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Theory and Methodology

A general framework for distance-based consensus in ordinal ranking models ☆

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Abstract

The problem of aggregating a set of ordinal rankings of n alternatives has given rise to a number of consensus models. Among the most common of these models are those due to Borda and Kendall, which amount to using average ranks, and the ℓ^1 and ℓ^2 distance models. A common criticism of these approaches is their use of ordinal rank position numbers directly as the values of being ranked at those levels. This paper presents a general framework for associating value or worth with ordinal ranks, and develops models for deriving a consensus based on this framework. It is shown that the ℓ^p distance models using this framework are equivalent to the conventional ordinal models for any $p \geqslant 1$. This observation can be seen as a form of validation of the practice of using ordinal data in a manner for which it was presumably not designed. In particular, it establishes the robustness of the simple Borda, Kendall and median ranking models.

Keywords: Ordinal; Preference; Ranking; Consensus

1. Introduction

The problem of combining a set of ordinal rankings to obtain a consensus has been the subject of study by numerous authors for more than two centuries. Problems of this nature arise frequently in a variety of areas including the evaluation of consumer preferences, allocation of priorities to R & D projects, and the prioritization of candidates in a preferential voting situation; see, for example, Black (1958), Brightwell and Cook (1978), Cook and Seiford (1978), Davis et al. (1972), and Riker (1961). While ordinal data can be collected in different formats, a popular and often used method is the vector format. Specifically, each voter ℓ provides a vector ranking

$$A^{\ell} = \left(a_i^{\ell}\right) = \left(a_1^{\ell}, a_2^{\ell}, \dots, a_n^{\ell}\right)$$

of n alternatives (e.g. projects). For example, the ranking (4, 3, 1, 2) for 4 alternatives a, b, c, d means that a rank of 4 is assigned to alternative a, 3 to alternative b, and so on.

Several different approaches have been suggested for aggregating voter responses into a compromise or consensus ranking. There are, for example, a number of ad hoc runoff procedures which have arisen from

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parliamentary settings. Methods such as the English, American and West Australian elimination systems are discussed by Keesey (1974). A popular method for deriving a consensus is to define a distance function d on the set of all rankings, and then determine that ordinal ranking $\tilde{B}$ which is as close as possible to $\{A^\ell\}$ in the minimum distance sense. Two particular distance functions, the ℓ^1 and ℓ^2 norms, have been discussed at length in the literature; see Cook and Seiford (1978, 1982). In the case of the ℓ^1 norm, for instance, the distance d between two ranking vectors A^1 and A^2 is given by

$$d(A^1, A^2) = \sum_{i=1}^{n} |a_i^1 - a_i^2|.$$

The ℓ^2 norm has the attractiveness of being associated with the class of least squares models, and is a formalization of the well known Borda (1781) marks or Kendall (1962) scores. We point out that the Borda and Kendall representations of rankings (in one case 1 is used as the highest rank position, while in the other case n is the highest) are equivalent for purposes of ℓ^1 and ℓ^2 based distances. Thus, we do not distinguish between these two representations. For a general discussion of distance based models, see Cook and Kress (1992).

The use of consensus techniques such as the ℓ^1 and ℓ^2 distance models, or the simpler rank sum approaches of Borda and Kendall, automatically implies that a numerical, i.e., an interval scale interpretation is being associated with the rank order (ordinal scale) information. Hence, the basic premise behind these approaches is that there is a utility or worth associated with the rank positions. A common criticism of using an approach such as a sum of *ranks* is that the rank positions are *themselves* being treated as those utilities.

The issue of consensus derivation from the perspective of distance models would then appear to involve addressing two questions. First, what utilities or worths *should* be attached to rank positions? Second, what is the appropriate way of combining these worths to best reflect the consensus among the rankings? The various models (distance models, Borda and Kendall scores) have generally been directed toward the second of these questions. We wish to point out at the outset that the present paper does not address the issue of whether or not distance models, and specifically the methods of Borda and Kendall lead to *good decisions* in a consensus sense. Neither is it claimed that ordinal data is itself sufficient to lead to appropriate decisions. The purpose here is to work within these existing frameworks, and to suggest methods to better implement such frameworks.

