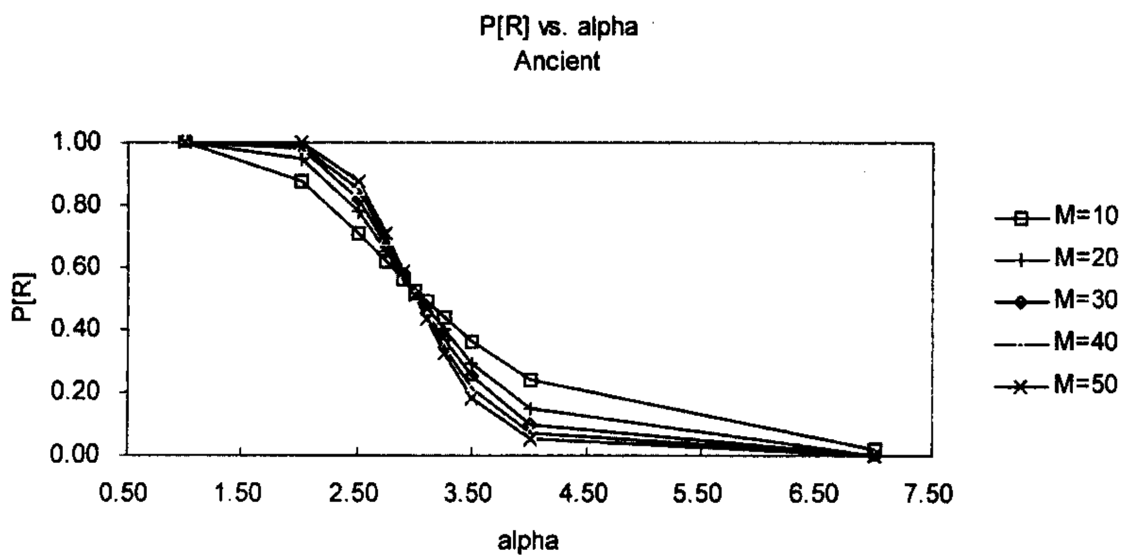

Figure 1 Win probabilities for $R - P[R]$.

Table 2 Win probability of Red, for different (f_b, f_r)

(f_b, f_r)	α	$M = 10$	$M = 20$	$M = 30$	$M = 40$	$M = 50$
(1, 1)	9.5	0.42	0.41	0.39	0.38	0.37
	9.25	0.44	0.44	0.44	0.43	0.43
	9	**0.47**	**0.48**	**0.48**	**0.48**	**0.49**
	8.5	0.52	0.56	0.58	0.59	0.61
	8	0.58	0.64	0.67	0.70	0.72
(0.3, 0.3)	9.5	0.49	0.47	0.46	0.45	0.44
	9.25	0.51	0.49	0.48	0.48	0.47
	9	**0.52**	**0.52**	**0.51**	**0.51**	**0.51**
	8.5	0.56	0.56	0.57	0.58	0.59
	8	0.59	0.61	0.63	0.65	0.66
(0.5, 0.3)	7	0.40	0.36	0.34	0.31	0.30
	6.6	0.44	0.42	0.40	0.39	0.38
	6.12	**0.49**	0.49	**0.49**	**0.50**	**0.50**
	5.8	0.53	0.55	0.56	0.57	0.58
	5	0.63	0.69	0.73	0.76	0.79

target are killed. If the latter has happened, then the shooter selects a new (live) target and a new engagement commences. Therefore, this battle represents an engagement where all live combatants are actively involved in combat and targets may be engaged by more than one shooter.

Unlike the Ancient battle this type of battle is characterised by the effect of concentrated fire. This effect has come to light with the introduction of modern, long-range weapons. Therefore, this type of battle is sometimes called the *Modern battle*[16], a name that we adopt here too for convenience. This type of combat is described by the well known Lanchester's Square Law. The stochastic model for this battle, and the corresponding deterministic model, are shown in Appendix C.

Analysis

In the modern battle the initial sizes of the forces on both sides have a direct effect on the win probability $P[R]$. This dependence on the initial force sizes can also be seen in the deterministic-parity point:

$$\alpha^* = \frac{2n_0N - n_0^2}{2m_0M - m_0^2} \tag{4}$$

which is an explicit function of M and N.

Table 2 presents the win probabilities $P[R]$ for three battle termination conditions given by the break points pairs (1, 1), (0.3, 0.3) and (0.5, 0.3). These probabilities are given for various values of individual exchange ratios α in the neighbourhood of the deterministic-parity point α^*, and for various values of initial Blue force $M(N = 3M)$. For each pair of breakpoint values the row corresponding to the deterministic-parity point α^* is highlighted. If $f_b = f_r$, then the deterministic-parity point is $\alpha^* = 9$. In this case, it can be seen from the table that, at this parity point, $P[R]$ approaches 0.5 as the battle size M increases. However it is interesting to note that while $P[R] < 0.5$ when the break points are 1, the reverse is true when these break points are 0.3. It can be verified, numerically, that $P[R] = 0.5$, for $M \geqslant 10$, around $f_b = f_r = 0.55$. This phenomenon is discussed later on.

The case where $f_r = 0.3$ and $f_b = 0.5$ demonstrates the significant effect of endurance on the outcome of the battle. Evidently, the odds for the less determined Red to win the battle are reduced considerably. For values of α greater than 6.12 (the deterministic-parity point of the (0.5, 0.3) battle), this effect becomes more significant as the battle size increases. For example, if $\alpha = 7$ then for $M = 10, P[R]$ is reduced from 0.67 in the (0.3, 0.3) battle to 0.4 in the (0.5, 0.3) battle. If $M = 50$, then the probability decreases from 0.81–0.3.

