A Many-On-Many Stochastic Duel Model for a Mountain Battle

Moshe Kress
Center for Military Analyses, P.O. Box 2250 (28), Haifa 31021, Israel

Terrain plays a major role in mountain battle. The advancing (attacking) force is usually restricted to move in a single column—along a narrow, winding, and steep road. The defending force, on the other hand, which is static, can select its positions such that most of its firepower can be effective against the front unit(s) of the attacking force. This combat situation is modeled as a special type of the many-on-many stochastic duel. This duel is a series of many-on-one subduels where at each such subduel the defending force units simultaneously engage the single exposed front unit of the attacking force. This special type of many-on-many stochastic duel demonstrates the possibility of practical applications of stochastic duel theory.

1. INTRODUCTION

Mountain warfare is characterized by the significant effect of the terrain on the ability of troops in general and armored vehicles, like tanks or armored personnel carriers (APC), in particular, to maneuver or even to advance [2, 5]. Roads are narrow, steep, and winding, and off-road movement is practically impossible. The advancing (attacking) force is compelled therefore to move in a single column where, in some situations, only the front unit (tank) becomes exposed to enemy fire [2]. The defending force, which is static throughout the battle, can take advantage of the topography and select its positions such that most of its firepower can be effective against the exposed unit of the attacking force. Such situations were typical, for example, in the 1982 Lebanon War [2].

This type of combat situation (which may apply to other scenarios as well) can be generally described as an ongoing duel between all the defending units and the exposed part of the attacking force. Once the front unit of the attacking column is hit and becomes disabled, another unit advances from behind and resumes the fire exchange with the defenders. While this fire exchange is in progress, the attacking force is static and therefore its advancement is halted.

Such a fire-exchange situation may be modeled as a many-on-many stochastic duel (SD) where the defender shoots simultaneously with all his live weapons, while the attacker shoots sequentially—from one weapon at a time. This particular type of a many-on-many SD is represented therefore as a sequence of many-on-one duels [4].

This article uses the results of an earlier one [4] to develop a specific model for a mountain duel scenario. The notation is described in Section 2. The basic

Naval Research Logistics, Vol. 39, pp. 437–446 (1992)
 CCC 0894-069X/92/040437-10$04.00

model is developed in Section 3. Formulas for the expected duration of battle are developed in Section 4. In Section 5, this model is restricted to the negative exponential distribution case to allow ease of computation. Finally, the restricted model is used to compute results of example scenarios in Section 6. These computations are analyzed to produce some tactical insights with regard to the assumptions made.

2. NOTATION AND DEFINITIONS

Following the commonly used approach in the SD literature (see, e.g., [1, 3, 4]), we consider here the interkill time distribution which is derived from the interfiring time distribution and the round-to-round kill probabilities. Specifically, if the interfiring time distribution between the (n-1)th round and the nth round is $H_n(t)$ and the kill probability at the nth round is $\phi(n)$, then the interkill time distribution $F(t)$ is given by

$$F(t) = \sum_{n=1}^{\infty} \left(\prod_{i=1}^{n} [*H_i(t)] \right) \phi(n), \tag{1}$$

where the asterisk denotes the convolution operator.

Let $F(t)$ and $G(t)$ denote the interkill time pdf for each one of the units in the Blue (attacking) force and the Red (defending) force, respectively. We assume that these interkill times are independent among the units. Following the situation description above, we assume that the battle between Blue (B) and Red (R) is a series of independent many-on-one duels (see [4]).

Once the exposed front unit of B is killed, the battle is resumed in the form of a new duel between the remaining R units and the fresh *B* unit. A given force (side) loses the battle when its attrition reaches a certain threshold. This threshold is called *critical attrition*.

Suppose that the initial force sizes are m and n for B and R, respectively. The critical attrition of R is $a(\mathrm{R})$, $1 \le a(\mathrm{R}) \le n$. We may assume, w.l.o.g., that the critical attrition of B is m, its initial size. The *critical size* of each side is the minimum number of units with which it is still willing to fight. Clearly, we have that (critical size) = (initial size) − (critical attrition) + 1. The critical size of B is 1 and that of R is denoted by $i(\mathrm{R})$. The notation is summarized in Table 1.