The present paper presents a general model for representing the *value* associated with rank positions in an ordinal setting. Although only a simple ordering condition is imposed on the value parameters, we are able to show that with ℓ^1 and ℓ^2 distance functions one achieves the *same optimal orderings* in this general framework as in the simple models where rank position data as in Cook and Seiford (1978, 1982) are used. This result is important in that it *validates*, in a distance-based consensus sense, the practice of attaching numerical meaning to ordinal rank position data. In Section 2 the two standard distance based models of consensus (ℓ^1 and ℓ^2 norms) are presented. Section 3 develops a general framework based upon attaching a value w_i to rank positions i. It is then shown that the above consensus models in the presence of this general framework are equivalent to the conventional rank position-based models. Section 4 discusses extensions and future directions.

2. Consensus via distance functions

Cook and Seiford (1978) examine the problem of deriving a consensus among a set of ordinal rankings. Specifically, let $\{A^\ell = (a_i^\ell)\}_{\ell=1}^m$ be a set of m voter-specified ordinal rankings, with a_i^ℓ being the rank assigned by voter ℓ to alternative $i = 1, 2, \ldots, n$. In the *linear* ordering case (no ties permitted), a_i^ℓ takes on values in the set $\{1, 2, \ldots, n\}$. In this case the authors have shown that in the presence of a certain set of axioms, there exists a unique distance function d_1 defined on the space of all linear orderings A. For any two rankings A^1 and $A^2 \in A$, this distance function is given by

$$d_1(A^1, A^2) = \sum_{i=1}^{n} |a_i^1 - a_i^2|. \tag{2.1}$$

Based on (2.1), a *consensus* or compromise ranking (i.e., a ranking which best reflects the combined opinions of the voters) can be defined as that linear ordering $\hat{B}$ which solves the ℓ^1 minimization problem

(P_1)

$$M_1 = \min_B \sum_{\ell=1}^{m} d_1(A^\ell, B) = \min_B \sum_{\ell=1}^{m} \sum_{i=1}^{n} |a_i^\ell - b_i|. \quad (2.2)$$

Cook and Seiford (1978) have shown that (2.2) can be written as a linear assignment problem.

In the more general case where ties are permitted, the set of allowable rank positions Ψ is given by

$$\Psi = \{1, 1.5, 2, 2.5, \ldots, \tfrac{1}{2}(2n-1), n\}. \quad (2.3)$$

Armstrong et al. (1982) show that (2.1) also defines the unique distance function for the general quasi-linear case. Consensus among a set of *quasi-linear* rankings $\{A^\ell\}$ is then that quasi linear ranking $\hat{B}$ which solves

(P_1')

$$M_1' = \min_B \sum_{\ell=1}^{m} \sum_{i=1}^{n} |a_i^\ell - b_i|, \quad (2.4)$$

with the understanding that $a_i^\ell, b_i \in \Psi$.

An alternative distance-based consensus model discussed in Cook and Seiford (1982) involves the ℓ^2 norm

(P_2')

$$d_2(A^1, A^2) = \sum_{i=1}^{n} (a_i^1 - a_i^2)^2. \quad (2.5)$$

Whereas (2.2) (or (2.4) for the quasi-linear case) is a formalization of the *median* of a set of rankings, (2.5) gives rise to the *mean* ranking. Being a type of minimum variance model, it has a direct connection to the original Borda (1781) 'method of marks', hence to Kendall (1962) 'scores'.

A common criticism of the Kendall and Borda consensus models, and, therefore, of the distance-based methods P_1' and P_2', is their reliance on the treatment of ordinal rank position data as if it were numerical. Since such ordinal data carries no presumption whatsoever about the degree of importance of being ranked k-th versus $(k+1)$st, models such as P_1' and P_2' are seen to be very limiting in that *differences* in worths of consecutive rank positions are all treated as being *equal*. Hence, we are crediting rankings with possessing more information than is intended. As an alternative to this approach, and as an attempt to respond to such a criticism, we present in the following section a general representation of rankings in terms of the *value* or *utility* associated with rank positions.

