Solving Markov Decision Processes

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Naval Postgraduate School

Monterey, CA

Searching for a Hidden Hostage

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Objective: Find the hostage as quickly as possible.

Triaging & Treating Patients on a Battlefield

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- For each patient, you can perform triage to roughly assess their condition, or perform treatment.

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Objective: Save as many lives as possible.

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Objective: Find the best way to use available blood to satisfy demand.

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5. Taking an action will affect what you observe next.

Part 1

MDPs: Modeling Decision-Making

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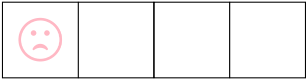
An **MDP** is defined by defining:

- the state set $\mathbb{X}$;
- the action set $A(x)$ for each state x;
- for each state x and action a,
  - the one-step cost $c(x, a)$, and
  - the transition probability distribution $p(\cdot|x, a)$ on the next state.

5. Taking an action will affect what you observe next

via a probability distribution $\quad p(\cdot|x, a)$

Searching for a Hidden Hostage: MDP



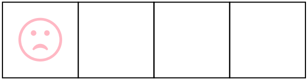
Time Elapsed: 0 hours

Only 1 location can be searched at a time.

1. P_i = posterior probability that the hostage is in location i
2. C_i = time needed to search location i
3. α_i = probability that search of location i gives a true positive
4. No false positives.

Searching for a Hidden Hostage: MDP

State = (1/4 1/4 1/4 1/4)



Time Elapsed: 0 hours

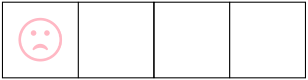
Action =

Only 1 location can be searched at a time.

1. Initially, $P_1 = P_2 = P_3 = P_4 = 1/4$.
2. Times to Search: $C_1 = 1,\ C_2 = 1,\ C_3 = 3,\ C_4 = 2$
3. True Positive Chances: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 2/3$
4. No false positives.

Searching for a Hidden Hostage: MDP

State = (1/4 1/4 1/4 1/4)



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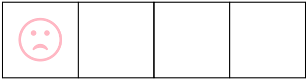
Searching for a Hidden Hostage: MDP

State = (3/10 1/10 3/10 3/10)



Time Elapsed: 1 hours

Action =

Only 1 location can be searched at a time.

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2. Times to Search: $C_1 = 1,\ C_2 = 1,\ C_3 = 3,\ C_4 = 2$
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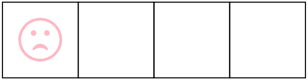

Time Elapsed: 1 hours

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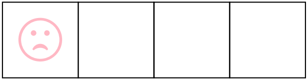
Searching for a Hidden Hostage: MDP

State = (1/8 1/8 3/8 3/8)



Time Elapsed: 2 hours

Action =

Only 1 location can be searched at a time.

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2. Times to Search: $C_1 = 1,\ C_2 = 1,\ C_3 = 3,\ C_4 = 2$
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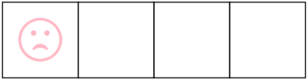
Searching for a Hidden Hostage: MDP

State $= (\quad 1/8 \quad 1/8 \quad 3/8 \quad 3/8 \quad)$



Time Elapsed: 2 hours

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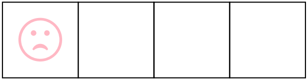
Searching for a Hidden Hostage: MDP

State = (1/6 1/6 1/2 1/6)



Time Elapsed: 4 hours

Action =

Only 1 location can be searched at a time.

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2. Times to Search: $C_1 = 1,\ C_2 = 1,\ C_3 = 3,\ C_4 = 2$
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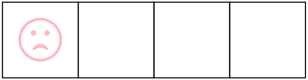
Searching for a Hidden Hostage: MDP

State = (1/6 1/6 1/2 1/6)



Time Elapsed: 4 hours

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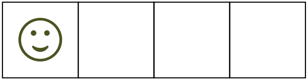
Searching for a Hidden Hostage: MDP

State = (1 0 0 0)



Time Elapsed: 5 hours

Action =

Only 1 location can be searched at a time.

