Integral Equations and the Method of Moments
(Chapter 3)

EC4630 Radar and Laser Cross Section

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E-field Integral Equation (EFIE)

The EFIE is used derived from the radiation integrals and the boundary conditions. Consider a PEC with surface current $\vec{J}_s(\vec{r}\,')$. The scattered field at an arbitrary observation point (not limited to the far field) is

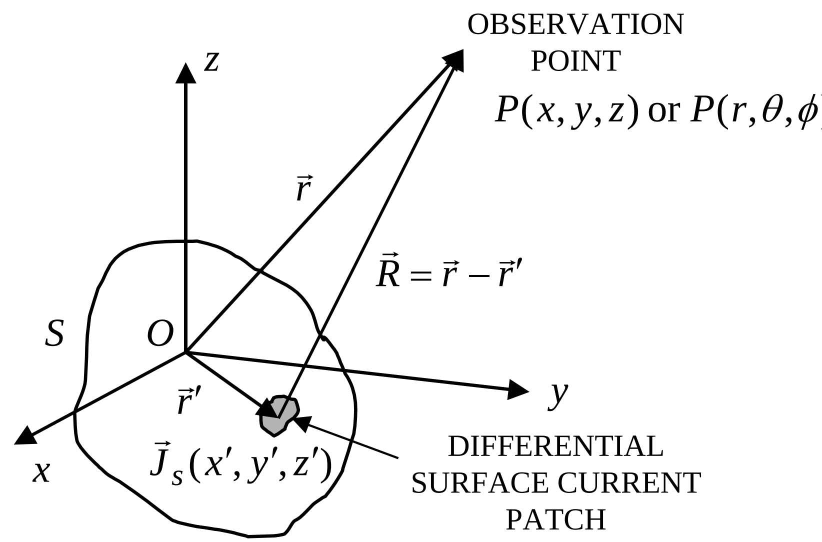
$$\vec{E}_s(r) = -j\omega\vec{A}(\vec{r}) - \frac{j}{\omega\mu_o\varepsilon_o}\nabla\left(\nabla\bullet\vec{A}(\vec{r})\right)$$

$$\vec{A}(\vec{r}) = \mu_o \iint_S \vec{J}_s(\vec{r}\,')G(\vec{r},\vec{r}\,')ds'$$

$$G(\vec{r},\vec{r}\,') = \frac{\exp(-jkR)}{4\pi R}$$

$$\vec{R} = \vec{r} - \vec{r}\,' = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$$

$$R = \left|\vec{R}\right|$$



The ∇ operator is taken with respect to the unprimed (observation) coordinates. Let P be on the surface, where the tangential component of the *total* $\vec{E}$ must be zero

$$\left[\vec{E}_i(\vec{r}) + \vec{E}_s(\vec{r})\right]\Big|_{\tan} = 0 \quad\rightarrow\quad \vec{E}_i(\vec{r})\Big|_{\tan} = -\vec{E}_s(\vec{r})\Big|_{\tan} = \left[j\omega\vec{A}(\vec{r}) + \frac{j}{\omega\mu_o\varepsilon_o}\nabla\left(\nabla\bullet\vec{A}(\vec{r})\right)\right]\Bigg|_{\tan}$$

E-field Integral Equation (EFIE)

Writing the equation in terms of the current gives the EFIE (called an integral equation because the unknown appears in the integrand)

$$\vec{E}_i(\vec{r})\Big|_{\tan} = \left[j\omega\mu_o \iint_S \vec{J}_s(\vec{r}\,')G(\vec{r},\vec{r}\,')ds' + \frac{j}{\omega\varepsilon_o}\nabla\left(\nabla\bullet\iint_S \vec{J}_s(\vec{r}\,')G(\vec{r},\vec{r}\,')ds'\right)\right]\Bigg|_{\tan}$$

A form more suitable for numerical solution has the derivatives in terms of the primed coordinates (i.e., ∇')

$$\vec{E}_i(\vec{r})\Big|_{\tan} = \left[j\omega\mu_o \iint_S \vec{J}_s(\vec{r}\,')G(\vec{r},\vec{r}\,')ds' + \frac{j}{\omega\varepsilon_o}\iint_S \nabla'\bullet\vec{J}_s(\vec{r}\,')\,\nabla'G(\vec{r},\vec{r}\,')ds'\right]\Bigg|_{\tan}$$

The method of moments (MoM) is a technique used to solve for the current:

1. Expand $\vec{J}_s(\vec{r}\,')$ into a series with unknown expansion coefficients
2. Perform a testing (or weighting) procedure to obtain a set of N linear equations (to solve for N unknown coefficients)
3. Solve the N equations using standard matrix methods
4. Use the series expansion in the radiation integral to get the fields due to the current

Fourier Series Similarity to the MoM

The method of moments (MoM) is a general solution method that is widely used in all of engineering. A Fourier series approximation to a periodic time function has a similar solution process as the MoM solution for current. Let $f(t)$ be the time waveform

$$f(t) = \frac{a_o}{T} + \frac{2}{T}\sum_{n=1}^{\infty}\left[a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\right]$$

For simplicity, assume that there is no DC component and that only cosines are necessary to represent $f(t)$ (true if the waveform has the right symmetry characteristics)

$$f(t) = \frac{2}{T}\sum_{n=1}^{\infty} a_n \cos(\omega_n t)$$

The constants are obtained by multiplying each side by the testing function $\cos(\omega_m t)$ and integrating over a period

$$\int_{-T/2}^{T/2} f(t)\cos(\omega_m t)\,dt = \frac{2}{T}\int_{-T/2}^{T/2}\left(\sum_{n=1}^{\infty} a_n \cos(\omega_n t)\right)\cos(\omega_m t)\,dt = \begin{cases} 0, & m \neq n \\ a_n, & m = n \end{cases}$$

This is analogous to MoM when $f(t) \to J_s(r')$, $a_n \to I_n$, $J_n \to \cos(\omega_n t)$, and $W_m \to \cos(\omega_m t)$. (Since $f(t)$ is not in an integral equation, a second variable t' is not required.) The selection of the testing functions to be the complex conjugates of the expansion functions is referred to as Galerkin's method.

