**Stirling's formula**

1. **Wallis' formula for \( \pi \).**

\[
C_n = \int_0^{\pi/2} \cos^n \theta \, d\theta,
\]

where \( n = 0, 1, 2, \ldots \).

By integration by parts (\( u = \cos^{n-1} \theta, \, dv = \cos \theta \, d\theta \), and a little "ingenuity")

\[
C_n = \frac{n-1}{n} C_{n-2}, \quad C_0 = \frac{\pi}{2}, \quad C_1 = 1.
\]

By induction:

\[
C_0 = \frac{\pi}{2}, \quad C_1 = 1
\]

\[
C_2 = \frac{1}{2} \frac{\pi}{2}, \quad C_3 = \frac{2}{3}
\]

\[
C_4 = \frac{3}{4} \frac{\pi}{2}, \quad C_5 = \frac{4}{5} \frac{2}{3}
\]

\[
C_6 = \frac{5}{6} \frac{3}{4} \frac{\pi}{2}, \quad C_7 = \frac{6}{7} \frac{4}{5} \frac{2}{3}
\]

\[
\vdots
\]

\[
C_{2n} = \frac{\pi}{2} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}
\]

\[
= \frac{\pi}{2} \frac{(2n)!}{4^n (n!)^2},
\]
\[ C_{2n+1} = \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)} = \frac{4^n (n!)^2}{(2n+1)!} \]

Since
\[ 0 < \cos^{n+1} \theta < \cos^n \theta, \quad 0 < \theta < \frac{\pi}{2}, \]
then
\[ C_{n+1} < C_n < C_{n-1} \]
and so
\[ 1 < \frac{C_n}{C_{n+1}} < \frac{C_{n-1}}{C_{n+1}} = 1 + \frac{1}{n} \rightarrow 1. \]

Thus
\[ \frac{C_{2n}}{C_{2n+1}} = \frac{\pi}{2} \frac{(2n)!}{4^n (n!)^2} \frac{(2n+1)!}{(2n+1)!} \rightarrow 1 \]

or
\[ \frac{\pi}{2} = \lim_{n \to \infty} \frac{16^n (n!)^4}{(2n)! (2n+1)!} \frac{C_{2n}}{C_{2n+1}} \]

\[ \frac{\pi}{2} = \lim_{n \to \infty} \frac{16^n (n!)^4}{(2n)! (2n+1)!} \]

\[ \pi = \lim_{n \to \infty} \frac{16^n (n!)^4}{[(2n)!]^2 (n + \frac{1}{2})} \]
\[
\lim \frac{(C^n(n!)^4 - 1)}{\eta [2(n!)^2]^{1 + \frac{1}{2n}}}
\rightarrow 1.
\]

\[
\sqrt{\pi} = \lim \frac{4^n(n!)^2}{\sqrt{n(2n)!}}.
\]

2. **Lower and upper bounds for**

\[
A_n := \int_1^n \ln x \, dx = n \ln n - n + 1
\]

are gotten from the trapezoidal and midpoint rules with step \( h = 1 \):

\[
T_n := \frac{1}{2} \left[ \ln 1 + \ln 2 + \ldots + \ln (n-1) + \ln n \right]
= \ln n! - \ln \sqrt{n}
< A_n, \ n > 1,
\]

since \( f(x) = \ln x \) is concave, and

\[
M_n := \ln \frac{3}{2} + \ln \frac{5}{2} + \ldots + \ln \left( n - \frac{1}{2} \right)
= \ln \frac{3}{2} \cdot \frac{5}{2} \ldots \frac{2n-1}{2}
\]
\[ M_n = \ln \frac{3 \cdot 5 \cdot 7 \ldots (2n-1)}{2 \cdot 2 \cdot 2 \ldots 2} \]
\[ = \ln 2 \cdot \frac{(2n)!}{4^n n!} \]
\[ > A_n, \ n > 1. \]