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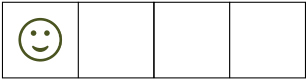
Searching for a Hidden Hostage: MDP

State = (1 0 0 0)



Time Elapsed: 5 hours

Action =

MDP Formulation:

1. Decision Epoch = end of a search
2. State = $(P_1, P_2, P_3, P_4) = \mathbf{P}$
3. Action = location to search
4. $c(\mathbf{P}, i) = C_i$
5. $p(\cdot|\mathbf{P}, i) : \begin{cases} \text{find hostage} & \text{with probability } \alpha_i P_i \\ \text{update } \mathbf{P} \text{ with Bayes' Rule} & \text{with probability } 1 - \alpha_i P_i \end{cases}$

Control via a Policy

Q: Given the state, what action should be taken?

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i.e., at each decision epoch t,

specify which action $\varphi_t(x)$ to take if state x is observed.

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specify which action $\varphi_t(x)$ to take if state x is observed.

e.g., in the hostage search problem,

search the location i maximizing $\frac{\alpha_i P_i}{C_i}$.

Optimal Policies

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Expected total cost of a policy φ:

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A policy is **optimal** if, for every initial state x, it

minimizes $v_T(x, \varphi)$ over all policies φ.

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Assume:

- number of states is finite
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1. Define $V_0(x) = 0$ for all states x.
2. For $t = 1, 2, \ldots, T$,

$$\varphi^*_{T-t}(x) = \arg\min_{a \in A(x)} \left\{ c(x, a) + \sum_{y \in \mathbb{X}} p(y|x, a) V_{t-1}(y) \right\}$$

$$V_t(x) = \min_{a \in A(x)} \left[c(x, a) + \sum_{y \in \mathbb{X}} p(y|x, a) V_{t-1}(y) \right]$$

Computing an Optimal Policy

Assume:

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Theorem:

The policy φ^* is optimal for the planning horizon T.

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Possible Interpretations of β:

1. Reflects time-value of money.
2. After each decision, the problem terminates with probability $(1 - \beta)$.

Infinite Horizon with Discounting

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> **Theorem:**
>
> There exists an optimal policy that is **stationary** (doesn't depend on what epoch t it is).

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2. For $t = 1, 2, \ldots$,

$$\varphi_t^*(x) = \underset{a \in A(x)}{\arg\min} \left\{ c(x,a) + \beta \sum_{y \in \mathbb{X}} p(y|x,a) V_{t-1}(y) \right\}$$

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Computing an Optimal Policy: Infinite Horizon

Q: How can a stationary optimal policy be **computed**?

Theorem:

For some finite T, the stationary policy that always takes actions according to $\varphi_T^*(\cdot)$ is optimal.

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Question:

Are there any guarantees on when value iteration will produce an optimal policy?

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Part 2

Efficiency of Computing Optimal Policies

Assessing the Efficiency of Algorithms

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- "Linear Search": 7 steps.
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What if we were searching for "aardvark"?

Assessing the Efficiency of Algorithms

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- "Linear Search": 7 steps.
- "Binary Search": 3 steps.

What if we were searching for "aardvark"?

What if the computer is very fast? Very slow?

Assessing the Efficiency of Algorithms

A1: Look at the number of steps needed in the worst case.

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If N is large, Linear Search may need many more steps.

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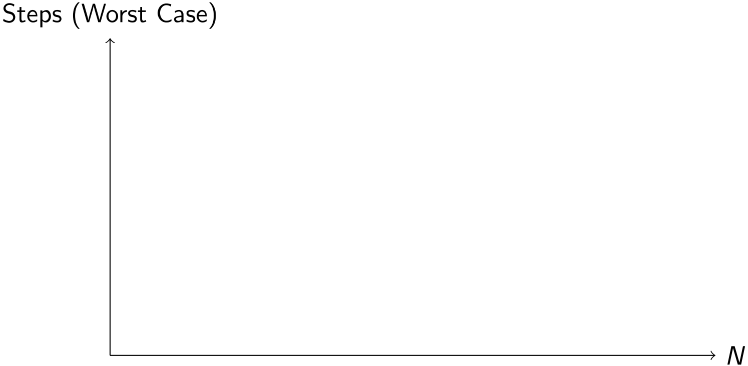

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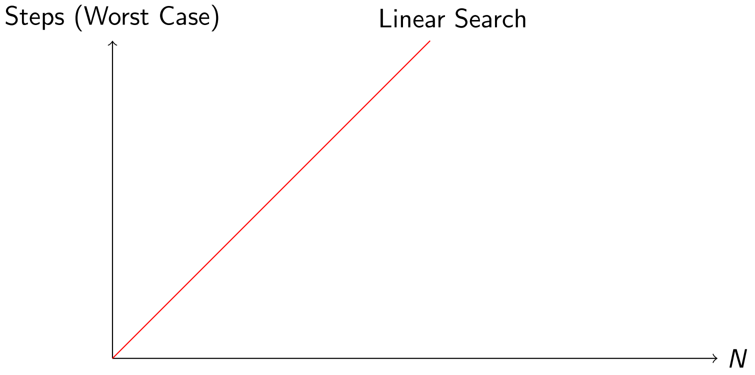

