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Dr. Givoli,

Beny, Frank, and I would like to submit the attached manuscript to be considered for publication in Wave Motion. The title and abstract are as follows:

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Thank you for your kind consideration.

Respectfully,

John Dea
High-Order Non-Reflecting Boundary Conditions for the Linearized 2-D Euler Equations: No Mean Flow Case

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Abstract

Higdon-type non-reflecting boundary conditions (NRBCs) are developed for the 2-D linearized Euler equations with Coriolis forces. This implementation is applied to a simplified form of the equations, with the NRBCs applied to all four sides of the domain. We demonstrate the validity of the NRBCs to high order. We close with a list of areas for further research.

1 Introduction

To perform mesoscale atmospheric modeling on a computer, one immediately runs into the problem of defining the computational domain. At some point, there has to be an edge to the computational domain, but the physical atmosphere lacks any edges. How, then, can we define a computational boundary where no physical boundary exists? The answer of course is to define a non-reflecting boundary condition (NRBC). How best to define such a boundary has been an active area of research for approximately 30 years. Ideally, an NRBC will be stable, accurate, fast, and easy to implement; realistically, one must generally choose two or three of those criteria, at best.

There are typically two approaches to NRBC development. The first is to prescribe the behavior at the boundaries in such a way as to reduce any spurious reflections. Early examples include the Sommerfeld-condition-based work of Orlanski [25] and the Padé approximations of Engquist and Majda [3, 4]. This approach was expanded by Higdon [12]-[18] and subsequently automated by Givoli, Neta, and van Joolen [6]-[9], [28]-[31]. The Orlanski scheme and the Engquist-Majda scheme are less accurate.

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than their successors; however, the Higdon scheme and its offshoots suffer from very high computational overhead.

The second approach is to surround the domain with a more dispersive computational medium, so that incoming waves enter the absorbing layer and diffuse to zero before their reflections re-enter the original domain. Examples include the Perfectly Matched Layer (PML) developed by Bérenger [1], applied to the linearized shallow water equations by Navon et. al. [23] and to the linearized Euler equations by Hu [19, 20, 21], and the sponge layer used by Giraldo and Restelli [5]. This approach requires additional storage and computation time for the expanded domain, and some reflections are still evident when the theoretically-exact absorbing layer is applied to a discrete computational domain. Furthermore, the absorbing layer surrounding the computational medium precludes the possibility of incoming waves in a nested modeling environment; the incoming waves will be diffused to zero before they enter the computational domain.

Here we apply the Higdon scheme to the linearized Euler equations. We take advantage of the Givoli-Neta-van Joolen automation and make subsequent improvements to reduce the computational overhead. This method removes approximately 55% of the Sommerfeld condition’s reflection error with only a modest increase to the computational time.

The rest of the paper is organized as follows: In Section 2 we outline the problem under consideration, the linearized Euler equations in 2-D with no advection, solved in an infinite domain with NRBCs on all four sides. Section 3 details the NRBCs and their application to the linearized Euler equations. In Section 4 we derive the Klein-Gordon equation from the linearized Euler equations with no mean flow. We discuss the finite difference discretization for the NRBCs and the interior scheme in Sections 5 and 6, and we provide a numerical example in Section 7. We then list some areas for further research (Section 8) and summarize our results (Section 9).

2 Problem Statement

Consider the linearized Euler equations in an open domain. For simplicity we assume that the domain has a flat bottom and that there is no advection, although this assumption may be removed in future studies. A Cartesian coordinate system \((x, y)\) is introduced, as shown in Fig. 1.
The nonlinear Euler equations are

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) + \partial_y (\rho v) &= 0 \\
\partial_t u + u \partial_x u + v \partial_y u + \frac{1}{\rho} \partial_x p &= fv \\
\partial_t v + u \partial_x v + v \partial_y v + \frac{1}{\rho} \partial_y p &= -fu \\
\partial_t p + u \partial_x p + v \partial_y p + \gamma p(\partial_x u + \partial_y v) &= 0,
\end{align*}
\]