We need to show this. More generally let
\[ i(x, 2h) = \int_{x-h}^{x+h} f(t) \, dt \]
\[ = F(x+h) - F(x-h) \]
with \( F \) "the" antiderivative of \( f \), so that \( F' = f \). Note in passing that \( i(x, 2h) \) is an odd function of \( h \) (\( x \) is fixed!). By Taylor's theorem with derivative form of the remainder (obtained by several applications of the Cauchy mean value theorem)
\[ i(x, 2h) = F(x) + F'(x)h + \frac{1}{2} F''(x)h^2 + \frac{1}{6} F'''(c_0)h^3 \]
\[ - F(x) + F'(x)h - \frac{1}{2} F''(x)h^2 + \frac{1}{6} F'''(c_1)h^3 \]
\[ = 2h \xi(x) + \frac{1}{3} \frac{\xi''(c_0) + \xi''(c_1)}{2} h^3 \]

with \( c_0 \) between \( x \) and \( x+h \), and \( c_1 \) between \( x-h \) and \( x \). By the intermediate value theorem, and replacing \( h \) by \( h/2 \),

\[ i(x, h) = \int_{x - \frac{h}{2}}^{x + \frac{h}{2}} \xi(t) \, dt \]
\[ = h \xi(x) + \frac{1}{24} \xi''(c) h^3 \]

\[ m(x, h) = \text{midpoint approximation to } i(x, h) \]

Adding up all these elemental approximations, and using the intermediate value theorem again, we get the general result that
\[ \int_a^b f(x) \, dx = M(h) + \frac{1}{24} f''(c) \, h^2 \]

with another \( c \) strictly between \( a \) and \( b \). For this we need \( f' \) continuous on \([a, b]\), \( f'' \) continuous on \((a, b)\). The corresponding result for the trapezoidal rule is

\[ \int_a^b f(x) \, dx = T(h) - \frac{1}{12} f''(c) \, h^2 \]

with yet another elusive \( c \).

Thus if \( f'' \) is of constant sign on \((a, b)\), \( M(h) \) and \( T(h) \) bracket the integral. And if \([a, b]\) is short enough, so the two \( c \)s are close together, the midpoint error is about half that of the trapezoidal rule. In our case \( a = 1, \ b = 2, \ f(x) = \ln x, \ f'(x) = \frac{1}{x}, \) and \( f''(x) = -\frac{1}{x^2} \) is constantly \( < 0 \).

Thus, getting back to Stirling,
\[ T_n < A_n < M_n, \quad n > 1, \]
that is
\[ \ln \frac{n!}{\sqrt{n}} < \ln n^n - n + 1 < \ln 2 \frac{(2n)!}{4^n n!}. \]

Since \( e^x \) is increasing, exponentiation gives
\[ \frac{n!}{\sqrt{n}} < e \left( \frac{n}{e} \right)^n < 2 \frac{(2n)!}{4^n n!}, \]
\[ n > 1. \]

This is getting "kinda close" to Stirling's formula
\[ n! = \sqrt{2\pi n} \left( \frac{n}{e} \right)^n, \quad n \to +\infty \]

Use of left and right Euler approximations is even more suggestive, but "less accurate". They give
\[ \ln n = \ln 1 + \ln 2 + \ldots + \ln (n-1) \]
\[ = \ln (n-1)! \]
and
\[ R_n = \ln 2 + \ln 3 + \ldots + \ln n \]
\[ = \ln n!, \]
so
\[ L_n < A_n < R_n, \, n \geq 1, \]
and
\[ (n-1)! < e\left(\frac{n}{e}\right)^n < n!, \]
that is
\[ e\left(\frac{n}{e}\right)^n < n! < e\left(\frac{n+1}{e}\right)^n. \]
We need to show the \( \sqrt{2\pi n} \) behavior, and that's harder. Do we have
\[ e < \sqrt{2\pi n} \leq n e? \]
3. This is where some work, and Wallis, come in. Let
\[ a_n := A_n - \ln n, \, n \geq 1. \]
Then \( a_1 = 0 \) and \( \sum a_n \) is strictly increasing. We show that \( \sum a_n \)
is bounded. So it will follow that \( a = \lim a_n \) exists. Finding \( a \) is the crux of the matter. Look at

\[
a_{k+1} - a_k = (A_{k+1} - A_k) - (T_{k+1} - T_k)
\]

\[
= \int_{k}^{k+1} \ln x \, dx - \frac{1}{2} \left[ \ln k + \ln (k+1) \right]
\]

\[
< (M_{k+1} - M_k) - (T_{k+1} - T_k) =
\]

\[
= \ln (k+1) - \frac{1}{2} \left[ \ln k + \ln (k+1) \right]
\]

\[
= \ln \left(1 + \frac{1}{2k}\right) - \frac{1}{2} \ln \left(1 + \frac{1}{k}\right)
\]

\[
= \frac{1}{2} \ln \left(1 + \frac{1}{2k}\right) - \frac{1}{2} \left[ \ln \left(1 + \frac{1}{k}\right) - \ln \left(1 + \frac{1}{2k}\right) \right]
\]

\[
= \frac{1}{2} \ln \left(1 + \frac{1}{2k}\right) - \frac{1}{2} \ln \left(1 + \frac{1}{2(k+1)}\right)
\]