E-field Integral Equation (EFIE)

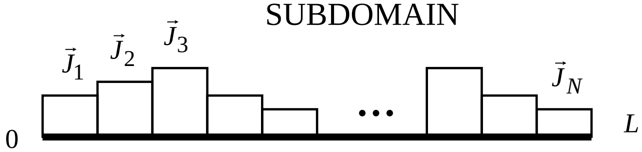
Current series expansion: $\vec{J}_s(\vec{r}\,') = \sum_{n=1}^{N} I_n \vec{J}_n(\vec{r}\,')$

$\vec{J}_n$ = basis functions (known, we get to select)

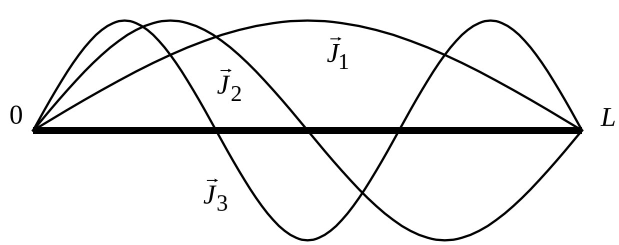
I_n = complex expansion coefficients

Types of basis functions: <u>entire domain</u> versus <u>subdomain</u> are illustrated below for a wire

SUBDOMAIN



ENTIRE DOMAIN



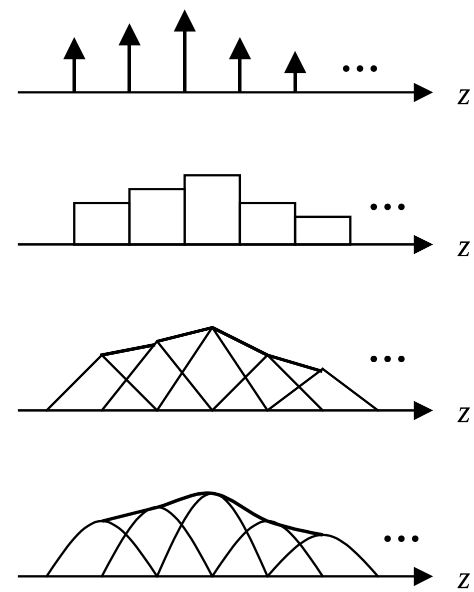
Common subsectional basis functions

δ FUNCTIONS (POINT APPROXIMATION)

PULSES (STAIRCASE APPROXIMATION)

TRIANGLES (LINEAR APPROXIMATION)

PIECEWISE SINUSOIDS (CURVED APPROXIMATION)



Testing Procedure

Define weighting (testing) functions $\vec{W}_m(\vec{r})$

- We get to choose, but generally select the complex conjugates of the expansion functions (Galerkin's method): $\vec{W}_m(\vec{r}) = \vec{J}_m^*(\vec{r})$
- The testing functions are defined at P (i.e., we are testing the field at observation points on the surface)
- To test we multiply the EFIE by each testing function and integrate over the surface (inner product)

$$\iint_{S_m} \vec{W}_m(\vec{r}) \bullet \vec{E}_i(\vec{r})\Big|_{\tan} ds = \qquad \text{for } m = 1, 2, \ldots, N$$

$$\iint_{S_m} \vec{W}_m(\vec{r}) \bullet \left\{ \sum_{n=1}^{N} I_n \times \iint_{S_n} \left[j\omega\mu_o \vec{J}_n(\vec{r}\,')G(\vec{r},\vec{r}\,') - \frac{j}{\omega\varepsilon_o} \nabla' \bullet \vec{J}_n(\vec{r}\,') \nabla' G(\vec{r},\vec{r}\,') \right]_{\tan} ds' \right\} ds$$

- Physically, this is equivalent to measuring the effect of current $\vec{J}_n(\vec{r}\,')$ at $\vec{r}\,'$ at the location of the test function $\vec{W}_m(\vec{r})$ at $\vec{r}$

Testing Procedure

Now take the summation outside:

$$\overbrace{\iint_{S_m} \vec{W}_m(\vec{r}) \bullet \vec{E}_i(\vec{r})\Big|_{\tan} ds}^{\equiv V_m} =$$

$$\sum_{n=1}^{N} I_n \underbrace{\left(\iint_{S_m} \vec{W}_m(\vec{r}) \bullet \left\{ \iint_{S_n} \left[j\omega\mu_o \vec{J}_n(\vec{r}\,')G(\vec{r},\vec{r}\,') - \frac{j}{\omega\varepsilon_o} \nabla' \bullet \vec{J}_n(\vec{r}\,')\nabla' G(\vec{r},\vec{r}\,') \right]_{\tan} ds' \right\} ds \right)}_{\equiv Z_{mn}}$$

or,

$$V_m = \sum_{n=1}^{N} I_n Z_{mn} \qquad \text{for } m = 1, 2, \ldots, N$$

The N equations can be put in matrix form: $\mathbf{V} = \mathbf{Z}\,\mathbf{I}$

$\mathbf{V} = N$ by 1 excitation vector (elements with units of volts)
$\mathbf{Z} = N$ by N impedance matrix (elements with units of ohms)
$\mathbf{I} = \mathbf{Z}^{-1}\mathbf{V} = N$ by 1 current vector (elements with units of amps)

MoM Matrix and Vectors

Impedance matrix elements:

$$Z_{mn} = \iint_{S_m} ds \iint_{S_n} ds' \left[j\omega\mu_o \vec{W}_m(\vec{r}) \bullet \vec{J}_n(\vec{r}') - \frac{j}{\omega\varepsilon_o} \nabla' \bullet \vec{J}_n(\vec{r}') \nabla \bullet \vec{W}_m(r') \right] G(\vec{r}, \vec{r}')$$

Excitation vector elements: $$V_m = \iint_{S_m} \vec{W}_m(\vec{r}) \bullet \vec{E}_i(\vec{r}) \Big|_{\tan} ds \quad \rightarrow \quad \begin{cases} V_m^{\theta}, & \vec{E}_i = \hat{\theta} E_{i\theta} \text{ (TM)} \\ V_m^{\phi}, & \vec{E}_i = \hat{\phi} E_{i\phi} \text{ (TE)} \end{cases}$$

The measurement vector gives the field at an observation point due to each basis function. These elements are obtained by using each basis function in the radiation integral (see the book for details):

$$R_n^p = \iint_{S_n} \vec{J}_n(\vec{r}') \bullet \hat{p}\, E_p(\vec{r}')\, ds' \text{ where } p = \theta \text{ or } \phi,\ \hat{p} = \begin{cases} \hat{\theta}, \text{ for TM} \\ \hat{\phi}, \text{ for TE} \end{cases}$$