Table 1. Notation

Side	Blue (B)	Red (R)
Posture	Attack	Defense
Interkill distribution	$F(t)$	$G(t)$
Initial size	m	n
Critical attrition	m	$a(\mathrm{R})$
Critical size	1	$i(\mathrm{R})$

3. THE MODEL

As was mentioned above, the battle comprises a series of independent many-on-one duels. In the first duel, one B unit (the front unit) is facing n R units. If B kills $a(\mathrm{R})$ units of R without being killed, then the battle is over and B is declared to be the winner. The probability of that event is denoted by $P[\mathrm{B}/n]$. Let $q(r/n)$ denote the probability that B lost the first duel (lost its front unit) while inflicting on R exactly r casualties, $r = 0, \ldots, a(\mathrm{R}) - 1$. That is, the probability that the single front unit of B killed r units of R before being killed. At the end of this duel, B and R are left with $m - 1$ and $n - r$ live units, respectively. From [4, Eq. (8)] we have that

$$q(r/n) = \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} G^c\left(\sum_{j=1}^{i} y_j\right) \prod_{i=1}^{r+1} f(y_i) \left(\left[G^c\left(\sum_{j=1}^{r} y_j\right)\right]^{n-r} - \left[G^c\left(\sum_{j=1}^{r+1} y_j\right)\right]^{n-r}\right) dy_1 \cdots dy_{r+1}, \tag{2}$$

where $G^c(x)$ denotes the complementary distribution function and $f(x)$ is the density function associated with $F(x)$.

The probability that R wins its first duel is, therefore,

$$P_1[\mathrm{R}] = \sum_{r=0}^{a(\mathrm{R})-1} q(r/n). \tag{3}$$

To simplify notation in the subsequent formulations, we define the state probabilities in terms of the number of remaining (live) R units. That is,

$$p(k/n) = q(n - k/n).$$

Hence, the probability that R wins the first duel with exactly k of its units still alive $[k \geq i(\mathrm{R})]$ is given by

$$P_1[\mathrm{R}, k] = p(k/n). \tag{4}$$

and

$$P_1[\mathrm{R}] = \sum_{k=i(\mathrm{R})}^{n} P_1[\mathrm{R}, k]. \tag{5}$$

Similarly,

$$P_2[\mathrm{R}, k] = \sum_{j=k}^{n} P_1[\mathrm{R}, j] p(k/j). \tag{6}$$

In general,

$$P_i[\mathrm{R}, k] = \sum_{j=k}^{n} P_{i-1}[\mathrm{R}, j]p(k/j), \qquad i = 2, \ldots, m$$

$$= p(k/n), \qquad \text{for } i = 1, \tag{7}$$

is the probability that R is still alive, with $k \geq i(\mathrm{R})$ units, at the end of the ith duel. The probability that R wins the whole battle is

$$P[\mathrm{R}] = P_m[\mathrm{R}] = \sum_{j=i(\mathrm{R})}^{n} P_m[\mathrm{R}, j]. \tag{8}$$

Given that R won the battle, the expected number of its remaining live units is

$$E(\mathrm{R}) = P^{-1}[R] \sum_{j=i(\mathrm{R})}^{n} jP_m[\mathrm{R}, j]. \tag{9}$$

The conditional probability that B wins a certain duel i, $i = 1, \ldots, m$ (and hence the whole battle) is (see [4, Eq. (9)] modified to account for the fact that $i(\mathrm{R})$ may be greater than one)

$$P[B/k] = \int_0^{\infty} \cdots \int_0^{\infty} \prod_{i=1}^{v} f(y_i) \prod_{i=1}^{v-1} G^c\left(\sum_{j=1}^{i} y_j\right)\left[G^c\left(\sum_{j=1}^{v} y_j\right)\right]^{i(\mathrm{R})} dy_1 \cdots dy_v, \tag{10}$$

where $v = k - i(\mathrm{R}) + 1$. Notice that v is the number of Red units Blue has to kill in order to win the duel (and the battle), given that Red started off that duel with k units.