Table 3 presents the deterministic-parity points and the corresponding win probabilities $P[R]$ for the case where $M = 10, N = 30$ and the break points range between 0.1 (low endurance) to 1 (total 'maintenance of objective'). It can be seen that the Red force is better off, in terms of the win probability at deterministic-parity, as it demonstrates a higher level of determination in battle. However, if Blue manifests a high level of determination too then the advantage is reversed to his favour. For example, if both sides manifest an equally low endurance ($f_b = f_r = 0.1$) then, at deterministic-parity, Red has a slight advantage ($P[R] = 0.57$). This situation is reversed ($P[R] = 0.47$) if both sides are highly determined ($f_b = f_r = 1$).

Figure 2 presents the expected casualties ratio C for the case $(f_b, f_r) = (1, 1)$. One can see the correlation between C and α which is almost linear for each value of M. The slope of the line increases as M increases. The linear relation between C and α is even more notable in the cases $(f_b, f_r) = (0.5, 0.3)$ and $(f_b, f_r) = (0.3, 0.3)$, which graphs are omitted here for the sake of brevity. This result can be partly explained by the fact that Red becomes more vulnerable as α increases. As a result, if the battle does not terminate at an early stage (for example, after 30% of attrition), its duration is long enough so that the rapid increase in the force ratio between the live Blue combatants and the live Red combatants can be observed.

Table 3 Deterministic-parity points (α^*) and win probabilities ($P[R]$). $M = 10$ and $N = 30$

	$f_b = 0.1$		$f_b = 0.3$		$f_b = 0.5$		$f_b = 0.7$		$f_b = 1$	
f_r	α^*	$P[R]$	α^*	$P[R]$	α^*	$P[R]$	α^*	$P[R]$	α^*	$P[R]$
0.1	9	0.57	3.35	0.48	2.28	0.45	1.88	0.43	1.71	0.41
0.3	24.16	0.60	9	0.52	6.12	0.49	5.04	0.47	4.59	0.44
0.5	35.53	0.61	13.24	0.54	9	0.50	7.42	0.48	6.75	0.45
0.7	43.11	0.61	16.06	0.54	10.92	0.51	9	0.49	8.19	0.46
1	47.37	0.62	17.65	0.55	12	0.52	9.89	0.50	9	0.47

The 3:1 rule

Table 4 below presents the 60–90% range of $P[R]$ in terms of the individual exchange ratio α. For each one of the three break points pairs, these values represent the range of α for which the 3:1 rule is valid if the attacker requires a minimum of 0.6 win probability $P[R]$, and is content with a maximum of 0.9 probability. These ranges are given for the five battle sizes $M = 10, 20, 30, 40, 50$.

Table 4 Ranges of α for 0.6–0.9 win probabilities for modern battle

	$(f_b, f_r) = (1, 1)$		$(f_b, f_r) = (0.3, 0.3)$		$(f_b, f_r) = (0.5, 0.3)$	
M	α (0.6)	α (0.9)	α (0.6)	α (0.9)	α (0.6)	α (0.9)
10	7.8	4	7.9	4	5.2	2.9
20	8.2	6.1	8.1	4.9	5.5	3.5
30	8.4	6.5	8.3	5.5	5.6	4
40	8.5	6.8	8.4	6	5.7	4.2
50	8.5	7	8.4	6.2	5.7	4.4

The fixed front battle

In many actual combat situations the attacker may wish to divide his force into two parts: an assault force that is actively involved in the engagement and a reserve force that is neither effective nor vulnerable. Such a deployment may be derived from doctrine, the terrain may dictate it or it may be a response to the enemy's deployment. The two stochastic models, that are presented in this section and in the following one, address this typical combat situation. In both models we assume that the defender (Blue) fights with all its force during the entire engagement. The difference between the two models lies in the way by which the attacker (Red) deploys and utilises his reserve force.

The first model is called the Fixed Front battle. In that combat situation the attacker assigns for the assault 2/3 of his initial force $N(= 3M)$, which is twice the size of the defender at the beginning of the engagement. This size of $2/3\ N$ is maintained during the engagement by continuously reinforcing the assault force by combatants from the reserve. Clearly, this reinforcement process is terminated when the attrition of Red is higher than $1/3\ N$. In that case, if the engagement is still in progress, that is if neither Blue nor Red has reached the terminating attrition level, then Red engages Blue with all of its live combatants—leaving no reserve units.

The stochastic model for that combat situation, and the corresponding deterministic model, are given in Appendix D.

Analysis

Recall that in the Modern battle the deterministic-parity point is the same ($=9$) for all the combat situations where the break points f_b and f_r are equal. This is not the case in the Fixed Front battle, as shown in Table 5. The deterministic-parity point α^* decrease as the break points f_b and f_r, $(f_b = f_r)$ decreases. That is, if the endurance of Blue and Red decreases at the same rate, then it is to the advantage of the defender Blue who can gain parity with a smaller individual exchange ratio α.