Here we used basic properties of logarithms: \( \ln(ab) = \ln a + \ln b \) and "increasingness" of \( \ln x \).
Now, summing some "collapsing" sums, we get
\[ a_n = \sum_{k=1}^{n-1} (a_{k+1} - a_k) \]
\[ < \sum_{k=1}^{n-1} (b_k - b_{k+1}) = b_1 - b_n \]
\[ = \frac{1}{2} \ln \frac{3}{2} - \frac{1}{2} \ln \left(1 + \frac{1}{2k}\right) \]
\[ < \frac{1}{2} \ln \frac{3}{2} , \]
where we put \( b_k = \frac{1}{2} \ln \left(1 + \frac{1}{2k}\right) \).
Thus we get
\[ a_n \to a \leq \frac{1}{2} \ln \frac{3}{2} , \]
but we don't know \( a \), yet. We'll save that for last. Let's see what we could do if we knew \( a \).

4. Since
\[ a_n = A_n - T_n \]
\[ = n \ln n - n + 1 - \ln \frac{n!}{\sqrt{n}} \]
then
It follows that

\[ \Delta_n = e^{1-a_n} = \frac{n!}{\sqrt{n}} \left( \frac{e}{n} \right)^n \]

so

\[ n! = \Delta_n \sqrt{n} \left( \frac{n}{e} \right)^n \]

and we really want to show that

\[ \Delta = e^{1-a} = \sqrt{2\pi} \]

Well we have

\[ a - a_n = \sum_{k=1}^{\infty} (a_{k+1} - a_k) - \sum_{k=1}^{n-1} (a_{k+1} - a_k) \]

\[ = \sum_{k=n}^{\infty} (a_{k+1} - a_k) \]

\[ < \frac{1}{2} \ln \left( 1 + \frac{1}{2n} \right) = \ln \left( 1 + \frac{1}{2n} \right)^{1/2} \]
so
\[ e^{a-a_n} < \left(1 + \frac{1}{2n}\right)^{\frac{1}{2}} < 1 + \frac{1}{4n}. \]

The last inequality is just
\[ \sqrt{1+2x} < 1+x, \] that is \[ 1+2x < (1+x)^2 = 1+2x+x^2, \] for \( x > 0. \)
Since \( a_n \to a \) then \( e^{a-a_n} \to e^0 = 1, \) so
\[ 1 < e^{a-a_n} < 1 + \frac{1}{4n} \]
is a darned good bound, for large \( n. \)

Now
\[ a_n = e^{1-a_n} \]
\[ = e^{1-a} e^{a-a_n} \]
\[ = \alpha e^{a-a_n}, \]
so
\[ \alpha < a_n < \alpha \left(1 + \frac{1}{4n}\right) \]
and, multiplying by \( \sqrt{n} \left(\frac{n}{e}\right)^n, \)
\[ \alpha \sqrt{n} \left(\frac{n}{e}\right)^n < n! < \alpha \sqrt{n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{4n}\right) \]
\[ \to 1 \]
5. That's beautiful, if we only knew \( \alpha \). In any case
\[
\frac{n!}{\alpha \sqrt{n} \left( \frac{n}{e} \right)^n}
\]
(Assuming \( \frac{a_n}{b_n} \to 1, n \to \infty \)).