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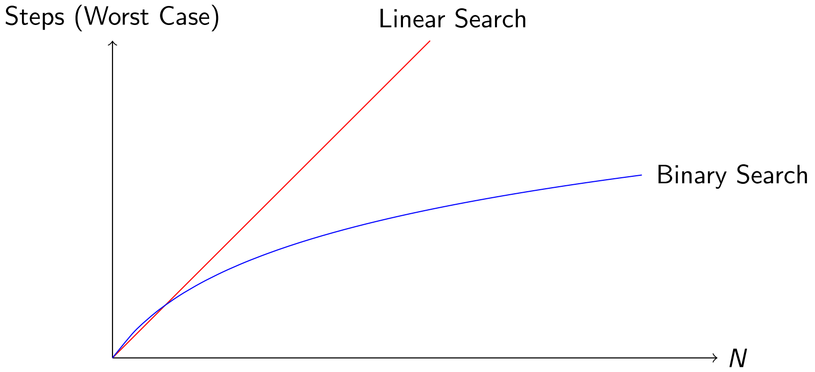

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Complexity of Algorithms for MDPs

- m = number of states in the MDP
- L = number of bits needed to encode the MDP data

What happens as $L \to \infty$? (e.g., costs get large)

1. weakly polynomial $\implies$ bound goes to ∞
2. strongly polynomial. $\implies$ bound is *unchanged*

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Efficiency of Value Iteration

Assume the discount factor β is a constant.

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Each iteration:

$$\varphi_t^*(x) = \arg\min_{a \in A(x)} \left\{ c(x,a) + \beta \sum_{y \in \mathbb{X}} p(y|x,a) V_{t-1}(y) \right\} \quad \sim m^3 \text{ steps}$$

$$V_t(x) = \min_{a \in A(x)} \left[c(x,a) + \beta \sum_{y \in \mathbb{X}} p(y|x,a) V_{t-1}(y) \right] \quad \sim m^3 \text{ steps}$$

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Theorem: (Tseng, 1990)
Value iteration needs at most[a]

$$\mathcal{O}\left(m[\log(m) + L]\right)$$

iterations to return an optimal policy.

[a] $f(x) = \mathcal{O}(g(x))$ if $\limsup_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| < \infty$

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Conclusion:
Value iteration is weakly polynomial.

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Proof that Value Iteration is Not Strongly Polynomial

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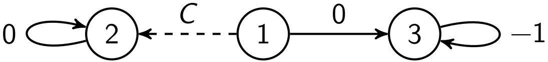

Proof that Value Iteration is Not Strongly Polynomial

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Value Iteration:

1. Define $V_0(x) = 0$ for all states x.
2. For $t = 1, 2, \ldots,$

$$\varphi_t^*(x) = \arg\min_{a \in A(x)} \left\{ c(x,a) + \beta \sum_{y \in \mathbb{X}} p(y|x,a) V_{t-1}(y) \right\}$$

$$V_t(x) = \min_{a \in A(x)} \left[c(x,a) + \beta \sum_{y \in \mathbb{X}} p(y|x,a) V_{t-1}(y) \right]$$

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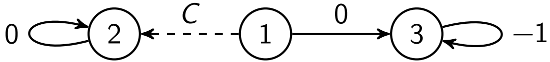

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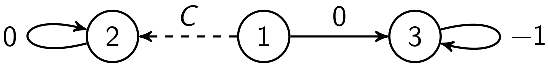

Iteration 1: $V_1(1) = \min\{C, 0\}$

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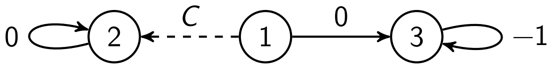

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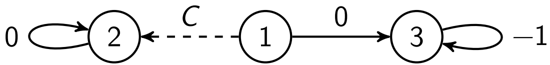

Iteration 1: $V_1(1) = \min\{C, 0\}$

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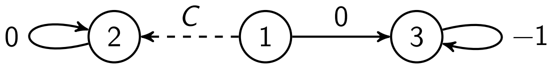