Table 6 presents the win probabilities $P[R]$, in the neighbourhood of the deterministic-parity points, for the three battle termination conditions $(1, 1)$, $(0.3, 0.3)$ and $(0.5, 0.3)$. Notice that the differences between the win probabilities in the two cases with equal break points, $(1, 1)$ and $(0.3, 0.3)$, are somewhat more significant here than is the case in the Modern battle. For example, consider the case where $M = 50$ and $\alpha = 8$. The ratio between the $(1, 1)$ case and the $(0.3, 0.3)$ case in the Fixed Front battle is

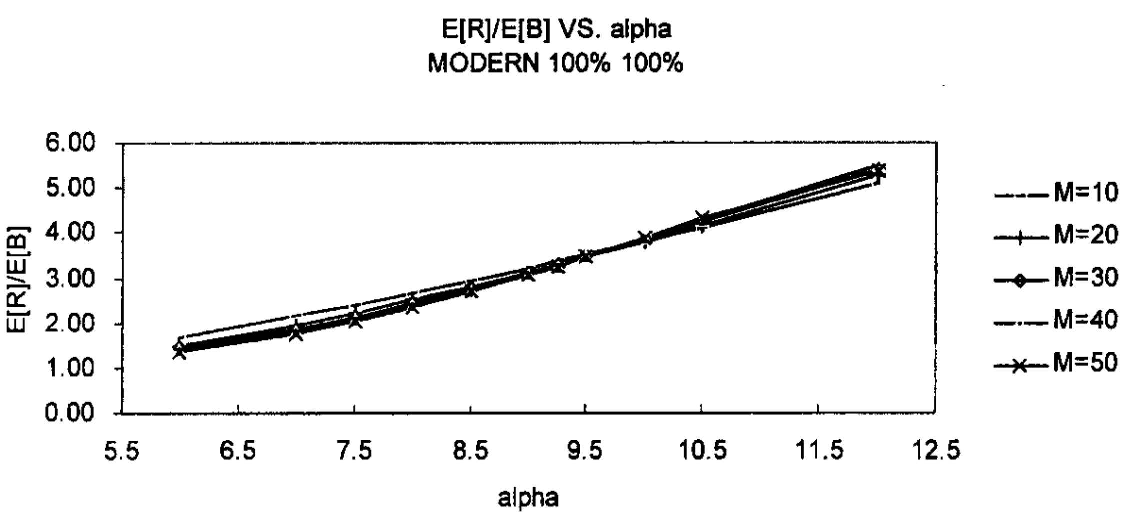

Figure 2 Expected casualties ratio, $(f_b, f_r) = (1, 1)$.

Table 5 Deterministic-parity points for equal break points

$f_b = f_r$	1	0.8	0.6	0.4	0.2
Deterministic-parity α	8	7.96	7.81	7.44	6.67

0.48/0.34 = 1.41. This ratio in the Modern battle is 0.72/0.66 = 1.09 (see Table 2). The greater effect of endurance on the Fixed Front battle is partially attributed to the enhanced variability of its deterministic-parity point, as compared to the Modern battle.

The 3:1 rule

Similarly to Table 4 in the previous section, Table 7 presents the 60–90% range of $P[R]$ in terms of the individual exchange ratio α. Comparing Table 7 with Table 4, it can be seen that the smallest and the largest figures in each table are not significantly different. The smallest figure, $(\alpha(90), M = 10, (f_b, f_r) = (0.5, 0.3))$ is 2.2 for the Fixed Front battle and is 2.9 for the Modern battle. The largest figure $(\alpha(60), M = 50, (f_b, f_r) = (1, 1))$ is 7.6 for the Fixed Front battle and 8.5 for the Modern battle. We may conclude that even though the Fixed Front battle is significantly different from the Modern battle, yet the implications of the 3:1 rule are similar. Specifically, if the battle is more than of minimal size, then the 3:1 rule is too strict if the individual exchange ratio α is smaller than 5, 4 and 3 for (1, 1), (0.3, 0.3) and (0.5, 0.3) break points, respectively. On the other hand, if these exchange ratios are larger than 8, 7 and 5 respectively, then a 3:1 force ratio may not be sufficient to facilitate victory with an adequate probability.

The variable front battle

In this battle the objective is to commit to the engagement a *fixed proportion* of the existing active force, specifically, 2/3

Table 6 Win probability of Red, for different (f_b, f_r)

(f_b, f_r)	α	$M = 10$	$M = 20$	$M = 30$	$M = 40$	$M = 50$
(1, 1)	9	0.35	0.32	0.29	0.27	0.25
	8.5	0.41	0.39	0.38	0.37	0.36
	8	**0.46**	**0.47**	**0.48**	**0.48**	**0.48**
	7.5	0.53	0.56	0.58	0.60	0.62
	7	0.59	0.65	0.69	0.72	0.75
(0.3, 0.3)	8	0.44	0.41	0.38	0.36	0.34
	7.5	0.48	0.46	0.44	0.43	0.43
	7.06	**0.51**	**0.51**	**0.51**	**0.51**	**0.50**
	6.5	0.56	0.58	0.59	0.60	0.61
	6	0.61	0.64	0.67	0.69	0.71
(0.5, 0.3)	5.75	0.36	0.31	0.28	0.25	0.23
	5.25	0.42	0.40	0.38	0.37	0.35
	4.8	**0.48**	**0.48**	**0.49**	**0.49**	**0.49**
	4.25	0.56	0.60	0.63	0.66	0.68
	4	0.60	0.66	0.70	0.73	0.76

Table 7 Ranges of α for 0.6—0.9 win probabilities—fixed front battle

	$(f_b, f_r) = (1, 1)$		$(f_b, f_r) = (0.3, 0.3)$		$(f_b, f_r) = (0.5, 0.3)$	
M	α (0.6)	α (0.9)	α (0.6)	α (0.9)	α (0.6)	α (0.9)
10	6.9	4.6	6.1	3.1	4	2.2
20	7.3	5.3	6.3	3.9	4.3	2.8
30	7.4	5.8	6.4	4.4	4.4	3.1
40	7.5	6.1	6.5	4.7	4.4	3.3
50	7.6	6.3	6.6	4.8	4.5	3.4

of the existing force. Since the size of the force decreases in the course of the engagement due to attrition, the front width of the assaulting force may also change. Hence, this model is labelled Variable Front battle. If the total size of Red is reduced below one and a half times that of Blue, then Red commits all of his live combatants into the engagement. Therefore, the battle comprises of two stages. In the first stage Red commits two thirds of his existing force into the engagement with Blue. If the battle is not terminated when terminating attrition levels are reached, the second stage commences when the live force size of Blue is equal to $\frac{2}{3}$ that size in Red, in which case Red commits its entire force to the engagement.