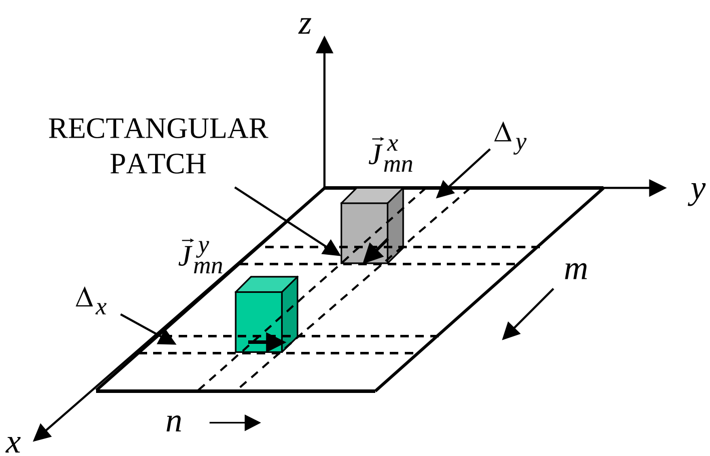
$$E_\theta = \frac{-jk\eta_o}{4\pi r} e^{-jkr} \sum_{n=1}^{N} R_n^{\theta} I_n$$

$$E_\phi = \frac{-jk\eta_o}{4\pi r} e^{-jkr} \sum_{n=1}^{N} R_n^{\theta} I_n$$

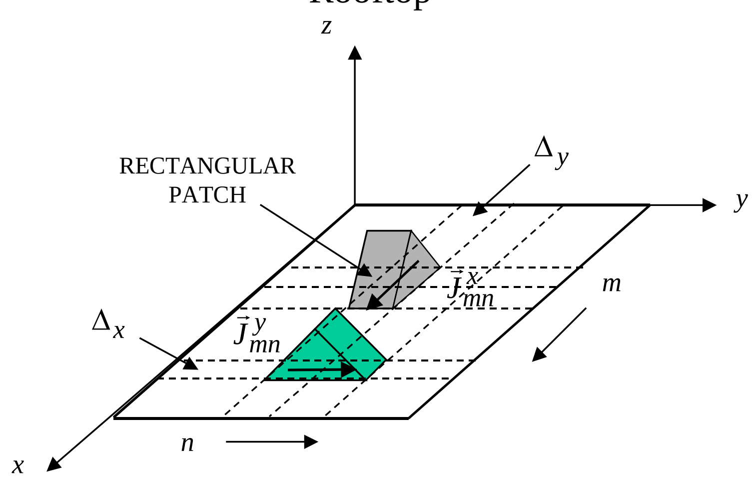
Basis Functions for Surfaces

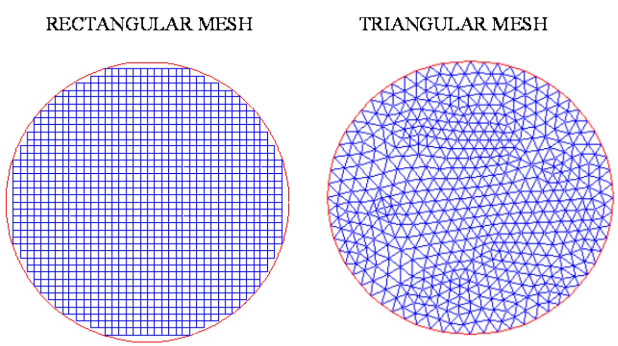
The two-dimensional extension of the step is a pedestal. The two-dimensional extension of the triangle is the rooftop. A minimum of two orthogonal components are required to represent an arbitrary current vector.

Pedestals



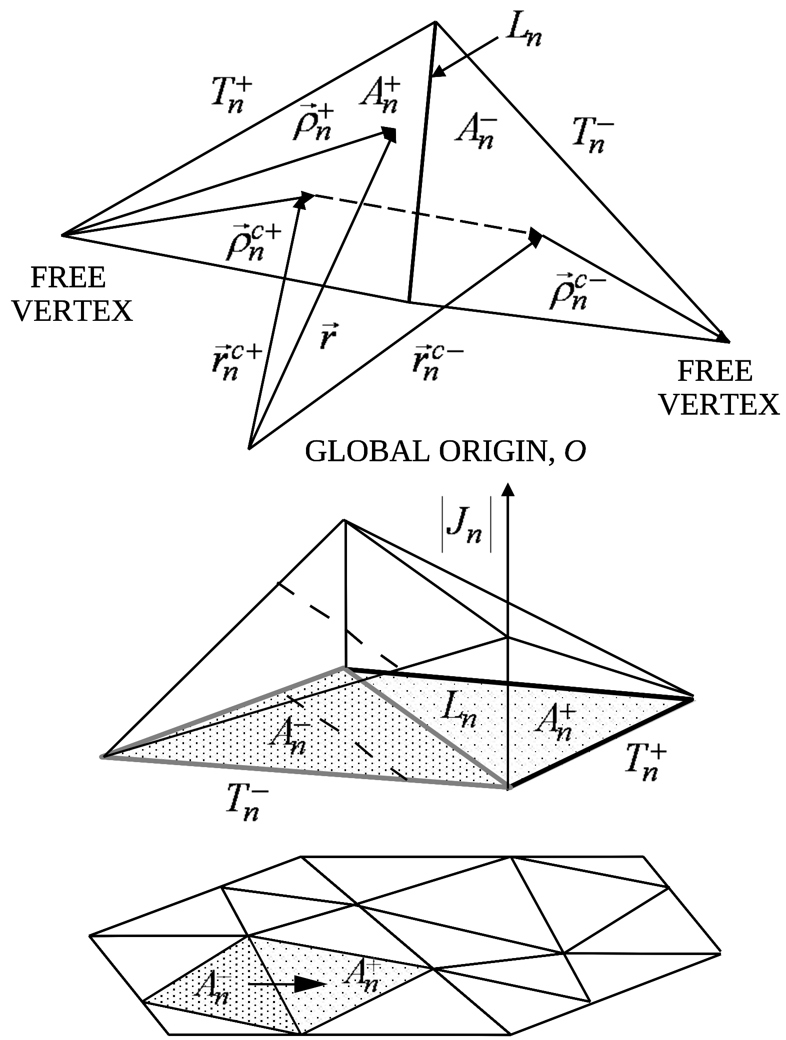
Rooftop





- Rectangular shaped subdomains are not suited to surfaces with curved edges
- The discretized shape does not represent the true edge contour -- computed edge scattered fields are not accurate
- Trianglular subdomains are more accurate.

RWG Basis Functions



The RWG (Rao, Wilton, Glisson) triangular subdomains are the most common.

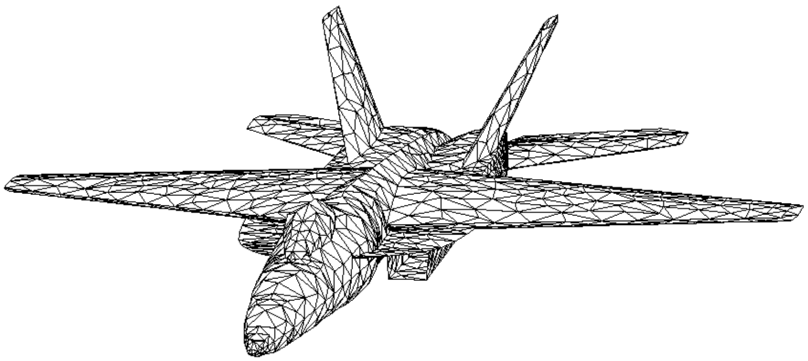
The surface is meshed into triangular subdomains.

