Homomorphic Encryption in Maude-NPA

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Beyond the Dolev-Yao Model

Crypto protocol analysis in the standard model free algebra (Dolev-Yao) well understood. But, it fails to account for algebraic identities of crypto operations:

- Diffie-Hellman,
- exclusive-or,
- homomorphism (one-sided distributivity)

used to break a protocol or to specify modern protocols. These operations are beginning to be understood in the bounded sessions case

- Decidability results for exclusive-or, exponentiation, homomorphisms, etc.

What is lacking:

1. more general understanding, especially for unbounded sessions,
2. tool support.
Our approach

- Use rewriting logic as general theoretical framework
  - protocols and intruder rules specified as rewrite rules
  - crypto properties as oriented equational properties and axioms
- Use narrowing modulo equational theories in two ways
  - as a symbolic reachability analysis method
  - as an extensible equational unification method
- Combine with state reduction techniques (grammars, optimizations, etc.)
- Implement in Maude programming environment
  - Rewriting logic gives us theoretical framework and understanding
  - Maude implementation gives us tool support
Maude-NPA

- A tool to find or prove the absence of attacks using backwards search
- Analyzes infinite state systems:
  - Active Dolev-Yao intruder
  - No abstraction or approximation of nonces
  - Unbounded number of sessions
- Intruder and honest protocol transitions using variant of strand space model: strands with a marker denoting the current state
  - Searches backwards through strands from final state.
  - Set of rewrite rules governs how search is conducted
  - Sensitive to past and future
Motivation

Our Plans

(1) Start by formalizing NPA techniques in rewriting logic (done)
   - Prove soundness and completeness theorems (done)
   - Implement in Maude (the Maude-NPA tool) (done)
(2) Include state reduction techniques present in NPA and new (done)
(3) Document and distribute the tool (done)
(4) Extend model to different types of equational theories
   - Explicit Encryption and Decryption (done)
   - Bounded Associativity (done)
   - AC-unification (done)
   - Diffie-Hellman Exponentiation (done)
   - Exclusive-or (done)
   - Homomorphism (one-sided distributivity) (current)
Explicit Encryption and Decryption

- Most formal models lack explicit decryption operator
- If a principal knows an encrypted message and a key, assume principal can decrypt message
  - Implicit assumption that principal never decrypts a message that wasn’t encrypted in the first place
  - Usually justified by assumption that principals can check format of decrypted message
- What if format checking isn’t implemented? Or what if it is, but you are trying to verify that it works properly?
- In that case, need to model both encryption and decryption explicitly, plus their cancellation, e.g. $d(K, e(K, Y)) = Y$. 
Some Examples of Algebraic Identities

Modular Exponentiation in Diffie-Hellman

• Basic DH protocol (each nonzero residue mod $P$ is a power of $g$)

1. $A \rightarrow B : g^{N_A} \mod P$
   B computes $(g^{N_A})^{N_B} \mod P$

2. $B \rightarrow A : g^{N_B} \mod P$
   A and B compute $(g^{N_B})^{N_A} = (g^{N_A})^{N_B} \mod P$ and get a shared secret key.

• Properties:

$$(g^X)^Y = g^{X*Y} = g^{Y*X} = (g^Y)^X$$

$$(X * Y) * Z = X * (Y * Z) \quad X * Y = Y * X$$

of modular exponentiation in order to faithfully represent this protocol
Some Examples of Algebraic Identities

Exclusive-Or

- Cheap and has provable security properties
  - If we send $X \oplus R$, where $R$ a random secret, observer learns no more about $X$ than before it saw message

- On the other hand, commutativity and cancellation properties make it tricky to reason about

$$X \oplus Y = Y \oplus X \quad \quad X \oplus X = 0$$
$$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (X \oplus Y) \oplus Z = X \oplus (Y \oplus Z) \quad X \oplus 0 = X$$
**Homomorphism**

- The electronic codebook (ECB) encryption splits a message into blocks and cyphers the blocks using the same key.