(1)

where we use the following shorthand for partial derivatives

\[
\begin{align*}
\partial_a &= \partial_a, \\
\partial_{ab} &= \frac{\partial^2}{\partial a \partial b},
\end{align*}
\]

and \( t \) denotes the time, \( u(x, y, t) \) and \( v(x, y, t) \) the unknown velocities in the \( x \) and \( y \) directions, \( \rho(x, y, t) \) the density, \( p(x, y, t) \) the pressure, \( f \) the constant Coriolis acceleration due to the Earth’s rotation, and \( \gamma = c_p/c_v \) the constant ratio of specific heats. Linearizing these equations about mean zero velocities, constant mean density \( \rho_0 \) and constant mean pressure \( p_0 \) (see, e.g., \([19]\) or \([22]\)), we get:

\[
\begin{align*}
\partial_t \rho + \rho_0 (\partial_x u + \partial_y v) &= 0 \\
\partial_t u + \frac{1}{\rho_0} \partial_x p &= fv \\
\partial_t v + \frac{1}{\rho_0} \partial_y p &= -fu \\
\partial_t p + \gamma p_0 (\partial_x u + \partial_y v) &= 0.
\end{align*}
\]

(2)

It can be shown that a single boundary condition must be imposed along the entire boundary to obtain a well-posed problem. At \( \vec{x} \to \infty \) the solution is known to be bounded and not to include any incoming waves. To complete the statement of the problem, initial values for \( u, v, p \) and \( \rho \) are given at time \( t = 0 \) in the entire domain.

We now truncate the infinite domain by introducing an artificial boundary \( \Gamma \), with \( \Gamma_N \) located at \( y = y_N \), \( \Gamma_W \) located at \( x = x_W \), \( \Gamma_S \) located at \( y = y_S \), and \( \Gamma_E \) located at \( x = x_E \) (see dotted lines in Figure 1). To obtain a well-posed problem in the finite domain \( \Omega \) we need, instead of the condition at infinity, a single boundary condition on each of the artificial boundaries \( \Gamma_{N,W,S,E} \). This should be a Non-Reflecting Boundary Condition (NRBC). We shall apply a high-order NRBC for the variables, as described in the following section.

### 3 Higdon-type NRBCs

On the artificial boundaries \( \Gamma \) we use one of the Higdon NRBCs [18]. These NRBCs were presented and analyzed in a sequence of papers [12], [14]-[17] for non-dispersive acoustic and elastic waves, and were extended in [18] for dispersive waves. Their main advantages are as follows:
1. The Higdon NRBCs are very general, namely they apply to a variety of wave problems, in one, two, and three dimensions and in various configurations.

2. They form a sequence of NRBCs of increasing order. This enables one, in principle (leaving implementational issues aside for the moment), to obtain solutions with unlimited accuracy.

3. The Higdon NRBCs can be used, without any difficulty, for dispersive wave problems and for problems in stratified media. Most other available NRBCs are either designed for non-dispersive media (as in acoustics and electromagnetics) or are of low order (as in meteorology and oceanography).

The scheme used here is different than the original Higdon scheme [18] in the following ways:

1. The discrete Higdon conditions were developed in the literature up to third order only, because of their algebraic complexity which increases rapidly with the order. Givoli and Neta [7] showed how to easily implement these conditions to an arbitrarily high order. The scheme is coded once and for all for any order; the order of the scheme is simply an input parameter.

2. The original Higdon conditions were applied to the Klein-Gordon linear wave equation and to the elastic equations. Here we show how to apply them to the linearized Euler equations (2).

3. The Higdon NRBCs involve some parameters which must be chosen. Higdon [18] discusses some general guidelines for their manual a priori choice by the user. Neta et. al. [24] showed how a simple choice for these parameters can dramatically simplify the calculations and enable implementation of NRBCs of much higher order with less computational overhead.

The Higdon NRBC of order $J$ is

$$H_J : \prod_{j=1}^{J} (\partial_t + C_j \partial_x) \eta = 0 \text{ on } \Gamma_E,$$  

where $\eta$ represents any one of the state variables $\rho, u, v, p$. Here, the $C_j$ are parameters which have to be chosen and which signify phase speeds in the $x$-direction. The boundary condition (3) is exact for all waves that propagate with an $x$-direction phase speed equal to any of $C_1 \ldots C_J$. This is easy to see from the reflection coefficient (see Givoli and Neta [7]). For the boundary $\Gamma_W$ we replace $\partial_x$ by $-\partial_x$. Likewise, on $\Gamma_N, S$ we use $\pm \partial_y$.