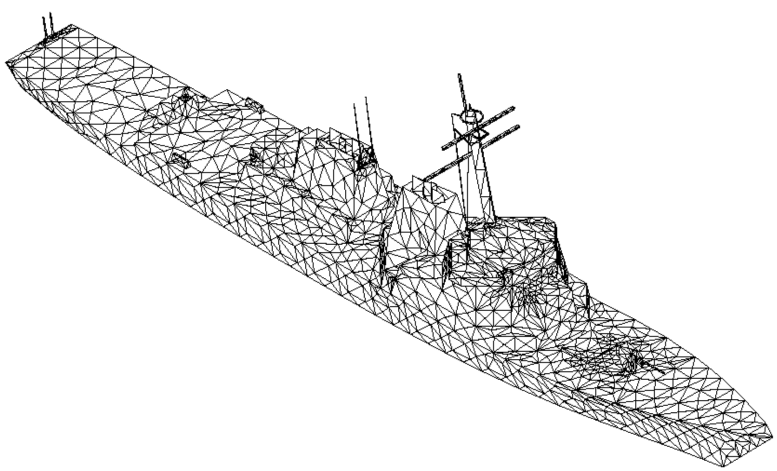
A "rooftop" basis function is associated with each interior edge:

$$\vec{J}_s(\vec{r}) = \begin{cases} \dfrac{L_n \rho_n^+}{2A_n^+}, \vec{r} \text{ in } T_n^+ \\[2ex] \dfrac{L_n \rho_n^-}{2A_n^-}, \vec{r} \text{ in } T_n^- \end{cases}$$

The current at a point in a subdomain is the vector sum of the current crossing the three edges, weighted by the current coefficients.

Surface Meshing





CAD (computer aided design) models can be used as the basis for surface meshing models

Many models are available cheaply from web sites (e.g., www.3dcadbrowser.com)

Common formats:

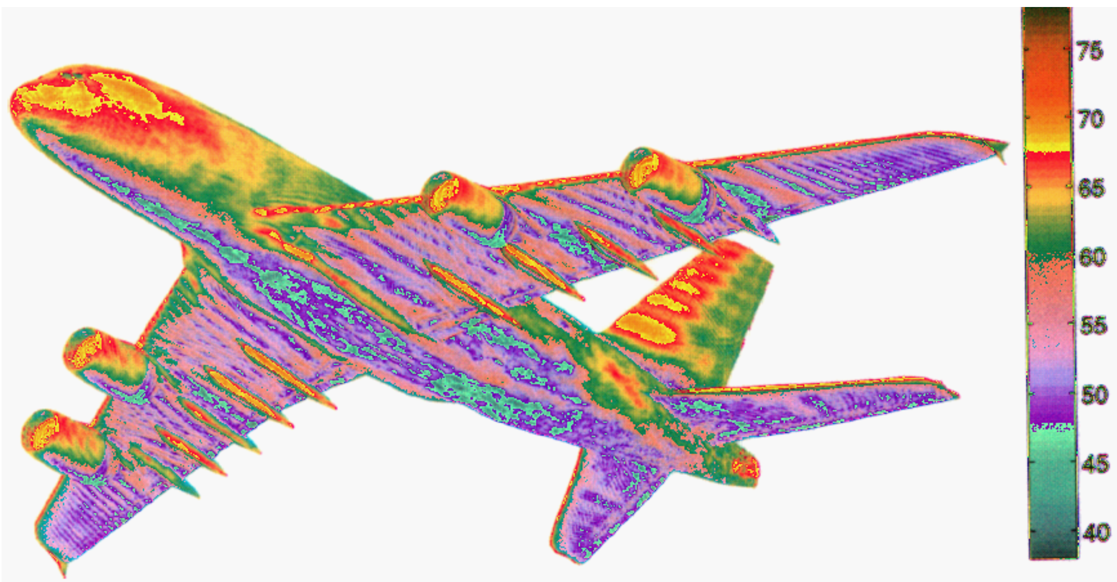
1. International Graphics Exchange Standard (*.iges or *.igs)
2. 3D design studio (*.3ds)
3. Stereolithography (*.stl)
4. Autocad (*.dwg)
5. Solidworks, Catia (*.step, *.stp)
6. Rhino (*.3dm)

CAD software packages have their own translators that are not always compatible. Further (substantial) modification is usually required to make model electrically realistic.

Current State of the Art

State of the art for computer processing for rigorous integral equation solvers (December 2009)

Induced current on a A380 aircraft in dBμA/m



- A380 aircraft
- 1.2 GHz
- 32 million unknowns
- 960 GB memory
- 11.5 hours

See Taboada, et al in *IEEE Antennas & Propagation Magazine*, V. 51, No. 6, December 2009

MoM Issues

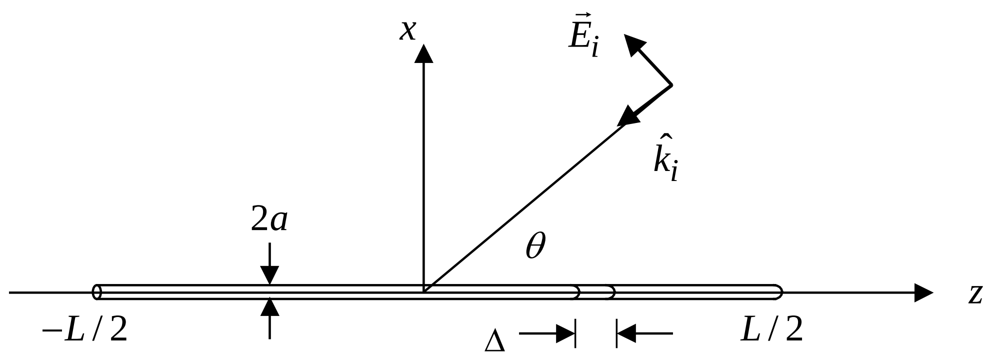
- The integral equations and the method of moments are "rigorous:"
  - Includes all scattering mechanisms and can be applied at any frequency
  - The current series converges to the true value as the number of basis functions is increased
- Subdomain basis functions are the most flexible and robust
- Triangle functions are the most common (good tradeoff between accuracy and complexity); the Rao-Wilton-Glisson (RWG) functions are the standard
- To obtain a converged result the subdomains are limited to approximately 0.1λ
  - This implies large matrices for electrically large targets
  - The large impedance matrices tend to become ill-conditioned (large spread in element order magnitudes; small values on the diagonal)
- Numerical integrations must be performed; the integrations must be converged
- The surface impedance approximation can be applied to obtain a modified EFIE
- The EFIE can be extended to volume currents and magnetic currents
- The EFIE suffers from "resonances" for closed bodies at low frequencies
- The magnetic field integral equation (MFIE) is the dual to the EFIE – it can be used in tandem with the EFIE for a more robust solution
- Other integral equations can be derived based on equivalence principles and induction theorems