Now Wallis' says
\[
\sqrt{\pi} = \lim_{n \to \infty} \frac{4^n (n!)^2}{\sqrt{n} (2n)!}.
\]

By the (unfinished) Stirling formula,
\[
\frac{4^n (n!)^2}{\sqrt{n} (2n)!} \sim \frac{4^n \frac{1}{2} \pi x^n}{\sqrt{n} \sqrt{2n} \sqrt[4]{2 \pi} (2n)^{2n}}.
\]

Thus
\[
\alpha = \sqrt{2 \pi},
\]

and Stirling is now finished.
(Wallace (Wally) Stirling was a president of Stamford University!)
Oh yes, $a = e^{1-a}$ so
\[ \ln x = 1-a, \quad a = 1-\ln x = 1-\ln \sqrt{2\pi}. \]
And $a > 0$ so $\ln \sqrt{2\pi} < 1$, that is $\sqrt{2\pi} < e$. Another relation between the two main numbers of Calculus. I computed
\[ \sqrt{2\pi} = 2.5066 \]
\[ e = 2.71828 \]
\[ a = 0.08106 \]
\[ \frac{1}{2} \ln \frac{3}{2} = 0.2027. \]
The trapezoidal bound is
\[ n! < e \sqrt{n} \left( \frac{n}{e} \right)^n, \]
not bad. The midpoint bound is, asymptotically,
\[ e \left( \frac{n}{e} \right)^n \leq 2\sqrt{2} \left( \frac{n}{e} \right)^n \]
so I guess
\[ e < 2\sqrt{2}, \]
and
\[ 2\sqrt{2} \approx 2.824. \]
6. More fiddling, for ultimate elegance.

\[ e^{T_n} = \frac{n!}{\sqrt{n}} < \frac{e^n}{e^{n/e}} < e^{H_n} = \frac{(2n)!}{4^n n!} \]

Use Wallis's,

\[ \frac{(2n)!}{4^n n!} = \frac{n!}{\sqrt{n!}} \]

to write \( e^{H_n} \) in terms of \( n! \):

\[ e^{H_n} = \frac{2}{\sqrt{\pi n}} n! \]

So

\[ \frac{n!}{\sqrt{n}} < e^{n/e^n} < \frac{2}{\sqrt{\pi n}} n! \]

that is

\[ \frac{e^{n/e^n}}{2/\sqrt{\pi n}} \leq n! < e^{n/e^n} \]

So \( H_n \) and Wallis's show that the \( \sqrt{n} \) behavior is "sharp" and only the constant is in question.

\[ \frac{e}{2/\sqrt{\pi}} \approx 2.4090 < 2.7183 \approx e, \]

\[ \sqrt{2\pi} \approx 2.5066. \]
7. Can we perhaps combine $T_n$ and $S_m$ to do better? We take a clue from

$$T(h) = \int_a^b f(x) \, dx + \frac{1}{12} f''(c_t) h^2,$$

$$M(h) = \int_a^b f(x) \, dx - \frac{1}{24} f''(c_m) h^2.$$

If we had $c_t = c_m$ (which we don't!) then the error term can be eliminated to get

$$S(h) := \frac{1}{3} T(h) + \frac{2}{3} M(h) = \int_a^b f(x) \, dx.$$

This is Simpson's rule, mentioned in most Calculus texts. It's error is $O(h^4)$, assuming $f$ has four continuous derivatives.

(But Richardson-Romberg is much more powerful; $S(h)$ is only its first step!) So what can we do with

$$S_{n+1} := \frac{1}{3} T_n + \frac{2}{3} M_{n+1}?$$
Note that $S_n$ is a weighted average of $T_n$ and $M_n$, so it lies strictly between $T_n$ and $M_n$ and hopefully much closer to $A_n$. No doubt we could even find out on which side of $A_n$ it lies, and even find another good approximation ($O(4^k)$) which lies on the opposite side. Idea: apply Richardson-Romberg acceleration to both $T_n$ and $M_n$. So let's just play a little.

$$e^{S_n} = e^{\frac{1}{3} T_n + \frac{2}{3} M_n}$$

$$= \left( e^{T_n} (e^{M_n})^2 \right)^{\frac{1}{3}}$$

$$= \left( \frac{n!}{\sqrt{n}} \frac{4}{\pi n} (n!)^2 \right)^{\frac{1}{3}}$$

$$= \left( \frac{4}{\pi} \right)^{\frac{1}{3}} \frac{n!}{\sqrt{n}}$$

$$= \left( \frac{4}{\pi} \right)^{\frac{1}{3}} \sqrt{2 \pi} \left( \frac{e}{n} \right)^n$$