Givoli and Neta [7] and Dea et. al. [2] summarize several observations about these NRBCs, which we omit here for brevity.
4 Equivalence of Linearized Euler Equations and Klein-Gordon Equation

Higdon showed in [18] that this NRBC formulation is compatible with the Klein-Gordon (dispersive wave) equation
\[ \frac{\partial^2}{\partial t^2} \eta - C_0^2 \nabla^2 \eta + f^2 \eta = 0 , \]  
(4)

Hence, if we can show that (2) is equivalent to (4), we can claim that this NRBC formulation will be stable here.

Differentiate (2d) with respect to \( t \)
\[ \frac{\partial}{\partial t} p + \gamma p_0 (\partial_x u + \partial_y v) = 0 . \]  
(5)

Now differentiate (2b) with respect to \( x \) and (2c) with respect to \( y \) and add
\[ \partial_x u + \partial_y v + \frac{1}{\rho_0} (\partial_{xx} p + \partial_{yy} p) = f (\partial_x v - \partial_y u) . \]  
(6)

Now substitute (6) into (5)
\[ \frac{\partial}{\partial t} p - \frac{\gamma p_0}{\rho_0} (\partial_{xx} p + \partial_{yy} p) + f \gamma \rho_0 (\partial_x v - \partial_y u) = 0 . \]  
(7)

Differentiate (2b) with respect to \( y \) and (2c) with respect to \( x \) and subtract
\[ \partial_y u - \partial_x v = f (\partial_y v + \partial_x u) . \]  
(8)

Combine terms to get
\[ \partial_x u + \partial_y v = - \frac{1}{f} \partial_t (\partial_x v - \partial_y u) . \]  
(9)

Combine (2d) and (9) to get
\[ \partial_t p = \frac{\gamma p_0}{f} \partial_t (\partial_x v - \partial_y u) . \]  
(10)

Integrate (10) with respect to time to get
\[ f (p - p_0) = \gamma p_0 (\partial_x v - \partial_y u) . \]  
(11)

Finally, substitute (11) into (7)
\[ \frac{\partial}{\partial t} p - \frac{\gamma p_0}{\rho_0} \nabla^2 p + f^2 (p - p_0) = 0 , \]  
(12)

which gives us the Klein-Gordon equation for the pressure perturbation \( p - p_0 \) with wave speed \( \sqrt{\gamma p_0/\rho_0} \).
5 Discretization of NRBCs

The Higdon condition $H_J$ is a product of $J$ operators of the form $\partial_t + C_j \partial_x$. Consider the following finite difference approximations (see e.g. [26]):

$$\partial_t \approx \frac{I - S_t^-}{\delta t}, \quad \partial_x \approx \frac{I - S_x^-}{\delta x}.$$  \(13\)

In (13), $\delta t$ and $\delta x$ are, respectively, the time-step size and grid spacing in the $x$ direction, $I$ is the identity operator, and $S_t^-$ and $S_x^-$ are backward shift operators defined by

$$S_t^- \eta_{pq}^n = \eta_{pq}^{n-1}, \quad S_x^- \eta_{pq}^n = \eta_{pq}^{n-1}.$$  \(14\)

Here and elsewhere, $\eta_{pq}^n$ is the FD approximation of $\eta(x, y, t)$ at grid point $(x_p, y_q)$ and at time $t_n$. We use (13) in (3) to obtain:

$$\left[\prod_{j=1}^{J} \left( \frac{I - S_t^-}{\delta t} + C_j \frac{I - S_x^-}{\delta x} \right) \right] \eta_{Eq}^n = 0.$$  \(15\)

Here, the index $E$ corresponds to a grid point on the boundary $\Gamma_E$. On the other open boundaries, the normal derivatives and shift operators should be adjusted accordingly.

Givoli and Neta [7] showed how to implement the Higdon NRBCs to any order using a simple algorithm. Their algorithm requires the summation of $O(3^J)$ terms. However, if we make the simplification

$$C_j \equiv C_0 \quad \forall \ j \in 1 \ldots J,$$  \(16\)

then we can simplify the summation to

$$Z \equiv \left( aI + bS_t^- + cS_x^- \right)^J \eta_{Eq}^n$$

$$= \sum_{\beta=0}^{J} \sum_{\gamma=0}^{J} \frac{J!}{\alpha! \beta! \gamma!} a^\alpha b^\beta c^\gamma S_t^{-\beta} S_x^{-\gamma} \eta_{Eq}^n = 0,$$  \(17\)

where $a = 1 - c$

$b = -1$

$c = -C_0 \frac{\delta t}{\delta x}$

$\alpha = J - \beta - \gamma$.