B winning the battle at its ith duel implies that it has lost $i - 1$ of its units. Blue's win probability at the ith duel is

$$P_i[\mathrm{B}] = \sum_{k=i(\mathrm{R})}^{n} P_{i-1}[R, k]P[\mathrm{B}/k], \qquad \text{for } i = 2, \ldots, m$$

$$= P[\mathrm{B}/n], \qquad \text{for } i = 1. \tag{11}$$

Blue's win probability is therefore

$$P[B] = \sum_{i=1}^{m} P_i[B]. \tag{12}$$

A major factor in battle is time. A mission is usually constrained by a predetermined completion time according to which the force commander has to plan and execute the operation. For example, the attacking force (B) may not accomplish its mission—even if it has won the battle, if that win occurred too late. Therefore, it is interesting and relevant to consider the time factor in our mountain duel context. This factor is considered next.

4. THE EXPECTED BATTLE TIME

The total battle time is composed of two factors: (1) the shooting time, and (2) the interduel maneuver time. First consider the shooting time.

Let $E(k, s)$ denote the expected length of time of a single many-on-one duel where R started off with k units, won the duel, and was left with s, $i(\mathrm{R}) \le s \le k$ live units. Similarly, $E(k, \mathrm{B})$ is the expected duration of a single duel where R started off with k units and lost. That duel is, of course, the last duel in the battle.

Given that B won the ith duel, and R started off the jth duel, $j = 1, \ldots, i$, with k_{j-1} live units, the mean total shooting time of the battle is

$$E_{\mathrm{B}}\{k_0, \ldots, k_{i-1}\} = \sum_{j=1}^{i-1} E(k_{j-1}, k_j) + E(k_{i-1}, \mathrm{B}), \tag{13}$$

where $k_0 = n$. This event occurs with probability

$$P[k_0, k_1, \ldots, k_{i-1}, \mathrm{B}] = P[\mathrm{B}/\mathrm{k}_{i-1}] \prod_{j=1}^{i-1} p(k_j/k_{j-1}). \tag{14}$$

Define $E_{\mathrm{B}}(i)$ as the conditional mean shooting time given that B won the ith duel

$$\begin{aligned} E_{\mathrm{B}}(i) &= (P_i[\mathrm{B}])^{-1} \sum_{k_{i-1}=i(\mathrm{R})}^{n} \cdots \sum_{k_1=k_2}^{n} E_{\mathrm{B}}\{k_0, \ldots, k_{i-1}\} \\ &\quad \times P[k_0, \ldots, k_{i-1}, \mathrm{B}], \qquad i = 2, \ldots, m \\ &= (P_1[\mathrm{B}])^{-1} E(n, \mathrm{B}) P[\mathrm{B}/n] = E(n, \mathrm{B}), \qquad \text{for } i = 1. \end{aligned} \tag{15}$$

In addition to the shooting time, B may spend a considerable amount of time in trying to maneuver past its stalled front units. Let D_i denote the total accumulated maneuver time of B, given that its win occurred at the ith duel. This implies that B has lost $i - 1$ of its units before winning the battle. We set $D_1 = 0$.

The total expected delay time of the winning force B is given now by

$$E = (P[\mathrm{B}])^{-1} \sum_{i=1}^{m} (E_{\mathrm{B}}(i) + D_i) P_i[\mathrm{B}]. \tag{16}$$

The values $E(k, s)$ can be computed by an appropriate integration of the transient state probabilities given in [4].

5. NEGATIVE EXPONENTIAL DISTRIBUTIONS

Suppose that the pdf of the interkill times are negative exponential with parameters a and β for R and B, respectively, then

$$p(k/n) = k\alpha\beta^{n-k} / \prod_{j=k}^{n} (\beta + j\alpha) \tag{17}$$

and

$$p[\mathrm{B}/k] = \beta^{k-i(\mathrm{R})+1} / \prod_{j=i(\mathrm{R})}^{k} (\beta + j\alpha). \tag{18}$$

From (4) and (6) we obtain that

$$P_2[\mathrm{R}, k] = \sum_{j=k}^{n} \left[j\alpha\beta^{n-j} / \prod_{t=j}^{n} (\beta + t\alpha) \right] \left[k\alpha\beta^{j-k} / \prod_{t=k}^{j} (\beta + t\alpha) \right] \tag{19}$$

or

$$P_2(\mathrm{R}, k] = \left[k\alpha^2\beta^{n-k} / \prod_{t=k}^{n} (\beta + t\alpha) \right] \sum_{j=k}^{n} (j/(\beta + j\alpha)). \tag{20}$$