3. A general model for distance-based consensus

Since strict linear or simple orders are a subset of the set of all quasi-linear or weak rankings (see Roberts, 1979), we treat only the general (quasi-linear) case. Specifically, let $\{A^\ell\}$ be a set of quasi-linear rankings where each member of a set of n alternatives occupies one of the rank positions in $\Psi = \{1, 1.5, 2, 2.5, \ldots, n\}$. It is noted that in the case of no ties, the alternatives are ranked only at positions $1, 2, 3, \ldots, n$. A set of alternatives θ which would normally occupy rank positions $k_1, k_1+1, \ldots, k_2$, would, if tied, be ranked at the *median* of these $k_2 - k_1 + 1$ positions. If the cardinality of θ is odd, all members of this set would receive a rank of k equal to the middle value of θ. Otherwise, k is the point halfway between the two middle values of θ.

In the models of Section 2, a rank position k is, in effect, credited with a *value* of k (or $n+1-k$, depending on the convention utilized). The rank position and the value of that position are, therefore, treated as being one and the same. In actuality, the value or worth of being ranked in position k is some (generally unknown) quantity w_k. Considering for the moment the pure linear ordering case, and using the convention that it is worth more to be ranked k-th than $(k+1)$st, it follows that whatever values we wish to assign to w_k, it should be true that $w_k > w_{k+1}$.[1] In the common case where the w_k are

[1] Note that in the development in Section 2 small rank numbers are worth more than large rank numbers. That is, a rank of 1 is preferred to a rank of 2, and so on. This was the convention used in Cook et al. (1978, 1982). We could just as easily have reversed the convention and used a rank n as the best, $n-1$ next best, etc. For purposes in this section we adopt the convention that worth of w_1 is *bigger* numerically than w_2, etc. The two directions are equivalent.

unknown, this restriction may be the only information available to the decision maker.

For the general quasi linear case define two sets of variables $\{w_k\}_{k=1}^{n}$ and $\{u_r\}$ with the latter being a set of $\frac{1}{2}n(n-1)$ variables

$$\{u_{k_1,k_1+1}\}_{k_1=1}^{n-1}, \{u_{k_1,k_1+1,k_1+2}\}_{k_1=1}^{n-2}, \ldots,$$
$$\{u_{k,k_1+1,\ldots,k_1+n-2}\}_{k_1=1}^{2}, u_{1,2,\ldots,n}.$$

The variable $u_{k_1,k_1+1,\ldots,k_1+r}$ represents the value or utility associated with the rank position occupied by $r+1$ alternatives which, if separately ranked, would be positioned at k_1, $k_1+1, \ldots, k_1+r$. Rather than assuming that the only valid rank positions in a weak ordering are the integer points and the midpoints of Ψ as in the case in the conventional model (see Armstrong et al., 1982), we assume *only* that a natural ordering condition exists between the w_k and u_r. Specifically, if *two* alternatives, for example, are tied for positions k and $k+1$, then the value $u_{k,k+1}$ of being ranked at this 'tied position' must be between the value w_k of being ranked at position k and the value w_{k+1} associated with position $k+1$. Formally, we require

$$w_k - u_{k,k+1} \geqslant z, \tag{3.1a}$$
$$u_{k,k+1} - w_{k+1} \geqslant z, \quad k=1,\ldots,n-1. \tag{3.1b}$$

Similarly, if $r+2$ alternatives are tied for positions $\{k, k+1, \ldots, k+r+1\}$, then the value $u_{k,k+1,\ldots,k+r+1}$ achieved by these $r+2$ alternatives lies between $u_{k,k+1,\ldots,k+r}$ and $u_{k+1,\ldots,k+r+1}$, i.e.,

$$u_{k,k+1,\ldots,k+r} - u_{k,k+1,\ldots,k+r+1} \geqslant z, \tag{3.2a}$$
$$u_{k,k+1,\ldots,k+r+1} - u_{k+1,k+2,\ldots,k+r+1} \geqslant z, \quad k=1,\ldots,n-r-1, \quad r=1,\ldots,n-2. \tag{3.2b}$$

$$w_n \geqslant z, \tag{3.3}$$

$$\sum_{k=1}^{n} w_k = \tfrac{1}{2}n(n+1), \tag{3.4}$$

where z is a specified small positive parameter. We refer to z as a *discrimination* parameter and note that by virtue of (3.4) z must be chosen in the range $0 < z \leqslant \frac{1}{2}$. Constraints (3.1a)–(3.3), therefore, specify that we discriminate between the worths of consecutive rank positions by some positive amount, and constraint (3.4) is a scaling convention (since in P_1, $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$). Clearly, in the absence of (3.4) there is no upper limit on the values of z that could be utilized. As will be shown, however, the optimal consensus ranking is independent of z in any event, meaning that this limit doesn't matter.