Such an attrition process may be typical in situations where the attack is a first stage in a larger operation, and the prospects for reinforcement in the course of that operation are poor. The tactics that Red may choose in such situations are to hold back and keep a substantial part of his force as fresh as possible, and to engage the defender B with only a (constant) fraction of his force for as long as possible. By adopting such tactics, the attacker trades off win probability of the first engagement with the ability to keep on with its mission, once this engagement is over.

Analysis

Two properties relating to the deterministic form of the Variable Front battle are proved in Appendix E.

Property E.1 The Variable Front battle can never reach its second stage unless the exchange ratio α is strictly greater than 6.

Property E.2 Suppose that a Variable Front battle reached its second stage. Then α^* is a deterministic-parity point if and only if it is a root of the quadratic equation

$$4[1-(1-f_h)^2]\alpha^2 - [33 - 6(1-f_b)^2 - 36(1-f_r)^2] + 54[1-(1-f_r)^2] = 0 \qquad (5)$$

If none of the roots of (5) is larger than 6 then it follows that this battle cannot both reach stage 2 and terminate at deterministic-parity. Therefore, the existence of deterministic-parity implies that the battle comprised only stage 1. Solving (5) for $(f_b, f_r) = (1, 1)$ result in the roots

$\alpha_1 = 2.25, \alpha_2 = 6$. For the cases where the break points are $(0.3, 0.3)$ and $(0.5, 0.3)$ the equation (5) has no real-valued roots. We conclude that for these three choices of break point pairs, deterministic-parity applies when the battle consists of the first stage only. Hence, in that regard, the Variable Front model can be treated as a Modern battle and therefore the deterministic-parity points for the three cases are easily shown to be 6, 6 and 4.08, respectively.

The win probabilities of Red, in the neighbourhood of the deterministic-parity point, for the three battle termination break points are given in Table 8. The respective expected casualties ratios C are similar to those of the Modern battle.

It is interesting to note the effect of Red's tactical constraints on its win probabilities. Consider a battle of size $M = 50$ and individual exchange ratio $\alpha = 4$. The break points are 0.5 and 0.3 for Blue and Red, respectively. In the Modern battle, in which Red can bring into effect all its combatants and to concentrate its fire, the win probability $P[R]$ is 0.95. In the Fixed Front battle, this probability is reduced to 0.76 and in the Variable Front battle, which constitutes the worst case from the stand point of Red, this probability is around 0.50.

The 3:1 rule

Considering Table 9 below, and following the analysis in the previous section, it can be seen that the 3:1 rule is valid for battle conditions that may be quite realistic in actual combat. For example this rule may be justified for the Variable Front battle if the battle is of small scale ($M = 10$) and the accepted attrition level is $(0.5, 0.3)$. In that case, even a relatively small individual exchange ratio of $\alpha = 2$ is sufficient to justify the 3:1 rule. For larger battles, however, one must assume a much more favourable individual exchange ratio, from the defender point of view, to justify this force ratio. This last conclusion applies to all types of combat that have been discussed.

Table 8 Win probability of Red, for different (f_b, f_r)

(f_b, f_r)	α	$M = 10$	$M = 20$	$M = 30$	$M = 40$	$M = 50$
	7	0.33	0.28	0.25	0.22	0.20
	6.5	0.40	0.37	0.35	0.34	0.32
	6	**0.47**	**0.48**	**0.48**	**0.48**	**0.49**
(1, 1)	5.5	0.55	0.60	0.62	0.65	0.67
	5	0.64	0.71	0.76	0.80	0.83
	7	0.44	0.39	0.36	0.34	0.32
	6.5	0.48	0.45	0.43	0.42	0.41
	6	**0.52**	**0.52**	**0.51**	**0.51**	**0.51**
(0.3, 0.3)	5.5	0.57	0.59	0.60	0.61	0.62
	5	0.63	0.66	0.69	0.72	0.74
	5	0.36	0.30	0.27	0.24	0.21
	4.5	0.42	0.40	0.38	0.36	0.35
	4.08	**0.49**	**0.49**	**0.49**	**0.50**	**0.50**
(0.5, 0.3)	3.5	0.60	0.64	0.68	0.70	0.72
	3	0.70	0.78	0.82	0.86	0.89

Table 9 Ranges of α for 0.6—0.9 win probabilities—variable front battle

	$(f_b, f_r) = (1, 1)$		$(f_b, f_r) = (0.3, 0.3)$		$(f_b, f_r) = (0.5, 0.3)$	
M	α (0.6)	α (0.9)	α (0.6)	α (0.9)	α (0.6)	(0.9) α
10	5.2	3.4	5.3	2.6	3.5	1.9
20	5.5	4.1	5.4	3.3	3.7	2.4
30	5.6	4.4	5.5	3.7	3.7	2.6
40	5.6	4.6	5.6	4.0	3.8	2.8
50	5.7	4.7	5.6	4.1	3.8	2.9

Summary and conclusions

In this paper the 3:1 rule of combat has been examined from a theoretical point of view—utilising Markov Stochastic Lanchester models. Four models have been presented and analysed; two simple models, the Ancient battle and the Modern battle, and two more intricate models that take into account tactical considerations of deployment and force utilisation.