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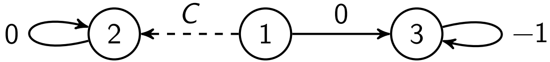

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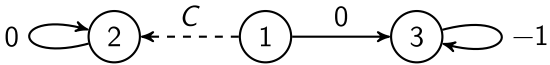

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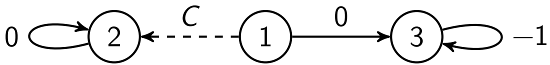

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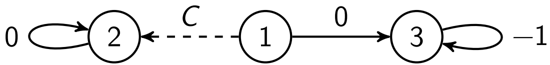

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Iteration t: $V_t(1) = \min\{C, \beta V_{t-1}(3)\}$

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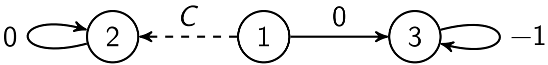

Observations:

1. $V_t(3) = -(1 + \beta + \beta^2 + \cdots + \beta^{t-1}) = -\frac{1-\beta^t}{1-\beta}$
2. Once "right" is selected, "left" will never again be selected.

$\vdots \qquad\qquad \vdots$

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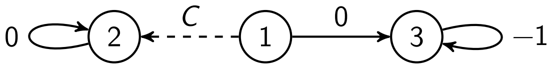

Idea: Given $N \geqslant 1$, select C so that:

1. "right" is optimal, and "left" is not.
2. On iterations $t = 1, \ldots, N$, "left" is selected.

$\vdots$ $\qquad\qquad$ $\vdots$

Iteration t: $\qquad V_t(1) = \min\{C, \beta V_{t-1}(3)\}$

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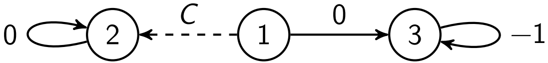

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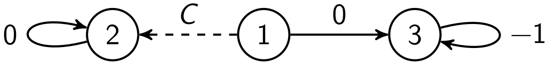

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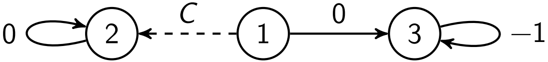

Iteration 1: $V_1(1) = \min\{C, 0\}$, $\varphi_1^*(1) = \text{"left"}$

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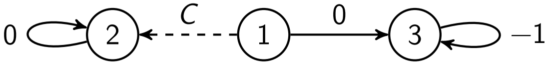

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Iteration 2:

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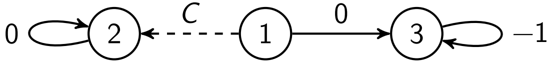

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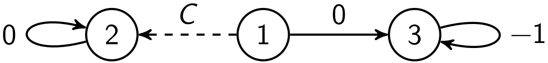

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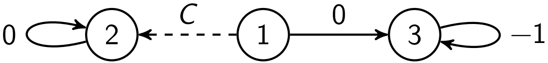

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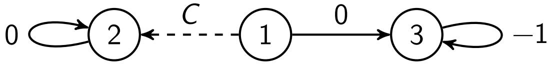

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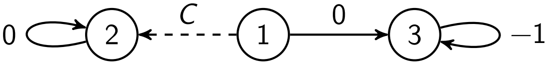

Iteration 1: $V_1(1) = \min\{C, 0\}, \quad \varphi_1^*(1) = \text{“left”}$

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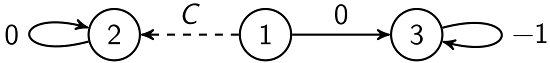
Iteration $N+1$:

Proof that Value Iteration is Not Strongly Polynomial

Given $N \geqslant 1$, select any C on the interval $\left(-\frac{\beta}{1-\beta}, -\frac{\beta(1-\beta^{N-1})}{1-\beta}\right)$.



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Proof that Value Iteration is Not Strongly Polynomial

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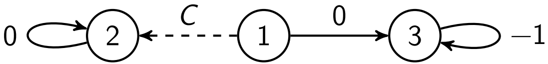

> **Theorem:** (H., 2016)
>
> For every $N \geqslant 1$, there is an MDP with $m = 4$ state-action pairs for which value iteration needs at least N iterations to return the optimal policy.