This summation consists of only $O(J^2)$ terms, reducing the computational time considerably.

6 Discretization in the Interior

We consider explicit FD interior discretization schemes for the linearized Euler equations (2) to be used in conjunction with the $H_J$ condition. The interaction between the $H_J$ condition and the interior scheme is a source of concern, since simple choices for an explicit interior scheme turn out to
give rise to instabilities. The effort to contrive a compatible discretization scheme was described in [2] for the linearized Euler equations without Coriolis. There, we used a one-sided differencing scheme for the interior, such that the discretized system was equivalent to the standard second-order centered-difference scheme for the scalar wave equation in $p$, which Higdon proved in [18] was compatible with the NRBC formulation. However, subsequent work has shown that adding the Coriolis terms to this scheme results in a system which cannot be converted to the Klein-Gordon equation. Hence, another approach is needed.

Let us reconsider a second-order centered-difference scheme,

$$\eta'(a) \approx \frac{\eta(a + \delta a) - \eta(a - \delta a)}{2\delta a}.$$  \hspace{1cm} (18)

where $\eta$ denotes any of our four state variables, and $a$ denotes any of our spatial or temporal variables. Using the shift operator notation from the preceding section, we define our difference approximations as

$$\Delta_a = \frac{S_a^+ - S_a^-}{2\delta a} \hspace{1cm} (a \in \{x, y, t\}).$$  \hspace{1cm} (19)

From this definition, we propose the following discretization scheme for (2):

$$\Delta_t \rho + \rho_0 (\Delta_x u + \Delta_y v) = 0$$

$$\Delta_t u + \frac{1}{\rho_0} \Delta_x p = f v$$

$$\Delta_t v + \frac{1}{\rho_0} \Delta_y p = -f u$$

$$\Delta_t p + \gamma p_0 (\Delta_x u + \Delta_y v) = 0.$$  \hspace{1cm} (20)

Apply $\Delta_y$ to (20b), $\Delta_x$ to (20c), $\Delta_t$ to (20d), and make the appropriate substitution. This gives us

$$\Delta_t \Delta_x p = \frac{\gamma \rho_0}{\rho_0} (\Delta_x \Delta_x p + \Delta_y \Delta_y p) - f \gamma p_0 (\Delta_x v - \Delta_y u).$$  \hspace{1cm} (21)

If $f = 0$, then this discretization is equivalent to a scalar wave discretization. Hence, in the absence of Coriolis forces, the discretization scheme (20) is compatible with the discrete Higdon NRBCs (as modified below).

Continuing our derivation, we apply $\Delta_y$ to (20b) and $\Delta_x$ to (20c), then subtract and combine terms to get

$$\Delta_t (\Delta_y u - \Delta_x v) = f (\Delta_y v + \Delta_x u).$$  \hspace{1cm} (22)

We then substitute (20d) into this result to get

$$\Delta_t (\Delta_y u - \Delta_x v) = -\frac{f}{\gamma \rho_0} \Delta_p.$$  \hspace{1cm} (23)

If we apply $\Delta_t$ to (21) and incorporate (23), we get

$$\Delta_t \left[ \Delta_t \Delta_p - \frac{\gamma \rho_0}{\rho_0} (\Delta_x \Delta_x p + \Delta_y \Delta_y p) + f^2 p \right] = 0.$$  \hspace{1cm} (24)
Thus, the quantity inside the brackets is constant from one time step to
the next. Since this equation applies to our initial state, then the quantity
within the brackets must initially be zero and thus remain zero always;
hence,
\[ \Delta_t \Delta_x p = \frac{\gamma_{p_0}}{\rho_0} (\Delta_x \Delta_x p + \Delta_y \Delta_y p) - f^2 p . \] (25)
If we expand our \( \Delta_{[x,y,t]} \) symbols into their corresponding shift operators
and apply them to the state variable \( p \), we see that (25) is actually
\[ \frac{p_{i,j}^{n+2} - 2p_{i,j}^n + p_{i,j}^{n+2}}{(2\delta t)^2} = \frac{\gamma_{p_0}}{\rho_0} \left( \frac{p_{i+2,j}^n - 2p_{i,j}^n + p_{i-2,j}^n}{(2\delta x)^2} + \frac{p_{i,j+2}^n - 2p_{i,j}^n + p_{i,j-2}^n}{(2\delta y)^2} \right) - f^2 p_{i,j}^n . \] (26)
This equation is the standard second-order centered-difference scheme for
the Klein-Gordon equation on a double-sized grid. Hence, the appropriate
discretization for the Higdon scheme is not (13) but
\[ \partial_t \simeq \frac{I - S_t^{-2}}{2\delta t} , \quad \partial_x \simeq \frac{I - S_x^{-2}}{2\delta x} . \] (27)