Within the context of these theoretical models, it has been shown that four factors determine the validity of the 3:1 rule: (a) The individual exchange ratio; (b) The size of the battle; (c) The acceptable attrition levels of both sides; and (d) The risk that Red is willing to take, represented by the acceptable range of the win probability $P[R]$. Arguably, the acceptable attrition level of Red is not independent of the win probability that Red is seeking to achieve.

Of the four models that have been analysed, the second one—The Modern battle—presents the most favourable posture from the point of view of Red. If Red can concentrate all its force against Blue and a single combatant of Blue is not *substantially* more effective than a single Red combatant (say, $\alpha < 3$), it has been shown that the 3:1 rule is too excessive from Red's point of view. In less favourable (but usually more realistic) situations in which Red cannot commit all his force to the assault, it has been shown that the 3:1 rule may be reasonable only for certain combinations of the factors described above. Specifically, if the battle is of small size, it is such that Blue has a higher endurance level than Red (50% acceptable attrition of Blue compared to 30% of Red) and an individual exchange ratio of at least 2, then for 90% confidence or higher, Red should attack with a 3:1 force ratio. In general, the individual exchange ratio α must be considerably larger (at least 5) to render the 3:1 (or higher) rule valid for that combat situations. The methodology that was used in this paper may be extended to other, richer and more involved, combat models that represent more realistic combat situations. Although this type of modeling offers very little in terms of predictive capabilities, it still provides a clear and relatively simple way for obtaining insights into the issue of determining the 'right' force ratio.

Appendix A—Markov Stochastic Lanchester models

General properties

Let (i, j) denote the state of the combat where Blue has i casualties, $i = 1, \ldots, m_0$, and Red has j casualties, $j = 1, \ldots, n_0$. Let $q(i, j; u, v; t, \Delta t), u = 0, \ldots, (m_0 - i)$, $v = 0, \ldots, (n_0 - j)$ denote the transition probability from state (i, j) at time t to state $(i + u, j + v)$ at time $t + \Delta t$. We assume that $q(i, j; m_0 - i, n_0 - j; t, \Delta t) = 0$, that is, a state of draw cannot be reached. Define the state probabilities

$$P(i, j, t + \Delta t) = \sum_{u=0}^{i} \sum_{v=0}^{j} q(i - u, j - v; u, v; t, \Delta t) \times P(i - u, j - v, t) \quad i = 0, \ldots, m_0, \; j = 0, \ldots, n_0 \quad (6)$$

with the initial condition $P(0, 0, 0) = 1$.

We also assume that multiple casualties of any type occur only with probability $o(\Delta t)$. Taking into account this assumption, we rewrite (6) as

$$\begin{aligned} P(i, j, t + \Delta t) = {} & q(i, j - 1; 0, 1; t, \Delta t) \cdot P(i, j - 1, t) \\ & + q(i - 1, j; 1, 0; t, \Delta t) \cdot P(i - 1, j, t) \\ & + q(i, j; 0, 0; t, \Delta t) \cdot P(i, j, t) + o(\Delta t) \end{aligned} \quad (7)$$

We further assume that for Δt small enough the transition probabilities $q(i, j; u, v; t, \Delta t)$ may be approximated by a linear function of Δt and that these transitions are stationary. That is

$$\begin{aligned} q(i, j; 0, 1; t, \Delta t) &= b(i, j)\Delta t \\ q(i, j; 1, 0; t, \Delta t) &= r(i, j)\Delta t \end{aligned} \quad (8)$$

The following set of differential equations is obtained:

$$\begin{aligned} dP(i, j, t)/dt = {} & -b(i, j) \cdot P(i, j, t) - r(i, j) \cdot P(i, j, t) \\ & + b(i, j - 1) \cdot P(i, j - 1, t) \\ & + r(i - 1, j) \cdot P(i - 1, j, t) \end{aligned} \quad (9)$$

and

$$\begin{aligned} dP(m_0, j, t)/dt &= r(m_0 - 1, j) \cdot P(m_0 - 1, j, t) \\ dP(i, n_0, t)/dt &= b(i, n_0 - 1) \cdot P(i, n_0 - 1, t) \end{aligned} \quad (10)$$

This set of equations is sometimes called the *Markov Stochastic Lanchester* model.[9] The transition probabilities $b(i, j)$ represent the effectiveness of the shooter, on one hand, and the vulnerability of the target, on the other. The same rule implies for $r(i, j)$

If the duration of the battle is not a parameter to be considered in the analysis, as it is the case in this paper, then the battle may be described by a Markov Chain with transition probabilities given by:

$$\begin{aligned} P_{(i,j)(i,j+1)} &= \frac{b(i, j)}{b(i, j) + r(i, j)} \\ q_{(i,j)(i+1,j)} &= \frac{r(i, j)}{b(i, j) + r(i, j)} \end{aligned} \quad (11)$$

We may now replace (9) by a set of difference equations that can easily be solved. That is,

$$\begin{aligned} P(i, j) = {} & q_{(i-1,j),(i,j)} \cdot P(i - 1, j) + P_{(i,j-1),(i,j)} \cdot P(i, j - 1) \\ & + (1 - q_{(i-1,j),(i,j)} - P_{(i,j-1),(i,j)}) \cdot P(i, j) \end{aligned} \quad (12)$$

The boundary equations are derived from (10).