Similarly, the probability that R won the battle with k $[k \geq i(\mathrm{R})]$ of its units still alive is

$$P_m[\mathrm{R}, k] = \frac{k\alpha^m\beta^{n-k}}{\prod_{t=k}^{n}(\beta + t\alpha)} \sum_{k_{m-1}=k}^{n} \cdots \sum_{k_1=k_2}^{n} \prod_{i=1}^{m-1} (k_i/(\beta + k_i\alpha)) \tag{21}$$

and

$$P[\mathrm{R}] = P_m[\mathrm{R}] = \sum_{k=i(\mathrm{R})}^{n} P_m[\mathrm{R}, k]. \tag{22}$$

It is easily seen that these probabilities depend only on the kill rate ratio $\theta = \beta/\alpha$ of the two forces and not on the individual values of these kill intensities. Also, these expressions can be generalized for the situation where the kill intensities change, as the battle develops from one duel to the next. This generalization, however, is not considered here.

We can write now $P[\mathrm{R}]$ as

$$P[\mathrm{R}] = \sum_{k=i(\mathrm{R})}^{n} \frac{k\theta^{n-k}}{\prod_{i=k}^{n}(\theta + i)} \sum_{k_{m-1}=k}^{n} \cdots \sum_{k_1=k_2}^{n} \prod_{i=1}^{m-1} k_i/(\theta + k_i). \tag{23}$$

The expected time length of a single duel where Red started off with k units, won the duel (killed the front Blue unit), and ended up with s live units, is

$$E(k, s) = \sum_{j=s}^{k} (\beta + j\alpha)^{-1}. \tag{24}$$

The only difference between a duel where R won while left with i(R) live units (its critical size) and a duel where B won is in the identity of the last killed unit. In the first duel the last kill was of a B unit, while in the second one the last kill was of a R unit. Since the expected length of time of the duel is not dependent on the identity of the last kill but rather on the total number of kills, it follows that

$$E(k, \mathrm{B}) = E(k, i(\mathrm{R})). \tag{25}$$

6. EXAMPLES

A company of Blue, which is moving along a mountain road, is engaged by a Red section ($n = 3$) which is deployed in defensive positions. Suppose that the critical attrition of Blue is 4 (about 30% of the force size), and therefore we can assume that $m = 4$. The critical attrition of R is $a(\mathrm{R}) = 2$, which implies that $i(\mathrm{R}) = 2$, too. The value of P[R] is computed for three values of the kill rate ratio θ: 0.5, 1, and 2.

Kill rate ratio—θ	Red's win probability P[R]
0.5	0.8
1	0.55
2	0.26

Notice that in this typical scenario, with the given attrition thresholds, if the individual kill rates are similar in both sides ($\theta = 1$), then the win probabilities for each side are close, too (0.45 for B and 0.55 for R).

The critical attrition of each side represents its endurance and the determination to win the battle; the higher its critical attrition, the higher is its endurance. Figures 1 and 2 present the change in P[R] as a function of B's endurance. The

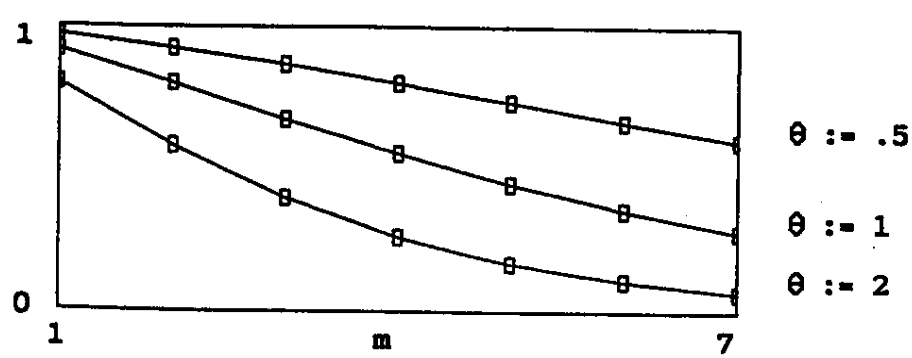

Figure 1. The win probability P[R] for $n = 3$.