With these ideas as a backdrop, we propose the following natural extension to model P_1':

(P_1'')

$$M_1'' = \min_{\{V_s\}} \sum_{\ell=1}^{m} \sum_{i=1}^{n} |V_{a_i}^{\ell} - V_{b_i}| \tag{3.5}$$

s.t. (3.1a)–(3.4),

where V_s equals one of the w_k or u_r in each case. If an alternative is not tied with anything else, the corresponding V is equal to some w_k. If the alternative in question is tied with r other alternatives, $V = u_{k,k+1,\ldots,k+r}$ for some k. The objective function (3.5) is a generalization of (2.2) in that it is not only necessary to determine the consensus ranking $\hat{B} = (\hat{b}_i)$, but also to find an appropriate set of worths w and u. One can view (3.5) in another way. Each voter ranking A^{ℓ} can be looked upon as that voter's estimate of an unknown "true" ranking B of the alternatives. Problem P_1'' can then be interpreted as a mechanism for assigning those values w_k,u_r to the rank positions which are such that voter disagreement is minimized (where disagreement is measured by an ℓ^1 norm). The minimization is, thus, a means of deriving weights that provide least absolute deviation.

What can be shown is that in this general framework where the values (w or u) associated with the different rank positions are restricted only to be a natural ordering as specified in (3.1a)–(3.4), model (3.5) reduces to the conventional model where only integer rank positions and the midpoints between them are allowed. This is given by the following theorem.

Theorem 3.1. *At the optimum of* (3.5) *all consecutive* w_k *are equally spaced by a distance* $2z$, *with all* u_r *located at midpoints between consecutive* w_k; *specifically,*

$$w_k = \tfrac{1}{2}(n+1) + (n+1-2k)z, \quad k=1,\ldots,n, \tag{3.6}$$

$$u_{k-1,k,k+1} = u_{k-2,k-1,k,k+1,k+2} = \cdots = u_{k-r,k-r+1,\dots,k+r} = w_k, \quad k=2,\dots,n-1, \quad r=1,\dots,k-1, \tag{3.7}$$

$$u_{k,k+1} = u_{k-1,k,k+1,k+2} = \cdots = u_{k-r,k-r+1,\dots,k+r+1} = \tfrac{1}{2}(w_k + w_{k+1}), \quad k=1,\dots,n-1, \quad r=0,\dots,k-1. \tag{3.8}$$

Proof. Consider the i-th term in (3.5) and assume that for some current optimal set of w_k, u_r, there exists a k_1 such that

$$w_{k_1} - w_{k_1+1} > 2z.$$

It is noted that if for all k it were true that $w_k - w_{k+1} = 2z$, then all gaps between consecutive rank positions would be equal to z and the result would follow. With no loss of generality, assume that rank position k_1 is smaller than the current position V_{b_i}, i.e. $w_{k_1} > V_{b_i}$ for alternative i. It is noted that at least one of

$$w_{k1} - u_{k_1,k_1+1} > z$$

or

$$u_{k_1,k_1+1} - w_{k_1+1} > z$$

must be true. With no loss of generality, assume the latter. Suppose the current set of w_k, u_r are now modified in the following way. For a sufficiently small value ϵ, increase all w_k and u_r at or below w_{k_1+1} (i.e., including w_{k_1+1}) by the amount $\epsilon/(n-k_1)$. At the same time, decrease all variables larger than w_{k_1+1}, but at or below w_{k_1}, by the amount ϵ. For ϵ sufficiently small, all constraints (3.1a)–(3.4) still hold, yet the overall value of M_1'' has decreased by a function of ϵ due to the decreases in w_{k_1} and u_r between w_{k_1} and w_{k_1+1}. Since this violates the assumption that the current set of weights is optimal, it must be the case that at the optimum

$$w_k - w_{k-1} = 2z$$

for all $k = 1,\dots,n-1$, and by virtue of the constraints, each u_r must either equal a w_k value or be located at a midpoint between consecutive w_k values. This implies that at the optimum of (3.5) all rank positions are equally spaced, and the expressions for w_k and u_r are as shown. □