Iteration N: $V_N(1) = \min\left\{C, -\frac{\beta(1-\beta^{N-1})}{1-\beta}\right\}$, $\varphi^*_N(1) = \text{“left”}$

Iteration $N+1$: $V_{N+1}(1) = \min\left\{C, -\frac{\beta(1-\beta^{N})}{1-\beta}\right\}$, $\varphi^*_{N+1}(1) = \text{“right”}$

Proof that Value Iteration is Not Strongly Polynomial

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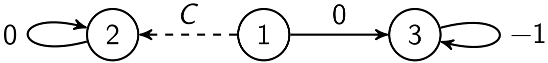

Conclusion:

Value iteration is not strongly polynomial.

$\vdots$ $\qquad\qquad\qquad\qquad$ $\vdots$

Iteration N: $\quad V_N(1) = \min\left\{C, -\frac{\beta(1-\beta^{N-1})}{1-\beta}\right\}, \quad \varphi_N^*(1) = \text{“left”}$

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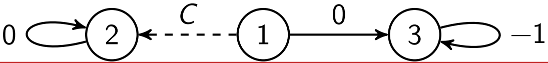
Proof that Value Iteration is Not Strongly Polynomial

Given $N \geqslant 1$, select any C on the interval $\left(-\frac{\beta}{1-\beta}, -\frac{\beta(1-\beta^{N-1})}{1-\beta}\right)$.



Conclusion:

Value iteration is not strongly polynomial.

(If it were, then value iteration should return the optimal policy for the example in at most a constant number of iterations.)

Iteration N: $V_N(1) = \min\left\{C, -\frac{\beta(1-\beta^{N-1})}{1-\beta}\right\}$, $\varphi^*_N(1) =$ "left"

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Stronger Conclusion:

The number of iterations needed by value iteration to return an optimal policy cannot be bounded above by any function of m only.

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t”

Iteration $N+1$: $V_{N+1}(1) = \min\left\{C, -\frac{\beta(1-\beta^{N})}{1-\beta}\right\}$, $\varphi^*_{N+1}(1) =$ “right”

Strongly Polynomial Algorithms for MDPs

Assume the discount factor β is a constant.

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A: Yes! **Policy Iteration:** First, select any policy φ_0^*.

1. Define $V_0(x) = v(x, \varphi_0^*)$ for all states x.
2. For $t = 1, 2, \ldots$,

$$\varphi_t^*(x) = \arg\min_{a \in A(x)} \left\{ c(x,a) + \beta \sum_{y \in \mathbb{X}} p(y|x,a) V_{t-1}(y) \right\}$$

$$V_t(x) = v(x, \varphi_t^*)$$

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> The policy iteration algorithm is strongly polynomial.

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Value Iteration vs. Policy Iteration

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Q: Is there a way to combine the good qualities of both?

A: One approach is **Modified Policy Iteration**: First, select $M \geqslant 0$.

1. Define $V_0(x) = 0$ for all states x.
2. For $t = 1, 2, \ldots$,

$$\varphi_t^*(x) = \arg\min_{a \in A(x)} \left\{ c(x,a) + \beta \sum_{y \in \mathbb{X}} p(y|x,a) V_{t-1}(y) \right\}$$

$$V_t(x) = T_{\varphi_t^*}^M V_{t-1}(x)$$

Value Iteration vs. Policy Iteration

$T^{M}_{\varphi^*_t}$ **"interpolates"** between VI and PI:

Value Iteration:

$$T^{0}_{\varphi^*_t} V_{t-1}(x) = \min_{a \in A(x)} \left\{ c(x,a) + \beta \sum_{y \in \mathbb{X}} p(y|x,a) V_{t-1}(y) \right\}$$

Policy Iteration:

$$\lim_{M \to \infty} T^{M}_{\varphi^*_t} V_{t-1}(x) = v(x, \varphi^*_t)$$

In Between: $M = 1, 2, \ldots$

$$T^{M}_{\varphi^*_t} V_{t-1}(x) = \mathbb{E}\left[\sum_{n=0}^{M} \beta^n c(x_n, \varphi^*_t(x_n)) + \beta^{M+1} V_{t-1}(x_{T+1}) \right]$$

Value Iteration vs. Policy Iteration

Value Iteration: Each iteration is cheap, but may require more iterations.

Policy Iteration: Each iteration is expensive, but may require fewer iterations.

> **Question:**
>
> Is modified policy iteration strongly polynomial, for some M?