Appendix B—The ancient battle

The stochastic model

At any point of time in the Ancient battle there are $s(i, j)$ duels in progress, where $s(i, j)$ is determined by the number of survivors on both sides. Formally, the 'concurrent' battle is described by (13) and the 'sequential' battle is described by (14).

$$\begin{aligned} & s(i, j) = h_b(i, j) = h_r(i, j) = \min\{M - i, N - j\} \\ & \quad i = 0, \ldots, m_0 - 1, j = 0, \ldots, n_0 - 1 \end{aligned} \quad (13)$$

$$\begin{aligned} & s(i, j) = h_b(i, j) = h_r(i, j) = 1 \\ & \quad i = 0, \ldots, m_0 - 1, j = 0, \ldots, n_0 - 1 \end{aligned} \quad (14)$$

In both cases: $s(i, j) = 0$ for $i = m_0$ or $j = n_0$

It follows that the stochastic model that represents this battle is

$$\begin{aligned} dP(i, j, t)/dt = {} & -\lambda s(i, j) \cdot P(i, j, t) - \mu s(i, j) \cdot P(i, j, t) \\ & + \lambda s(i, j - 1) \cdot P(i, j - 1, t) \\ & + \mu s(i - 1, j) \cdot P(i - 1, j, t) \\ & i = 0, \ldots, m_0, j = 0, \ldots, n_0 \end{aligned} \quad (15)$$

Following (11) in Appendix A, it can be easily seen that, at any point of time in the engagement, the probability that the next casualty will be Red is $p = p_{(i,j)(i,j+1)} = \lambda/(\lambda + \mu) = \alpha/(\alpha + 1)$. It follows that the win probability of Red $P[R]$ is given by

$$\begin{aligned} P[R] &= q^{m_0} \sum_{i=0}^{n_0-1} \binom{m_0 + i - 1}{m_0 - 1} p^i \\ &= \left(\frac{1}{\alpha + 1}\right)^{m_0} \sum_{i=0}^{n_0-1} \binom{m_0 + i - 1}{m_0 - 1} \left(\frac{\alpha}{\alpha + 1}\right)^i \end{aligned} \quad (16)$$

where $q = 1 - p$.

Brown[17] has shown that

$$P[R] \underset{m_0 \to \infty}{\longrightarrow} = \Phi\left[\frac{n_0 - \alpha \cdot m}{\sqrt{\alpha(m_0 - n_0)}}\right] \quad (17)$$

where Φ is the standard normal probability distribution function.

It can be seen that for m_0 large enough $P[R]$ approaches 1 if $n_0/m_0 > \alpha$, approaches 0 if $n_0/m_0 < \alpha$ and approaches 0.5 if $n_0/m_0 = \alpha$. For $n_0/m_0 \neq \alpha$, the rate of approach (to 0 or to 1) depends on size of $|n_0/m_0 - \alpha|$.

The deterministic model

The deterministic model depicting the Ancient battle, otherwise known also as Lanchester's First Linear Law, is given by

$$\begin{aligned} B'(t) &= -\rho \\ R(t) &= -\beta \end{aligned} \qquad (18)$$

where $B(0) = M$ and $R(0) = N$.

The deterministic-parity point here is

$$\alpha^* = \frac{f_r N}{f_b M} = \frac{n_0}{m_0} \qquad (19)$$

Proof of Lemma 1

Define:

$P_B(i, n_0)$ The probability that B wins the battle, having i casualties.

$P_R(m_0, j)$ The probability that R wins the battle, having j casualties.

The expected number of casualties can be determined from:

$$\begin{aligned} E[B] &= m_0 \sum_{j=0}^{n_0-1} P_R(m_0, j) + \sum_{i=0}^{m_0-1} i \cdot P_B(i, n_0) \\ E[R] &= n_0 \sum_{i=0}^{m_0-1} P_B(i, n_0) + \sum_{j=0}^{n_0-1} j \cdot P_R(m_0, j) \end{aligned} \qquad (20)$$

We have to prove that:

$$E[R] = \alpha \cdot E[B]$$

Proof Clearly, $E[R] = \alpha \cdot E[B]$ if and only if

$$\begin{aligned} n_0 \cdot \sum_{i=0}^{m_0-1} \binom{n_0+i-1}{n_0-1} p^{n_0} q^i + \sum_{j=1}^{n_0-1} j \cdot \binom{m_0+j-1}{m_0-1} p^j q^{m_0} \\ = \alpha \left[m_0 \cdot \sum_{j=0}^{n_0-1} \binom{m_0+j-1}{m_0-1} p^j q^{m_0} \right. \\ \left. + \sum_{i=1}^{m_0-1} i \cdot \binom{n_0+i-1}{n_0-1} p^{n_0} q^i \right] \end{aligned} \qquad (20)$$

Recall that $\alpha = p/q$. Therefore, it is left to prove that

$$\begin{aligned} & p^{n_0} \cdot n_0 \cdot \sum_{i=0}^{m_0-1} \binom{n_0+i-1}{n_0-1} q^i \\ & \quad - p^{n_0+1} \sum_{i=1}^{m_0-1} i \cdot \binom{n_0+i-1}{n_0-1} q^{i-1} \\ & = q^{m_0-1} \cdot m_0 \cdot \sum_{j=0}^{n_0-1} \binom{m_0+j-1}{m_0-1} p^{j+1} \\ & \quad - q^{m_0} \sum_{j=1}^{n_0-1} j \cdot \binom{m_0+j-1}{m_0-1} p^j \end{aligned} \qquad (21)$$

Or, equivalently, to show that

$$\begin{aligned} & p^{n_0} \cdot \left[n_0 + \sum_{i=1}^{m_0-1} q^{i-1} \cdot \binom{n_0+i-1}{n_0-1} (n_0 q - ip) \right] \\ & = q^{m_0-1} \cdot \left[m_0 + \sum_{j=1}^{n_0-1} p^j \cdot \binom{m_0+j-1}{m_0-1} (m_0 p - jp) \right] \end{aligned} \qquad (22)$$

It can be seen that both sides of the last equation sum up to:

$$\binom{m_0+n_0-1}{m_0-1} p^{n_0} q^{m_0-1} (n_0 + m_0 - 1) \qquad (23)$$

and the result follows.