Example: Scattering From a Thin Wire

Example 3.1: TM scattering from a thin wire of length L using pulse basis functions. A thin wire satisfies the condition radius, $a << \lambda$ and $a << L$. There is only an axial component of current and the problem reduces to one dimension:

$$\vec{J}_s(z') = \hat{z}\frac{I(z')}{2\pi a} = \left(\frac{\hat{z}}{2\pi a}\right)\sum_{n=1}^{N} I_n p_n(z'), \text{ where } p_n(z') = \begin{cases} 1, & z_n - \Delta/2 \le z' \le z_n + \Delta/2 \\ 0, & \text{else} \end{cases}$$

$\Delta = N/L$ is the segment length and $z_n = \dfrac{2n-(N+1)}{2}\Delta$ is the center of segment n.



Since $\nabla' \equiv \hat{z}\, d/dz'$ the EFIE reduces to

$$\hat{z}E_{i_z} = \hat{z}\frac{j}{\omega\varepsilon_o}\int_{-L/2}^{L/2} I(z')\left[k^2 + \frac{d^2}{dz^2}\right]\underbrace{\frac{e^{-jk|z-z'|}}{4\pi|z-z'|}}_{G(z,z')}dz'$$

Example: Scattering From a Thin Wire

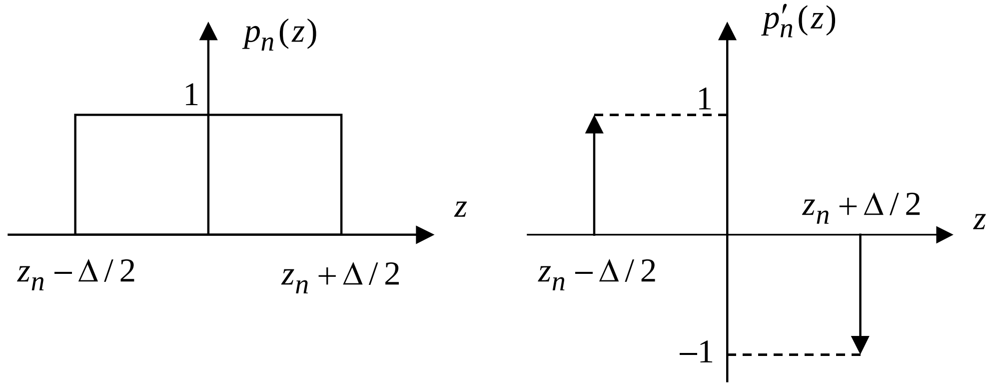
Use identities to convert the unprimed derivatives to primed ones

$$E_{i_z} = j\omega\mu_o \int_{-L/2}^{L/2} I(z')G(z,z')dz' + \frac{j}{\omega\varepsilon_o} \int_{-L/2}^{L/2} \left(\frac{\partial I(z')}{\partial z'}\right)\left(\frac{\partial G(z,z')}{\partial z'}\right) dz'$$

The derivative of the pulse

$$p_n'(z') \approx \underbrace{\delta\Big[z' - \overbrace{\big(z_n - \Delta/2\big)}^{z_n^-}\Big]}_{\equiv \delta_n^-} - \underbrace{\delta\Big[z' - \overbrace{\big(z_n + \Delta/2\big)}^{z_n^+}\Big]}_{\equiv \delta_n^+}$$

Pulse basis function Derivative



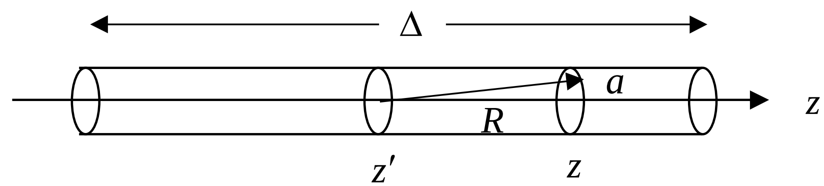
Example: Scattering From a Thin Wire

After the testing procedure, the impedance elements are found to be

$$Z_{mn} = jk\eta_o \int_{z_m^-}^{z_m^+} dz \int_{z_n^-}^{z_n^+} dz' \frac{e^{-jk|z-z'|}}{4\pi|z-z'|} - \frac{j\eta_o}{k^2} \int_{z_m^-}^{z_m^+} dz \int_{z_n^-}^{z_n^+} dz' \left(\delta_n^- - \delta_n^+\right)\left(\delta_m^- - \delta_m^+\right) \frac{e^{-jk|z-z'|}}{4\pi|z-z'|}$$

To avoid a singularity when *m*=*n* the testing is performed on the surface

$$\frac{e^{-jk|z-z'|}}{4\pi|z-z'|} \to \frac{e^{-jk\sqrt{(z-z')^2+a^2}}}{4\pi\sqrt{(z-z')^2+a^2}}$$



This method of handling the singularity is not very robust (see book for discussion).

The impedance integrals can be evaluated numerically. To reduce computation time one of the double integrals in each term can be approximated by using the simple "rectangular rule"

$$\int_{z_n^-}^{z_n^+} f(z')dz' \approx \Delta f(z_n)$$

which leads to Equations (3.56), (3.57) and (3.58) in the book (see the next page).