- Identical plaintext blocks are encrypted into identical ciphertext blocks (does not hide data patterns well). Sensitive to the property:

$$e(K, X; Y) = e(K, X); e(K, Y)$$
Outline

① Introduction to Rewriting Logic and Narrowing
② Equational Unification
③ Maude-NPA Integration & Demo
④ Conclusions & Future Work
Rewriting Logic in a Nutshell

A rewrite theory $\mathcal{R}$ is a triple $\mathcal{R} = (\Sigma, E, R)$, with:

- $(\Sigma, R)$ a set of rewrite rules of the form $t \rightarrow s$ (i.e., protocol transitions)
  e.g. $e(K, N_A; X) \rightarrow e(K, X)$
- $(\Sigma, E)$ a set of equations of the form $t = s$ (i.e., cryptographic properties)
  e.g. $d(K, e(K, Y)) = Y$

Intuitively, $\mathcal{R}$ specifies a concurrent system, whose states are elements of the initial algebra $T_{\Sigma/E}$ specified by $(\Sigma, E)$, and whose concurrent transitions are specified by the rules $R$. 
Let $R$ a set of rewrite rules and $E$ an equational theory

Rewriting: $t \rightarrow_{R,E} s$ if there is

- a non-variable position $p \in Pos(t)$;
- a rule $l \rightarrow r \in R$;
- a matching $\sigma$ (modulo $E$) such that $t|_p =_{E} \sigma(l)$, and $s = t[\sigma(r)]_p$.

Example:

- $R = \{X \oplus X \rightarrow 0, X \oplus 0 \rightarrow X, X \oplus X \oplus Y \rightarrow Y\}$
- $E = \{X \oplus Y = Y \oplus X, (X \oplus Y) \oplus Z = X \oplus (Y \oplus Z)\}$
- $(a \oplus b) \oplus a \rightarrow_{R,E} b$
Narrowing and Backwards Narrowing

Narrowing: $t \leadsto_{\sigma, R, E} s$ if there is
- a non-variable position $p \in Pos(t)$;
- a rule $l \rightarrow r \in R$;
- a unifier $\sigma$ (modulo $E$) such that $\sigma(t|_p) =_E \sigma(l)$, and $s = \sigma(t[r]_p)$.

Example:
- $R = \{ e(K, N_A; X) \rightarrow e(K, X) \}$
- $E = \{ d(K, e(K, Y)) = Y \}$
- $e(k, X) \leadsto \{ X \mapsto N_A; X' \}, R, E$ \quad $e(k, X')$
- $d(k, X) \leadsto \{ X \mapsto e(k, e(K, N_A; X')) \}, R, E$ \quad $e(K, X')$

Backwards Narrowing: narrowing with rewrite rules reversed
Narrowing can be used as a general deductive procedure for solving reachability problems of the form

\[(\exists \bar{x}) \ t_1(\bar{x}) \rightarrow t_1'(\bar{x}) \land \ldots \land t_n(\bar{x}) \rightarrow t_n'(\bar{x})\]

in a given rewrite theory.

- The terms \(t_i\) and \(t_i'\) denote sets of states.
- For what subset of states denoted by \(t_i\) are the states denoted by \(t_i'\) reachable?
- No finiteness assumptions about the state space.
- Sound and complete for topmost rewrite theories.
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A Little Background on Unification

- Given a signature $\Sigma$ and an equational theory $E$, and two terms $s$ and $t$ built from $\Sigma$:

- A unifier of $s$ and $t$ is a substitution $\sigma$ to the variables in $s$ and $t$ such that $\sigma s$ can be transformed into $\sigma t$ by applying equations from $E$ to $s$ and its subterms

- Example: $\Sigma = \{d/2, e/2, m/0, k/0\}$, $E = \{d(K, e(K, X) = X)\}$. The substitution $\sigma = \{X/e(K, Y)\}$ is a unifier of $d(K, X)$ and $Y$.

- The set of most general unifiers of $s$ and $t$ is the set $\Gamma$ such that any unifier $\sigma$ is of the form $\rho \tau$ for some $\rho$, and some $\tau$ in $\Gamma$.

- Example, $\{X/e(K, Y), Y/d(K, X)\}$ is the set of mgu’s of $e(K, X)$ and $Y$. 