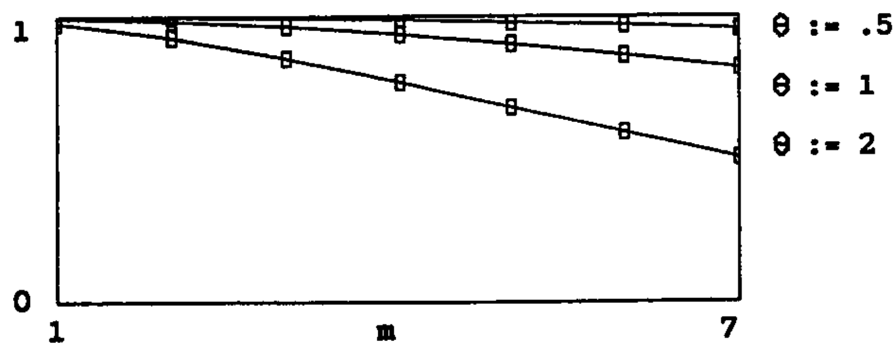

Figure 2. The win probability $P[\mathrm{R}]$ for $n = 6$.

critical attrition of B ranges between $m = 1$ (low determination) and $m = 7$ (high determination). In Figure 1, $n = 3$ (one section) and $a(\mathrm{R}) = 2$, and in Figure 2, $n = 6$ (two sections) and $a(\mathrm{R}) = 4$.

It can be seen that Blue's endurance affects its win probability, in particular if its individual kill rate is at least as high as Red's. This effect is especially notable when the Red force is small ($n = 3$). In this case, if Blue increases, for example, its critical attrition from 3 to 5, then its win probability $P[\mathrm{B}] = 1 - P[\mathrm{R}]$ for the case $\theta = 1$, increases from 0.33 to 0.55—an increases of over 60%.

Evidently, the endurance of each one of the opposing forces has a major impact on the outcome of the battle. Table 2 presents the relations between the two sides' attrition thresholds for the case where the defending Red force has $n = 3$ units and its endurance, represented by the critical attrition $a(\mathrm{R})$, ranges between 1 (surrender after one loss) and 2. For a given kill intensity ratio θ, and Red's critical attrition $a(\mathrm{R})$, the required critical attrition of Blue is obtained such that its win probability $P[\mathrm{B}]$ is at least 0.5. We observe that the required critical attrition of Blue is most sensitive to that of Red when $\theta = 1/2$. The sensitivity decreases when the kill intensity ratio turns around to be in its favor. It is important therefore for Blue to assess correctly the potential endurance of Red when the kill intensity odds are against him. If $\theta = 1/2$ and Red is determined [$a(\mathrm{R}) = 2$], then Blue must be prepared to lose nine of its units in order to reach parity in the odds to win. This information may change Blue's intent to fight in the first place.

Figure 3 presents the effect of Red's size on its win probability. Suppose, once again, that $m = 4$—a Blue company (11 pieces) which is prepared to sacrifice

Table 2. Critical attrition of Blue at parity

Kill intensity ratio θ	Red's critical attrition $a(\mathrm{R})$	Blue's critical attrition m
½	1	5
	2	9
1	1	3
	2	5
2	1	2
	2	3

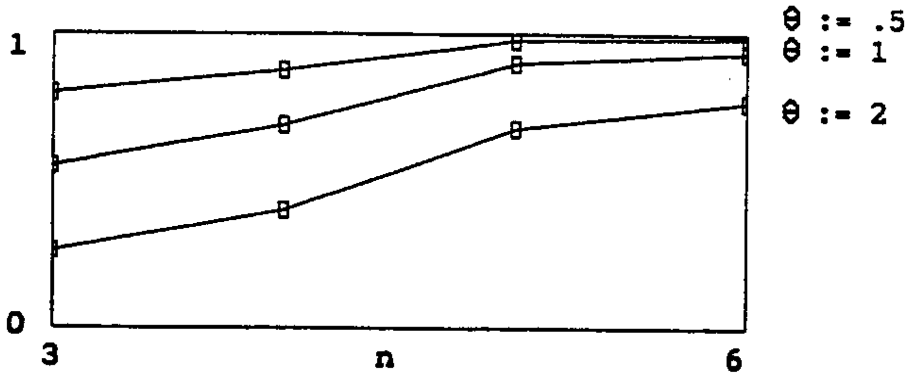

Figure 3. The effect of Red's size on $P[\mathrm{R}]$.

up to one third of its initial size. The size of the Red force ranges between $n = 3$ (a section) and $n = 6$ (two sections), and its critical attrition is $a(\mathrm{R}) = 2$ for $n = 3, 4$, and $a(\mathrm{R}) = 3$ for $n = 5, 6$.