2. For $t = 1, 2, \ldots,$

$$\varphi_t^*(x) = \arg\min_{a \in A(x)} \left\{ c(x, a) + \beta \sum_{y \in \mathbb{X}} p(y|x, a) V_{t-1}(y) \right\}$$

$$V_t(x) = T_{\varphi_t^*}^M V_{t-1}(x)$$

Value Iteration vs. Policy Iteration

Value Iteration: Each iteration is cheap, but may require more iterations.

Policy Iteration: Each iteration is expensive, but may require fewer iterations.

Theorem: (H. et al, 2014)

Modified policy iteration is not strongly polynomial for any M.

2. For $t = 1, 2, \ldots,$

$$\varphi_t^*(x) = \arg\min_{a \in A(x)} \left\{ c(x,a) + \beta \sum_{y \in \mathbb{X}} p(y|x,a) V_{t-1}(y) \right\}$$

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Theoretical Efficiency of Algorithms for MDPs

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(via linear programming)

Summary:

We showed that there is a **stark difference** between value iteration (and modified policy iteration) and policy iteration.

Policy iteration shown to be strongly polynomial in **2011**.

Value iteration and modified policy iteration shown to be not strongly polynomial in **2014**.

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(via linear programming)

Question:

Given an MDP, which algorithm should be used?

Policy iteration shown to be strongly polynomial in **2011**.

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Part 3

Computing Optimal Policies in Practice

Solving MDPs in Practice

Policy iteration should be used if possible.

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Otherwise, **approximate** methods are needed.

- approximate versions of value iteration, policy iteration

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At most N customers in the queue.

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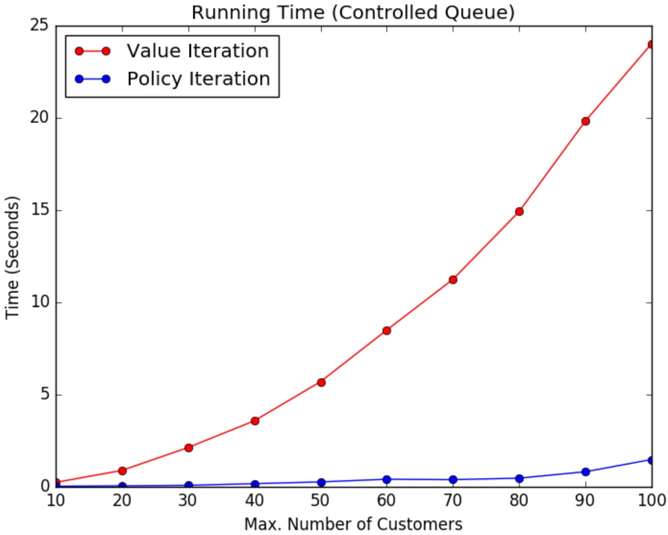
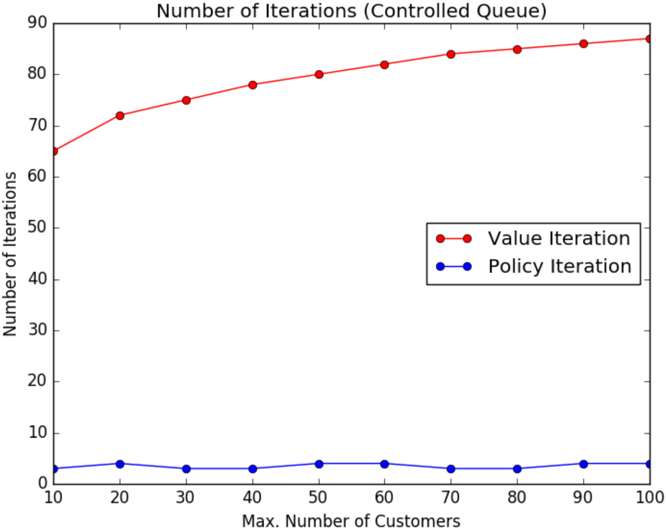
Objective: Control the service rate to minimize the expected discounted total cost. (discount factor $= 0.9$)

Running Time: Value Iteration vs. Policy Iteration



Number of Iterations: Value Iteration vs. Policy Iteration



Part 4

Future Research

Balancing Basic and Applied Research

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Basic:

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1. Joint maintenance and scheduling (e.g., in semiconductor manufacturing) (H. et al, 2018)
2. Inventory management (H. et al., 2018)
3. **Sequential decision-making in military operations**

Thank You!