**Equational Unification**

- Given the theory, can have:
  - at most one mgu (empty theory)
  - a finite number (AC)
  - an infinite number (associativity)

- Problem in general undecidable, so different algorithms devised for different theories

- Compare to syntactic unification:
  - \( f(a, X) = f(Y, b) \) has solution \( X \mapsto b, Y \mapsto a \)
  - \( f(a, X) = f(b, Y) \) has no solution
  - \( f(a, X) =_{AC} f(b, Y) \) has solution \( X \mapsto b, Y \mapsto a \)
  - \( X + 0 = X \) has no solution
  - \( X + 0 =_{ACU} X \), where 0 is the identity, has solution \( id \)
Equational Unification

- Equational unification is the solving of formulas $\exists \vec{x} \; t =_E t'$
- When $E$ convergent TRS, narrowing provides a complete $E$-unification procedure [Hullot80] e.g. cancellation $d(K, e(K, Y)) = Y$
- When $E = \Delta \uplus B$ and $\Delta$ convergent and coherent modulo $B$, narrowing provides a complete $E$-unification procedure [Jouannaud-Kirchner-Kirchner-83] e.g. exclusive-xor

\[
(g^X)^Y = g^{X*Y} = g^{Y*X} = (g^Y)^X
\]

\[
(X \ast Y) \ast Z = X \ast (Y \ast Z) \quad X \ast Y = Y \ast X
\]

- Narrowing provides semi-decidable $E$-unification procedure, since it may not terminate even for simple cases
Narrowing is very inefficient. Strategies have been studied in the literature.

When $E$ convergent TRS, basic narrowing strategy [Hullot80] is complete for normalized substitutions.

Cases where basic narrowing terminates have been studied in order to provide decidable $E$-unification procedures.

When $E = \Delta \uplus B$ and $\Delta$ convergent and coherent modulo $B$, no many strategies have been studied in practice and AC-narrowing is highly non-terminating.

Variant-narrowing [Escobar-Meseguer-Sasse] is the most promising strategy for equational unification. It is partially implemented in Maude-NPA.
Variant-Narrowing

1. Complete narrowing strategy modulo axioms $B$ with smaller search space than full $B$-narrowing.

2. Decidable narrowing-based $E$-unification procedure

   - Based on $E$-variant of [Comon-Delaune-RTA05], we define variant narrowing strategy:
     1. it only uses substitutions in normal form
     2. complete under very general assumptions on $B$ and $\Delta$
     3. if $\Delta$ has the finite variant property of [Comon-Delaune-RTA05],
        (a) compute all $E$-variants of a term in a space-effective way
        (b) obtain a finitary $E$-unification procedure
Equational Theories & Finite Variant Property

1. [Escobar-Meseguer-Sasse-TechRep07]
   \[ Lhs \rightarrow Rhs \]  where \( Rhs \) is a variable or a constant (bound 1)
   plus some extra conditions

2. [Comon-Delaune-RTA05]
   Exclusive Or (max. bound 1)

3. [Comon-Delaune-RTA05]
   Abelian group (max. bound 2)

4. [Comon-Delaune-RTA05]
   Diffie-Hellman (max. bound 4)

5. [Comon-Delaune-RTA05]
   Homomorphism (NOT)
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Protocol Verification: Maude-NPA

- Maude-NPA uses backwards search from an insecure state to find attacks or to prove unreachability of cryptographic protocols \((\Sigma, \Delta \cup B, R)\)

- Narrowing at **two levels** in Maude-NPA
  1. a theory \((\Sigma, \Delta \cup B, R)\): \((\Delta \cup B\)-narrowing with rules \(R\))
  2. for \(\Delta \cup B\)-unification \((B\)-narrowing with rules \(\Delta\))

- \(\Delta \cup B\)-unification for each backwards step using \(R\)
  1. Built-in unification algorithms desirable
  2. **Our hybrid approach**: built-in algorithms for \(B\), and a generic algorithm (variant narrowing) for \(\Delta\).
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Conclusions

• Equational unification is critical for cryptographic protocol analysis

• Equational unification can be:
  – unitary (empty theory),
  – finitary (AC),
  – infinitary (associativity)

• Equational unification in Maude-NPA:
  – built-in unification algorithms
    (AC –Core Maude–, homomorphism –Meta Level–),
  – our hybrid approach: built-in algorithms for $B$, and a generic algorithm (variant narrowing) for $\Delta$.

• When a theory $E$ has the finite variant property (modulo AC), variant narrowing provides an efficient equational unification algorithm
Future work

• Prototype implementation developed (already used in Maude-NPA)
• Homomorphism algorithm (from Narendran-Lynch) does not support AC properties but Maude-NPA relies on that for states ⇒ infrastructure for combining different unification procedures (AC & homomorphism)
• Homomorphism algorithm (from Narendran-Lynch) does not support sorts (order-sorted) ⇒ order-sorted filter (Hendrix-Meseguer)
• Is there variant-narrowing modulo homomorphism possible?? (e.g. homomorphism with exclusive-or)