Thus, distance-based consensus forces the gaps between consecutive rank positions to be as small as possible, namely to be equal to the minimum discrimination z. At one end of the spectrum if $z=0$, were permitted, all $w_k = \frac{1}{2}(n+1)$, i.e., all alternatives are tied, meaning that all distances = 0 and $M_1''=0$. At the other end of the scale when $z=\frac{1}{2}$, $w_n = 1$, $w_{n-1} = 2, \dots, w_1 = n$, and we have the familiar integer rank positions. This gives the following corollary.

Corollary 3.1. *When $z=\frac{1}{2}$, problems P_1' and P_1'' are identical.*

Proof of the following theorem is straightforward and is therefore omitted.

Theorem 3.2. *Any linear ranking $\hat{B} = (\hat{b}_i)$ which solves P_1' also solves P_1'' and vice-versa.*

This equivalence of P_1' and P_1'' establishes, in an ℓ^1 norm sense, the validity of attaching an interval scale, i.e., numerical interpretation to ordinal rank position data.

Other ℓ^p norms

From the proof of Theorem 3.1 it follows that for any p, the function

$$\sum_{\ell=1}^{m} \sum_{i=1}^{n} |V_{a_i^\ell} - V_{b_i}|^p$$

will always be minimized when all weights are separated by a minimal amount. Hence, for all p, and in particular for ℓ^2 and ℓ^∞ norms, the distance-based consensus models, where rank positions are treated as utilities, are *equivalent* to the more general utility structures. Hence, any ranking which solves the conventional ordinal problem also solves (3.5) for any ℓ^p norm.

4. Conclusions

In this paper we have established that the general representation of distance-based consensus in terms of the value or worth associated with ordinal rank positions is equivalent to the conventional representation where distance is a function of the rank positions only. Generally speaking, this implies that the practice, for example, of using the average or total of ranks (Borda and Kendall scores) is *valid* in that it is, in a least distance sense, the same as using interval scale utilities. An important implication of this result is that the long standing and seemingly ad hoc practice in many consumer preference models of treating rank position data as if they were interval scale values is *efficient* from a minimum distance point of view. In a somewhat similar setting where data envelopment analysis has been applied to consensus in a preferential voting framework, Cook and Kress (1990) have shown that in certain special circumstances such a model is equivalent as well to the Borda and Kendall methods.

The concepts presented herein can be extended in a number of directions. One possible direction is to introduce a *discrimination intensity function* $f(i, z)$ as a replacement for z in the aforementioned models. Such a function would permit one to have, for example, decreasing gaps (i.e., decreasing discrimination) between consecutive rank positions as one goes further down the ordinal scale.

As an example one might choose to replace z in the constraints of P_1'' by a function such as $f(k, z) = z/k$. In model P_1'', the (implicit) constraints

$$w_k - w_{k+1} \geqslant 2z$$

would then be replaced by

$$w_k - w_{k+1} \geqslant 2f(k, z).$$

Thus, we discriminate more between the values of being ranked in 1st versus 2nd place than is true of the values of being ranked in 2nd versus 3rd place, and so on. If this same idea were applied universally to the constraints of P_1'' (i.e., replacing z by $f(k, z)$) it can be shown that the minimizing objective function forces w_1 and w_n as close together as possible. This means that whatever value w_1 takes on, w_2 will be set at

$$w_2 = w_1 - 2f(1, 2), \ldots, w_{k+1} = w_k - 2f(k, z),$$

and

$$u_{k_{k+1}} = w_k - f(k, z).$$

Clearly, the maximum value of z is limited by constraint (3.4) as before.

Another possible direction is to prioritize the voters in order of their importance or contribution to the consensus. Consensus derivation in this situation needs further investigation.

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