The effect of Red's size on $P[\mathrm{R}]$ becomes more notable as θ increases. If $\theta = 2$, then doubling the Red force size from 3 to 6 results in tripling its win probability $P[\mathrm{R}]$.

For the case $n = 3$, $m = 4$, and $a(\mathrm{R}) = 2$, the conditional mean shooting time, given that Blue won the battle at the ith duel, $i = 1, \ldots, 4$, is obtained from (15). For example, for $i = 3$,

$$E_{\mathrm{B}}(3) = (P_3[\mathrm{B}])^{-1}\{E_{\mathrm{B}}(3, 3, 3)P(3, 3, 3, \mathrm{B}) + E_{\mathrm{B}}(3, 3, 2)P(3, 3, 2, \mathrm{B})$$

$$+ E_{\mathrm{B}}(3, 2, 2)P(3, 2, 2, \mathrm{B})\},$$

where, for example,

$$E_{\mathrm{B}}(3, 3, 2) = E(3, 3) + E(3, 2) + E(2, \mathrm{B})$$

and

$$P(3, 3, 2, \mathrm{B}) = P[\mathrm{B}/2]p(3/3)p(3/2).$$

Table 3 presents the expected total battle time (conditional on Blue's win) for three types of interduel maneuver time functions, and for various values of the kill rates α and β.

Table 3. Expected total battle time

				Interduel maneuver time		
α	β	θ	$P[\mathrm{B}]$	$D_i = 0$	$D_i = 10(i-1)$	$D_i = 10i\sqrt{(i-1)}$
0.1	0.05	½	0.2	13	31.5	53
0.2	0.1	½	0.2	6.5	25	46.5
0.1	0.1	1	0.45	10.5	27	46
0.05	0.05	1	0.45	21	37.5	56.5
0.05	0.1	2	0.74	15	28.5	44
0.1	0.2	2	0.74	7.5	21	36.5

It is seen that the mean total shooting time (the case where $D_i = 0$) is not significantly affected by the kill intensity ratio θ. For example, if the smaller of α and β is 0.1, then the expected total shooting time is 6.5 minutes for $\theta = \frac{1}{2}$, and 7.5 minutes for $\theta = 2$.

7. CONCLUSION

A tactical situation typical of mountain warfare was modeled as a many-on-many stochastic duel. The win probabilities for each side were obtained as a function of (a) the kill rate ratio, (b) the force size, and (c) the endurance which is expressed in terms of the acceptable critical attrition. In addition, the expected total time duration of the battle was obtained.

The model was applied in the case of negative exponential interkill time distributions. It was shown that, in addition to the obvious effect of kill rate and size, the endurance of each side plays also a major role in the determination of the odds to win.

The tactical insights obtained from that model demonstrate the possibility of practical applications of stochastic duel theory.

REFERENCES

[1] Ancker, C.J., Jr., *One-on-One Stochastic Duels*, Research Monograph, Military Applications Section of the Operations Research Society of America, 1982.
[2] Gabriel, R.A., "Lessons of War: The IDF in Lebanon," *Military Review*, **64**(8), 47–65 (1984).
[3] Gafarian, A.V., and Ancker, C.J., Jr., "The Two-on-One Stochastic Duel," *Naval Research Logistics Quarterly*, **31**, 309–324 (1984).
[4] Kress, M., "The Many-on-One Stochastic Duel," *Naval Research Logistics*, **34**, 713–720 (1987).
[5] Schepe, G., *Mountain Warfare in Europe*, National Security Series No. 2/83, Centre for International Relations, Queen's University, Kingston, Ontario (1983).

Manuscript received 6/17/91
Manuscript accepted 7/19/91