$$= (128 \pi)^{\frac{1}{3}} \left( \frac{e}{n} \right)^n$$
This is to be considered as an approximation to the exact value
\[ e^{A_n} = e^{\left( \frac{n}{\varepsilon} \right)^n}, \]
along with
\[ e^{T_n} = \frac{n!}{\sqrt{2\pi n}} \left( \frac{n}{e} \right)^n, \]
\[ e^{M_n} = \frac{2}{\sqrt{\pi}} \frac{n!}{\sqrt{n}} + 2\sqrt{2} \left( \frac{n}{e} \right)^n. \]
The corresponding constants are
\[ T_n : \sqrt{2\pi} \approx 2.50663 \]
\[ M_n : 2\sqrt{2} \approx 2.82843 \]
\[ S_n : (128\pi)^{1/6} \approx 2.71881 \]
\[ A_n : e = 2.71828 \]
As approximations to \( e \) the first two have about two digits of accuracy with that for \( M_n \) being a tad closer. But that for \( S_n \) has about four accurate digits, roughly twice as many!
8. Lower and upper bounds for Wallis:

\[
\frac{\pi}{2} = \frac{16^n (2n)!^4}{(2n)!^2 (2n+1)!} \cdot \frac{C_{2n}}{C_{2n+1}} > 1
\]

Simple to show:

\[
\frac{C_{2n}}{C_{2n+1}} = \frac{C_{2n-2}}{C_{2n-1}} \left(1 - \frac{1}{4n^2}\right)
\]

\[
\frac{C_0}{C_1} = \frac{\pi}{2} \cdot \frac{C_2}{C_3} = \frac{C_0}{C_1} \left(1 - \frac{1}{4}\right) = \frac{3\cdot C_0}{4\cdot C_1},
\]

So

\[
\frac{16^n (2n)!^4}{(2n)!^2 (2n+1)!} > \frac{\pi}{2},
\]

\[
\frac{4^n (2n)!^2}{\sqrt{n} (2n)! \sqrt{1 + \frac{1}{2n}}} > \sqrt{\pi},
\]

\[
S_{n+1} = \frac{4^n (2n)!^2}{\sqrt{n} (2n)!} < \sqrt{\pi}
\]

Since

\[
\frac{S_n}{S_{n-1}} = \frac{4n^2}{(2n)(2n-1)} \cdot \frac{\sqrt{n-1}}{\sqrt{n}} = \frac{\sqrt{1 - \frac{1}{2n}}}{1 - \frac{1}{2n}} < 1
\]

Since

\[
1 - \frac{1}{2n} < \left(1 - \frac{1}{2n}\right)^2 = 1 - \frac{1}{n} + \frac{1}{4n^2}.
\]
\[
\frac{C_0}{C_1} = \frac{\pi}{2}, \quad \frac{C_2}{C_3} = \frac{3}{4} \frac{\pi}{2}
\]

\[
\frac{C_4}{C_5} = \left(1 - \frac{1}{4 \cdot 2^2}\right) \frac{3}{4} \frac{\pi}{2} = \left(1 - \frac{1}{16}\right) \frac{3}{4} \frac{\pi}{2}
\]

\[
= \frac{15}{16} \frac{3}{4} \frac{\pi}{2}
\]

\[
\frac{C_6}{C_7} = \left(1 - \frac{1}{4 \cdot 3^2}\right) \frac{C_4}{C_5} = \left(1 - \frac{1}{36}\right) \frac{C_4}{C_5}
\]

\[
= \frac{35}{36} \frac{15}{16} \frac{3}{4} \frac{\pi}{2}
\]

\[
\frac{\pi}{2} = \frac{4}{3} \frac{16}{15} \frac{36}{35} \frac{64}{63} \frac{100}{99} \frac{144}{143} \ldots
\]

\[
= \prod_{u=1}^{\infty} \frac{(2u)^2}{(2u)^2 - 1} = \prod_{u=1}^{\infty} \frac{1}{1 - \frac{1}{4u^2}}
\]

\[
\text{product}
\]

\[
P_n = \prod_{k=1}^{n} \frac{(2k)^2}{(2k)^2 - 1} = P_{n-1} \frac{(2n)^2}{(2n)^2 - 1}
\]

\[
= P_{n-1} \frac{1}{1 - \frac{1}{4n^2}} > P_{n-1}
\]

\[
P_n \to \frac{\pi}{2}.
\]

\[
\ln p_n = \ln p_{n-1} + \ln \frac{1}{1 - \frac{1}{4n^2}}
\]

\[
= \ln p_{n-1} - \ln \left(1 - \frac{1}{4n^2}\right)
\]

\[0 < \cdot < 1\]
In a sum of positive terms
\[ \ln p_n = \sum_{1}^{n} - \ln (1 - \frac{1}{4k^2}) \]