Example: Scattering From a Thin Wire

Final result:

$$Z_{mn} = jk\eta_o \Delta \int_{z_m^-}^{z_m^+} dz \int_{z_n^-}^{z_n^+} \frac{e^{-jkR_m}}{4\pi R_m} dz' - \frac{j\eta_o}{k^2} \left\{ \frac{2e^{-jkR_1}}{4\pi R_1} - \frac{e^{-jkR_2}}{4\pi R_2} - \frac{e^{-jkR_3}}{4\pi R_3} \right\}$$

where

$R_m = \sqrt{(z_m - z')^2 + a^2}$, $R_1 = \sqrt{(z_m - z_n)^2 + a^2}$, $R_2 = \sqrt{(z_m - z_n - \Delta)^2 + a^2}$, and

$R_3 = \sqrt{(z_m - z_n + \Delta)^2 + a^2}$. The TM excitation elements are

$$V_m^\theta = \int_0^{2\pi} \int_{z_m^-}^{z_m^+} \left(\hat{\theta} e^{jkz\cos\theta} \right) \bullet \left(\hat{z} \frac{p_m(z)}{2\pi a} \right) a\,d\phi\,dz = \Delta \sin\theta \operatorname{sinc}\left(\frac{\Delta}{2} k\cos\theta \right) e^{jkz_m \cos\theta}$$

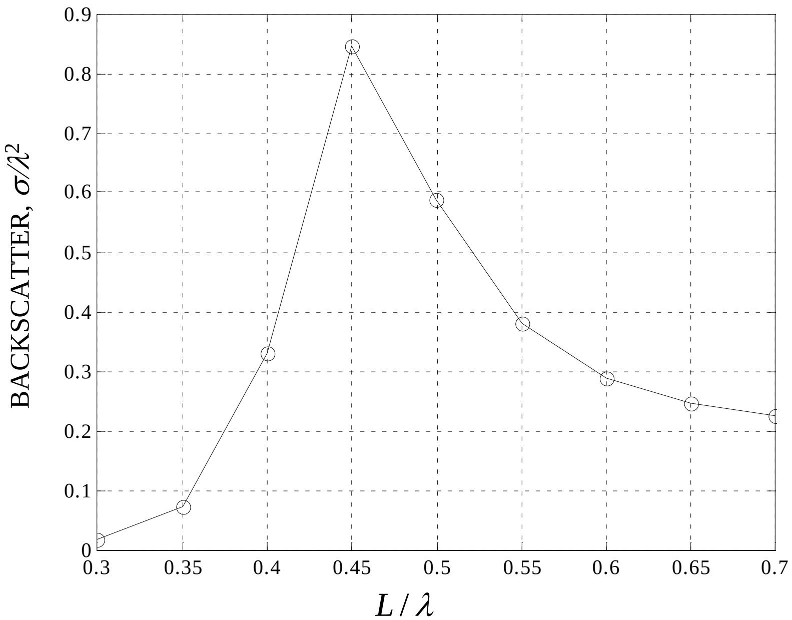
The receive elements for TM polarization are the same as the excitation elements

$$R_n^\theta = \int_0^{2\pi} \int_{z_n^-}^{z_n^+} \left(\hat{\theta} e^{jkz\cos\theta} \right) \bullet \left(\hat{z} \frac{p_n(z)}{2\pi a} \right) a\,d\phi\,dz = \Delta \sin\theta \operatorname{sinc}\left(\frac{\Delta}{2} k\cos\theta \right) e^{jkz_n \cos\theta}$$

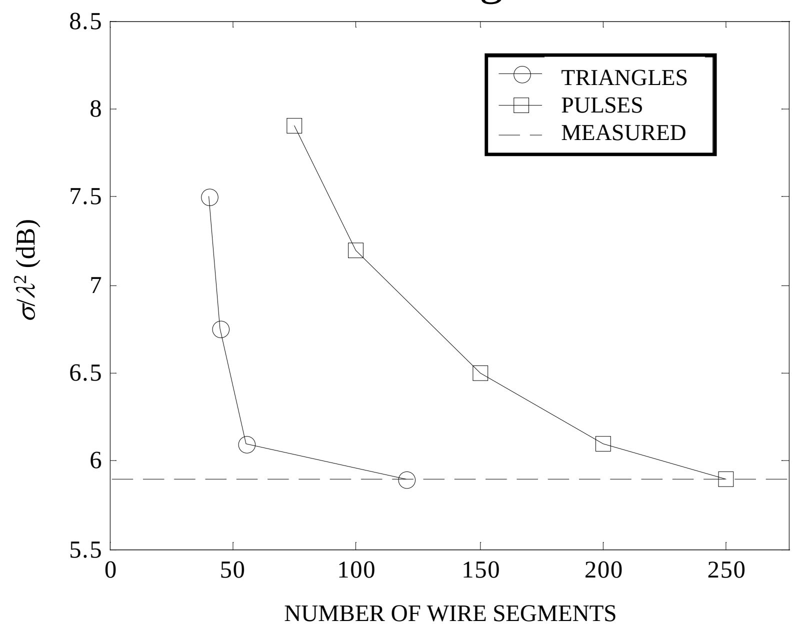
Example: Scattering From a Thin Wire

The field and RCS: $E_\theta = \dfrac{-jk\eta_o}{4\pi r} e^{-jkr} \mathbf{R}\mathbf{Z}^{-1}\mathbf{V} \quad \rightarrow \quad \sigma_{\theta\theta} = \dfrac{k^2\eta_o^2}{4\pi} \left| \mathbf{R}\mathbf{Z}^{-1}\mathbf{V} \right|^2$

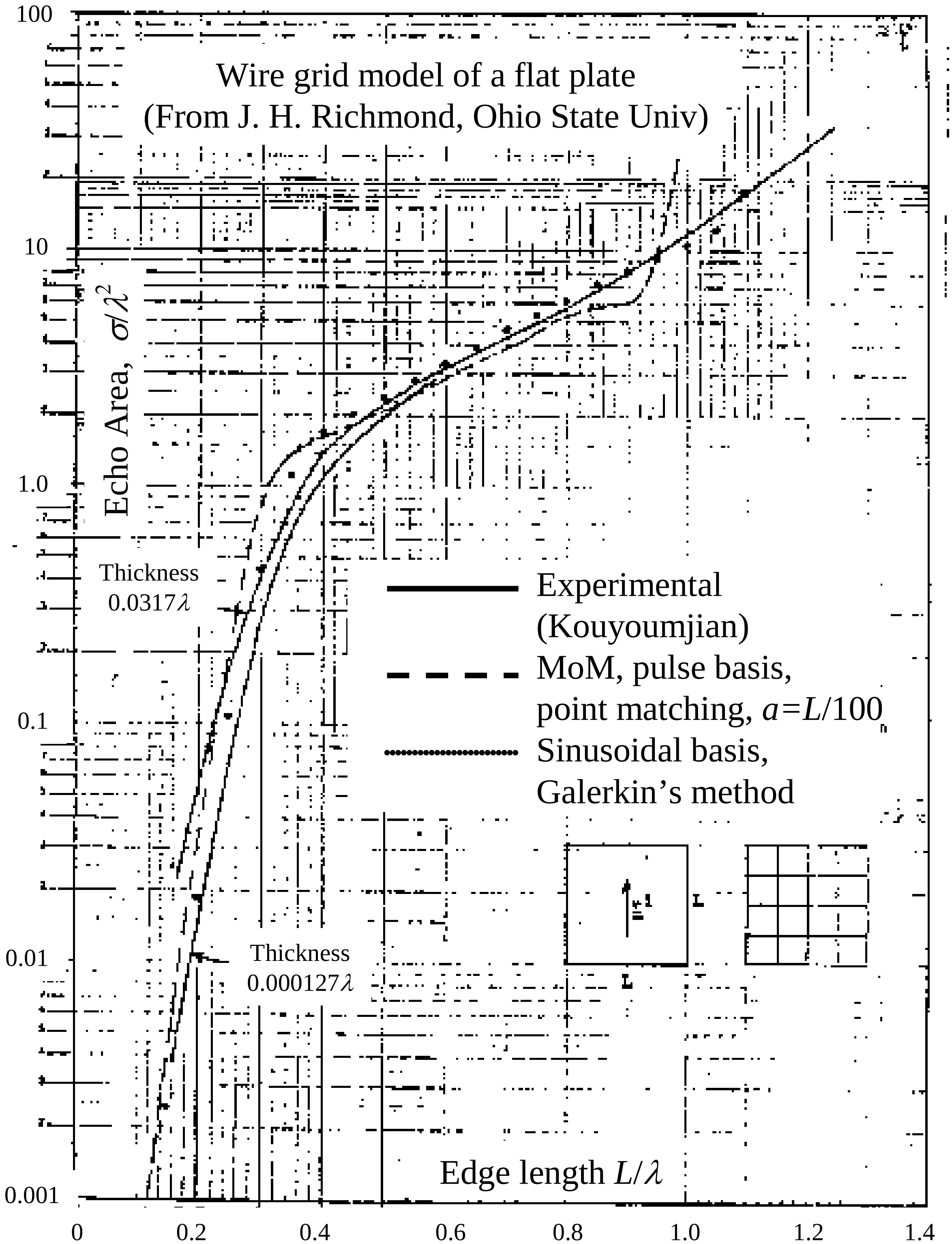
Wire backscatter (θ = 90 deg)