Appendix C—The modern battle

The stochastic Model

In this case the attrition functions, $h_b(i, j)$ and $h_r(i, j)$ are:

$$\begin{aligned} h_b(i, j) &= \begin{cases} M - i & i = 0, \ldots, m_0 - 1 \quad j = 0, \ldots, n_0 - 1 \\ 0 & \text{otherwise} \end{cases} \\ h_r(i, j) &= \begin{cases} N - j & i = 0, \ldots, m_0 - 1 \quad j = 0, \ldots, n_0 - 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \qquad (24)$$

The stochastic model is obtained by substituting $b(i, j) = \lambda h_b(i, j)$ and $r(i, j) = \mu h_r(i, j)$ in (11).

If the battle is at state (i, j), then the probability that the next casualty will be Red is

$$p_{(i,j)(i,j+1)} = \frac{\lambda(M-i)}{\lambda(M-i) + \mu(N-j)} = \frac{\alpha(M-i)}{\alpha(M-i) + N - j} \qquad (25)$$

Unlike the ancient battle, this probability is not kept constant throughout the battle; it varies as the number of live combatants, on both sides, changes.

It is shown by Brown[17] that if the battle terminates at total annihilation, that is, $M = m_0$ and $N = n_0$ then the win probability for Red is

$$P[R] = \alpha^{-m_0} \sum_{k=1}^{n_0} \frac{(-1)^{n_0-k} k^{m_0+n_0} \Gamma(k/\alpha + 1)}{(n_0-k)! k! \Gamma(k/\alpha + m_0 + 1)} \qquad (26)$$

The computation of the win probabilities for other attrition thresholds is done numerically by utilising (12).

The deterministic model

The Modern battle, or Lanchester's Square Law, is represented, in the deterministic case, by the pair of differential equations

$$\begin{aligned} B'(t) &= -\rho R(t) \\ R'(t) &= -\beta B(t) \end{aligned} \quad (27)$$

Appendix D—The fixed front battle

The stochastic model

The attrition functions for this type of battle are:

$$h_b(i,j) = \begin{cases} M-i & i=0,\dots,m_0-1 \\ & j=0,\dots,n_0-1 \\ 0 & \text{otherwise} \end{cases}$$

$$h_r(i,j) = \begin{cases} \frac{2}{3}N & i=0,\dots,m_0-1 \\ & j=0,\dots,\min\{n_0-1,N/3\} \\ (N-j) & i=0,\dots,m_0-1 \\ & j=N/3+1,\dots,n_0-1 \\ 0 & \text{otherwise} \end{cases} \quad (28)$$

Given that the battle is at a non-absorbing state $(i,\ j)$, the probability that the next casualty will be in the red side is

$$p(i,j) = \frac{\lambda(M-i)}{\lambda(M-i)+\mu\min\{2N/3, N-j\}} \quad (29)$$

The deterministic model

In its deterministic form, the Fixed Front model is represented by the following pair of differential equations:

$$\begin{aligned} B' &= -\rho\ \min\left(\frac{2}{3}R_0,\ R\right) \\ R' &= -\beta B \end{aligned} \quad (30)$$

where $B(0) = M$ and $R(0) = N$.

The deterministic-parity point α^* depends on the break point of R-f_r. It can be obtained by a straightforward algebraic manipulations of the state equations of the Fixed Front model. This deterministic-parity point is given by

$$\alpha^* = \begin{cases} \dfrac{4}{3}\cdot\left(\dfrac{N}{M}\right)^2\cdot\dfrac{f_r}{1-(1-f_b)^2} & f_r < \frac{1}{3} \\ \left(\dfrac{N}{M}\right)^2\cdot\dfrac{\frac{8}{9}-(1-f_r)^2}{1-(1-f_b)^2} & f_r \geqslant \frac{1}{3} \end{cases} \quad (31)$$

Since, in our case, $N = 3M$, it follows that

$$\alpha^* = \begin{cases} \dfrac{12f_r}{1-(1-f_b)^2} & f_r < \frac{1}{3} \\ \dfrac{8-9(1-f_r)^2}{1-(1-f_b)^2} & f_r \geqslant \frac{1}{3} \end{cases} \quad (32)$$

For the break points $(1,1)$, $(0.3, 0.3)$ and $(0.5, 0.3)$ the deterministic-parity points are 8, 7.06 and 4.8, respectively.