By Taylor
\[ \ln (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots, \quad -1 < x \leq 1 \]
\[ - \ln (1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots, \quad -1 \leq x < 1 \]
\[ - \ln \left(1 - \frac{1}{4k^2}\right) = \frac{1}{4k^2} + \frac{1}{2 \cdot 4^2 k^4} + \frac{1}{2 \cdot 4^3 k^6} + \cdots \]
\[ \ln p_n = \frac{1}{4} \sum_{1}^{n} \frac{1}{k^2} + \frac{1}{2 \cdot 4^2} \sum_{1}^{n} \frac{1}{k^4} + \cdots \]
\[ p_{\infty} = \frac{\pi}{2} \]
\[ \ln \frac{\pi}{2} = \frac{1}{4} \sum_{1}^{\infty} \frac{1}{k^2} + \frac{1}{2 \cdot 4^2} \sum_{1}^{\infty} \frac{1}{k^4} + \cdots \]
\[ = \sum_{1}^{\infty} \frac{\zeta(2n)}{n \cdot 4^n} \]

\underline{Riemann \ zeta \ function}
\[ \zeta(z) := \sum_{1}^{\infty} \frac{1}{k^z}, \quad \text{Re} z > 1 \]

\underline{Bernoulli \ numbers}
\[ B_{2n} = 2(-1)^{n-1} \frac{\zeta(2n)}{(2\pi)^{2n}} \]

\underline{Generating \ function}
\[ \frac{1}{\tanh(z/2)} = \sum_{0}^{\infty} \frac{B_{2n}}{(2n)!} z^{2n} \]
Special values

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Also $B_1 = -\frac{1}{2}$ and $B_{2n+1} = 0$, $n \geq 1$.

Solution to many induction problems. For $p = 0, 1, 2, \ldots$,

$$\sum_{k=1}^{n} k^p = p! \sum_{k=0}^{p} \frac{(-1)^k B_k}{k! (p+1-k)!}$$

In particular

$$\sum_{k=1}^{n} k^p = \frac{n^{p+1}}{p+1}, \ n \to +\infty.$$

Asymptotic behavior

$$B_{2n} \sim 4(-1)^{n-1} \sqrt{\pi n} \left(\frac{n}{\pi e}\right)^{2n}, \ n \to +\infty,$$

that is, with Stirling,

$$\frac{B_{2n}}{(2n)!} \sim \frac{2(-1)^{n-1}}{(2\pi)^{2n}}, \ n \to +\infty.$$

Thus the radius of convergence of the series for $\frac{x}{2 \coth \frac{x}{2}}$ is $2\pi$, as it should be. (Why?)
**Euler-Maclaurin Formula**

\[ T(h) = \text{trapezoidal rule, step } h \]

\[ = \int_a^b f(x)dx + \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} \int_a^b f^{(2m)}(x)dx \cdot h^{2m} \]

\[ h \to 0. \]

The series is only an asymptotic one. It need not converge!

(There is a remainder term, as with Taylor series.) But that doesn't matter for the

**Richardson-Romberg application**

All we need is \( \sqrt{\text{error}} \sim O(h^2) \)

\[ T(h) = \int_a^b f(x)dx + C_2 h^2 + C_4 h^4 + \ldots \]

Then

\[ 4 T\left(\frac{h}{2}\right) = 4 \int_a^b f(x)dx + C_2 h^2 + C_4 \frac{h^4}{4} + \ldots \]

\[ S(h) = \frac{4}{3} T\left(\frac{h}{2}\right) - \frac{1}{3} T(h) = \int_a^b f(x)dx + C_4 h^4 + \ldots \]

\[ \frac{1}{2}\left(T(h) + M(h)\right) \]

\[ S(h) = \frac{1}{3} T(h) + \frac{2}{3} M(h) = \int_a^b f(x)dx + C_4 h^4 + \ldots \]

\[ = \text{Simpson's rule.} \]

And so on, ad nauseum!