Convergence

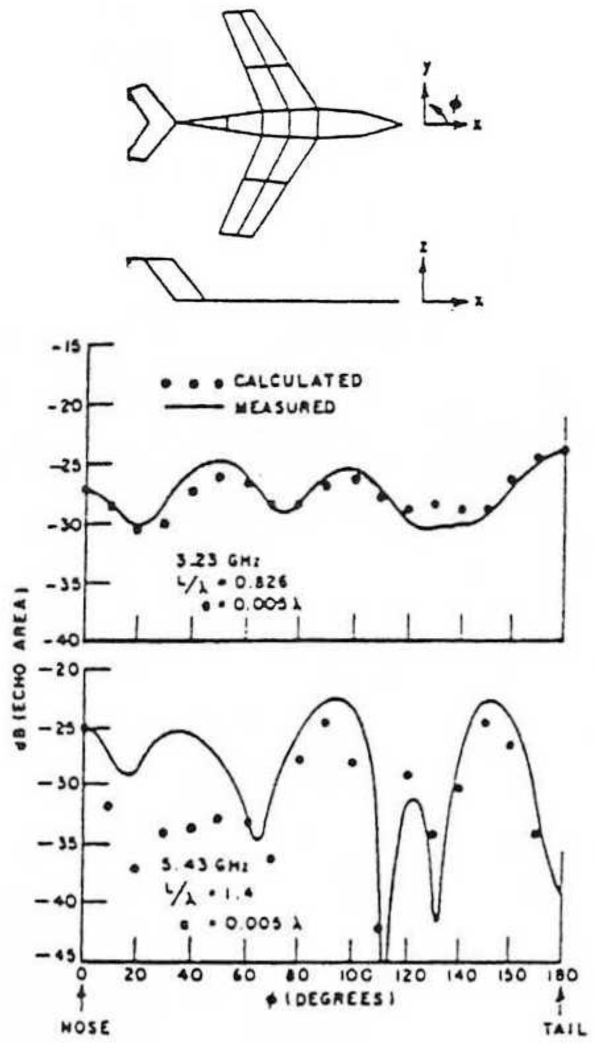


Wire Grid Approximation

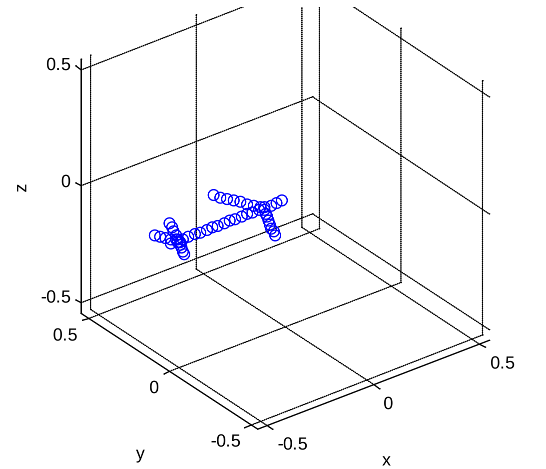


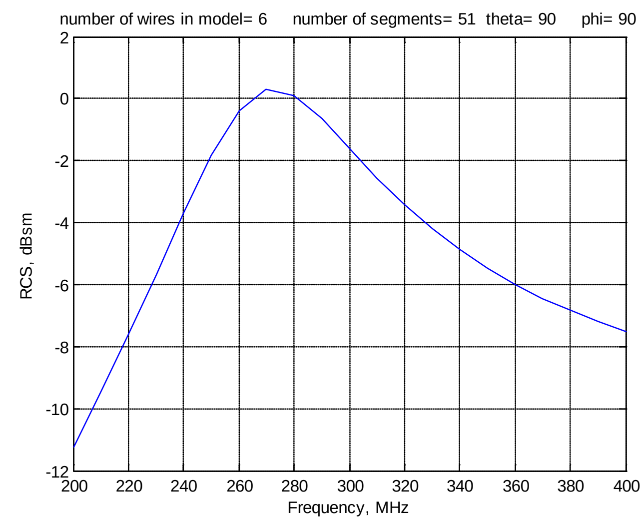
Wire Grid Models

From Lin and Richmond (Ref. 16 in the book)



Using "wiregrid.m" TE pol (pulse basis functions)