7 Numerical Example

Let us consider a simple numerical example. We look at a square domain
10 km on each side, subdividing it into a 100 × 100 computational domain
with the Higdon-like NRBCs on all four sides (see Fig. 1). Using a mean
atmospheric density of 1.2 kg/m³ and pressure of 1.01 × 10⁵ N/m² [11], a Coriolis
value of \( f = 7.292116 \times 10^{-5} \text{rad/s} \) [27], and zero advection, our initial
condition is a cosine bubble in the center of the domain:

\[ p_{0}^{x,y} = \begin{cases} p_0 \left( 1 + \cos \left( \frac{\pi d}{100} \right) \right) & : \quad d \leq r \\ p_0 & : \quad \text{otherwise} \end{cases} \] (28)
\[ \rho_{0}^{x,y} = \begin{cases} \rho_0 \left( 1 + \cos \left( \frac{\pi d}{100} \right) \right) & : \quad d \leq r \\ \rho_0 & : \quad \text{otherwise,} \end{cases} \]

where
\[ d = \sqrt{(x-x_C)^2 + (y-y_C)^2} \]
\[ r = 1 \text{ km} , \]
and \( x_C \) and \( y_C \) denote the center of the domain. For comparison, our
reference solution domain is 30 km wide and 30 km high, with the domain
of interest in the center. We define the normalized error norm for each state variable \eta as

\[ E_\eta = \sqrt{\frac{\sum_{i=1}^{N_x} \sum_{j=1}^{N_y} (\eta_J(i,j) - \eta_0(i,j))^2}{N_x N_y \max(\eta_0)}}, \]  

(29)

where \( N_x, N_y \) are the number of grid points in the \( x \) and \( y \) directions, respectively, \( \eta_J \) is a solution state variable using the \( J \)-order NRBC, and \( \eta_0 \) is the reference solution. We use \( \max(\eta_0) \) to normalize the four state variables’ error norms to approximately the same order of magnitude. Our time step is computed by

\[ \delta t = \frac{\sqrt{\delta x^2 + \delta y^2}}{2C_0}, \]

(30)

which equals the CFL limit, thus guaranteeing stability. Using the discretization scheme (20), we run the simulation up to \( t = 24 \), long enough for the primary wave to exit the computational domain with the wave trough just passing through the corners. Figures 2–4 show the state variable \( u \) at the end of the run for \( J = 1, 6, \) and 10, respectively. Table 1 shows the error norms (29) for each state variable as \( J \) goes from 1 to 10.

8 Areas for Further Research

The preceding example demonstrates, in a limited setting, that high-order Higdon NRBCs are compatible with the linearized Euler equations. However, there are far more areas to explore in this implementation. The following list shows some of the areas available for future research, some of which are currently under investigation by the authors:
1. Thorough investigation of the long-time stability for large \( J \).

2. Extending the scheme to the case of the linearized Euler equations with a nonzero mean flow (advection).

3. Extending the scheme to include the effects of gravity (in the \( xz \) plane).

4. Implementing the scheme with auxiliary variables, using finite differences and finite elements, using both the Givoli-Neta AV formulation [9] and the Hagstrom-Warburton variation [10].

5. Extending the scheme to permit incoming waves, for example, in a nested mesoscale model.

6. Experimenting with the use of the NRBC with the nonlinear Euler equations (1) in the computational domain. (Need to find a stable interior scheme-NRBC combination.)

### 9 Conclusion

In this paper, we have shown that Higdon-type NRBCs are compatible with the linearized Euler equations with Coriolis and zero mean flow. These NRBCs provide greater accuracy (reduced spurious reflection) than the basic Sommerfeld or Orlanski boundary conditions. A prototypical implementation was developed, and a numerical example demonstrating the capabilities of the scheme was provided.

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References


Figure(s) 1
Figure 3

X-direction velocity perturbation

X-direction velocity perturbation, reference solution

Truncated reference solution

Solution delta

Error norm=0.00098845