Appendix E—The variable front battle

The stochastic model

The attrition functions here are:

$$h_b(i,j) = \begin{cases} M-i & i=0,\dots,m_0-1 \\ & j=0,\dots,n_0-1 \\ 0 & \text{otherwise} \end{cases}$$

$$h_r(i,j) = \begin{cases} \frac{2}{3}(N-j) & \frac{2}{3}(N-j) > M-i \\ & i=0,\dots,m_0-1\ j=0,\dots,n_0-1 \\ N-j & \frac{2}{3}(N-j) \leqslant M-i \\ & i=0,\dots,m_0-1\ j=0,\dots,n_0-1 \\ 0 & \text{otherwise} \end{cases} \quad (33)$$

The transition probability from a non-absorbing state $(i,\ j)$ to state $(i+1, j)$ is

$$q_{(i,j),(i+1,j)} = \begin{cases} \dfrac{\frac{2}{3}\mu(N-j)}{\frac{2}{3}\mu(N-j)+\lambda(M-i)} & \text{if}\ \frac{2}{3}(N-j) > M-i \\ \dfrac{\mu(N-j)}{\mu(N-j)+\lambda(M-i)} & \text{if}\ \frac{2}{3}(N-j) \leqslant M-i \end{cases} \quad (34)$$

The deterministic model

The deterministic model that depicts the Variable Front battle combat situation is divided also into two stages:

$$\begin{cases} \left.\begin{aligned} B' &= -\tfrac{2}{3}\rho \\ R' &= -\beta B \end{aligned}\right\} & \text{if}\ \frac{2}{3}R > B \\ \left.\begin{aligned} B' &= -\rho R \\ R' &= -\beta B \end{aligned}\right\} & \text{if}\ \frac{2}{3}R \leqslant B \end{cases} \quad (35)$$

Two properties are proven next.

Property E.1 A Variable Front battle can never reach its second stage unless the exchange ratio α is (strictly) greater than 6.

Proof The state equation that corresponds to the first stage of the Variable Front battle is:

$$\frac{2}{3}(R_0^2 - R_t^2) = \alpha(B_0^2 - B_1^2) \tag{36}$$

Recall that $R_0 = 3B_0$. In order that the second stage of the battle will be reached there must exist $t^* < \infty$ such that $(2/3)\ R_{t^*} = B_{t^*}$. It follows that the state equation at t^* is

$$\frac{2}{3}\left(9B_0^2 - \frac{9}{4}B_{t^*}^2\right) = \alpha(B_0^2 - B_{t^*}^2) \tag{37}$$

and the condition must hold:

$$\alpha = \frac{6\left(9B_0^2 - \frac{1}{4}B_{t^*}^2\right)}{B_0^2 - B_{t^*}^2} \tag{38}$$

Property E.2 Suppose that a Variable Front battle reached its second stage. Then α^* is a deterministic-parity point if and only if it is a root of the quadratic equation:

$$4[1 - (1 - f_b)^2]\alpha^2 - [33 - 6(1 - f_b)^2 - (1 - f_r)^2]\alpha + 54[1 - (1 - f_r)^2] = 0 \tag{39}$$

Proof From (37) it follows that at the end of the first stage of the battle the live Blue and Red forces are:

$$B_t^* = \sqrt{\frac{\alpha - 6}{\alpha - \frac{3}{2}}}B_0$$
$$R_{t^*} = \frac{3}{2}B_{t^*} \tag{40}$$

The second stage of Variable Front battle follows the standard Square Law, hence, from (40) it follows that the state equation for the second stage of the battle is:

$$9B_0^2\left[\frac{1}{4}\frac{\alpha - 6}{\alpha - \frac{3}{2}} - (1 - f_r)^2\right] = \alpha B_0^2\left[\frac{\alpha - 6}{\alpha - \frac{3}{2}} - (1 - f_r)^2\right] \tag{41}$$

Canceling out B_0^2 and rearranging produce the result.

References

1 von Clausewitz C (1984). *On War.* (Translation and Eds, Howard M and Paret P). Princeton University Press: NJ, USA.
2 Epstein JM (1989). The 3:1 rule, the adaptive dynamic model, and the future of security studies. *Int Security* **13**: 90–127.
3 Mearsheimer JJ (1982). Why the Soviets can't win quickly in central Europe. *Int Security* **7**: 16–17.
4 Romjue JL (1984). *From Active Defense to AirLand Battle:* The Development of Army Doctrine, 1973–1982. Historical Office, U.S. Army Training and Doctrine Command, Fort Monroe, Va.
5 Liddel Hart BH (1960). *Deterrent or Defence: A Fresh Look at the West's Military Position.* Stevens: London.
6 Sun T (1971). *The Art of War.* (Translation Griffith SB) Oxford University Press: UK.
7 Mearsheimer JJ (1989). Assessing the conventional balance: The 3:1 rule and its critics. *Int Security* **13**: 54–89.
8 Epstein JM (1988). Dynamic analysis and the conventional balance in Europe. *Int Security* **12**: 154–165.
9 Ancker CJ (1995). A proposed foundation for a theory of combat. *Naval Res Logist* **42**: 311–343.
10 Barfoot CB (1974). Markov duels. *Opns Res* **22**: 318–330.
11 Barfoot CB (1989). Continuous-time Markov duels: theory and application. *Naval Res Logist* **36**: 243–253.
12 Koopman BO (1970). A study of the logical basis of combat simulation. *Opns Res* **18**: 855–882.
13 Taylor JG (1983). *Lanchester Models of Warfare.* Operations Research Society of America, Alexandria, Va.
14 Shudde RH (1971). Lanchester's theory of combat. In: Zehna PW (ed). *Selected Methods and Models in Military Operations Research.* Naval Postgraduate School, Monterey, California.
15 Kress M (1992). A many-on-many duel model for a mountain battle. *Naval Res Logist* **39**: 437–446.
16 DARCOM-P (1979). *Engineering Design Handbook, Part Two.* US Army Material Development and Readiness Command.
17 Brown RH (1962). The theory of combat: the probability of winning. *Opns Res* **11**: 418–425.
18 Lanchester FW (1916). *Aircraft in Warfare: The Dawn of the Fourth Arm.* Constable: London.

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