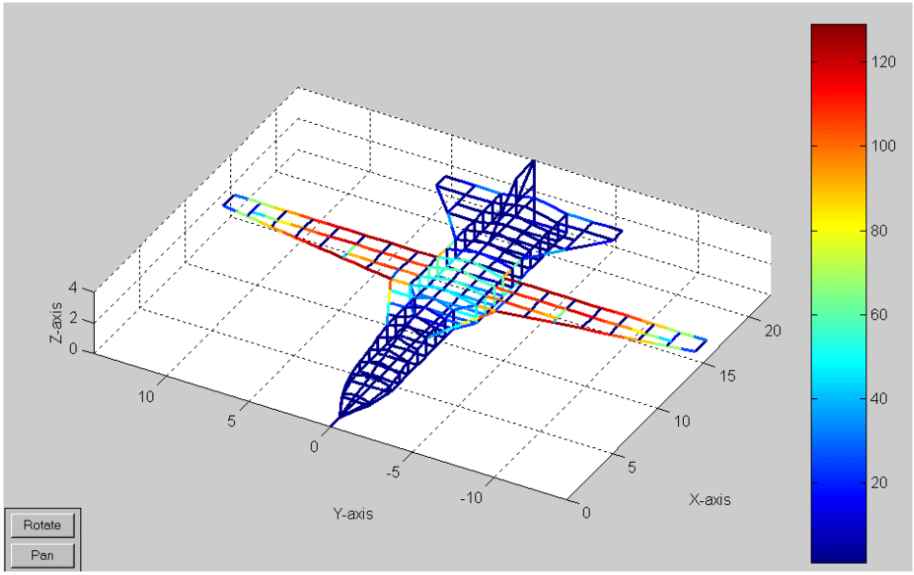


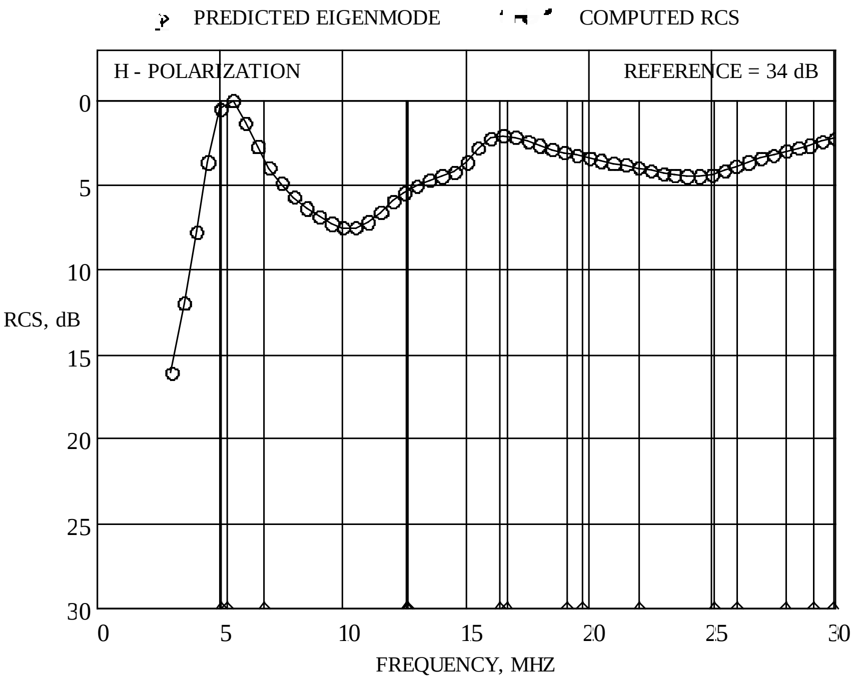


NEC Wire Grid Models

NEC = Numerical Electromagnetics Code (other versions: NECWIN, GNEC, SuperNEC)

Current on a wire grid model
of a F-111 at 5.07 MHz





(From Prof. Jovan Lebaric, *Naval Postgraduate School*)

RWG Codes by Makarov* (Lab 4)

The MM solution and calculation of the RCS is performed in a series of steps, by executing the following Matlab scripts. The scripts rwg1v2, rwg2, rwg3v2, and rcsv2 are run sequentially, in that order. Makarov's codes (names in caps) have been modified by Jenn (names in parenthesis)

RWG1 (rwg1v2)
- Uses the structure mesh file, e.g. platefine.mat, as input
- Creates the RWG edge element for every inner edge of the structure
- The total number of elements is EdgesTotal.
- Output calculated geometry data to mesh1.mat

RWG2 (rwg2)
- Uses mesh1.mat as an input.
- Output calculated geometry data to mesh2.mat

RWG3 (rwg3v2)
- Uses the mesh2.mat as an input.
- Specify the frequency
- Calculates the impedance matrix using function IMPMET
- Output Z matrix (EdgesTotal by EdgeTotal) and relevant data to impedance.mat

RCS (rcsv2)
- Uses both mesh2.mat and impedance.mat as inputs
- Specify angle resolution for computation
- Specify incoming direction of plane wave
- Specify polarization of plane wave
- Computed the "voltage" vector (RHS of MoM eqn)
- Solves MoM for the scattering problem of the structure
- Can be modified to compute bistatic RCS in the far field based on computed surface current (E & H computed in function POINT)
- Plot the monostatic RCS of the structure

All of these programs have been consolidated into one program:

rwgRCS

**Antenna and EM Modeling with MATLAB*, by Sergey Makarov, Wiley (Ref 10 in the book)

EFIE With Surface Resistivity

Using the resistive sheet boundary condition the EFIE becomes

$$\vec{E}(\vec{r})\Big|_{\tan} = \left[\vec{E}_i(\vec{r}) + \vec{E}_s(\vec{r})\right]_{\tan} = R_s\vec{J}_s$$

$$\vec{E}_i(\vec{r})\Big|_{\tan} = R_s\vec{J}_s + \left[j\omega\mu_o \iint_S \vec{J}_s(\vec{r}\,')G(\vec{r},\vec{r}\,')ds' + \frac{j}{\omega\varepsilon_o}\iint_S \nabla'\bullet\vec{J}_s(\vec{r}\,')\nabla' G(\vec{r},\vec{r}\,')ds' \right]\Bigg|_{\tan}$$

The testing procedure gives the impedance elements. The result is the previous PEC result $(Z_{PEC_{mn}})$ plus a new contribution due to the term outside of the square brackets

$$Z_{mn} = Z_{PEC_{mn}} + Z_{L_{mn}}, \text{ where } Z_{L_{mn}} = \iint_S \vec{W}_m \bullet \left[R_s(\vec{r})\vec{J}_n(\vec{r})\right] ds$$

Example: Thin resistive wire

$$Z_{L_{mn}} = (2\pi a)R_s \int_{-L/2}^{L/2} \left(\hat{z}\frac{p_m(z)}{2\pi a}\right)\bullet\left(\hat{z}\frac{p_n(z)}{2\pi a}\right) dz = \begin{cases} R_s\Delta/(2\pi a), & m=n \\ 0, & m\neq n \end{cases}$$

Note we can also use this result for a thin dielectric shell (see Example 3.6) where $R_s = 1/(j\omega\Delta\varepsilon t), \quad \Delta\varepsilon = \varepsilon - \varepsilon_o$ $(t << \lambda)$