# 00 – FIR Filtering Results Review

# & Practical Applications

# ❖ Mean Square Signal Estimation

*Distorted received signal*

$s$

*Transmitted signal*

$x \longrightarrow$ Optimal processor $\longrightarrow \widehat{d}$

*best estimate of transmitted signal s (as a function of received signal x)*

**Possible procedure:**

Mean square estimation, i.e.,

minimize $\xi = E\left\{ \left| s - \hat{d}(\underline{x}) \right|^2 \right\}$

leads to $\hat{d}(\underline{x}) = E[s \mid \underline{x}]$

(proof given in Appendix A)

- **conditional mean**!, usually nonlinear in $\underline{x}$ [exception when $\underline{x}$ and $s$ are jointly normal Gauss Markov theorem]
- Complicated to solve,
- Restriction to Linear Mean Square Estimator (LMS), estimator of $s$ is **forced** to be a linear function of measurements $\underline{x}$: $\rightarrow \hat{d} = \underline{h}^H \underline{x}$
- Solution via Wiener Hopf equations using orthogonality principle

## ❖ Orthogonality Principle

**Use LMS Criterion:** estimate $s$ by $\hat{d} = \underline{h}^H \underline{x}$
where weights $\{h_i\}$ minimize MS error:

$$\sigma_e^2 = E\left\{\left|\underline{s} - \hat{d}\left(\underline{x}\right)\right|^2\right\}$$

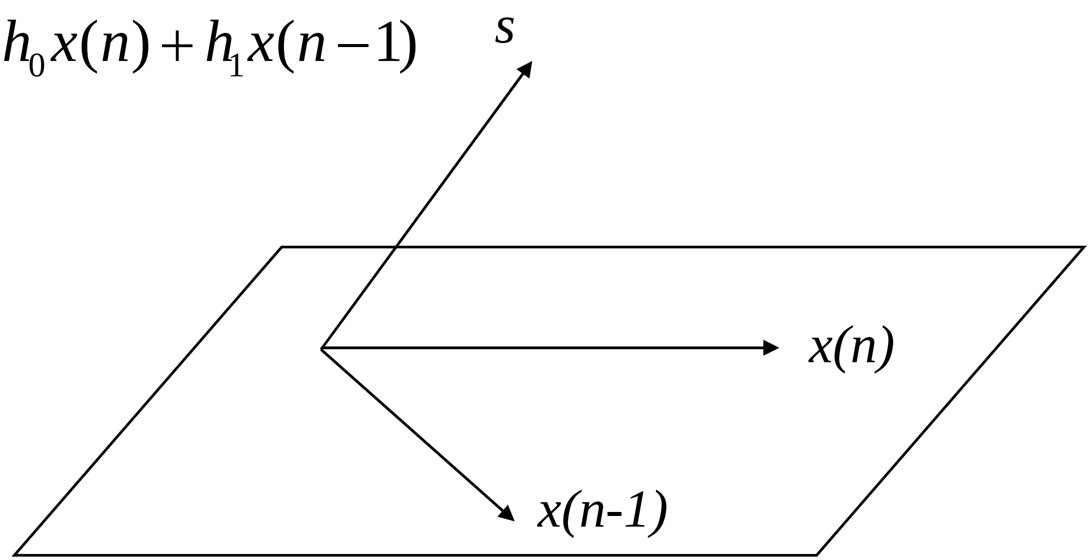
> *Theorem*: Let error $e = s - \hat{d}$
>
> $\underline{h}$ minimizes the MSE quantity $\sigma_e^2$ if $\underline{h}$ is chosen such that
>
> $$E\left\{e x_i^*\right\} = E\left\{x_i^* e\right\} = 0, \ \forall_i = 1, \cdots, N$$
>
> i.e., the error $e$ is orthogonal to the observations $x_i, i = 1 \ldots, N$ used to compute the filter output.

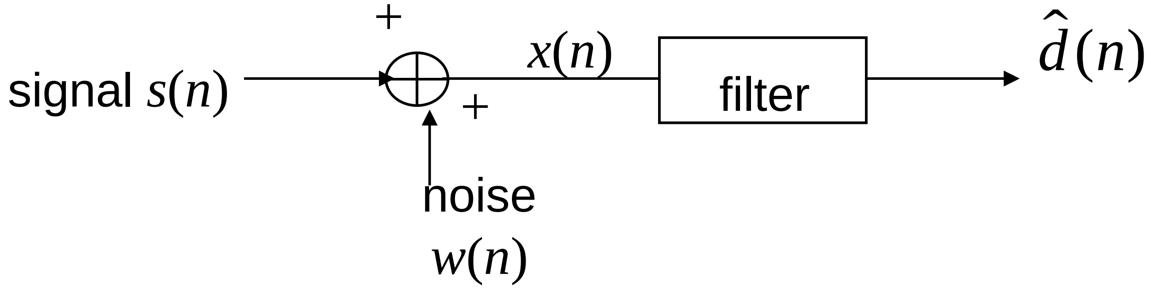
**Corollary:** minimum MSE obtained: $\sigma_{e_{\min}}^2 = E\left\{s e^*\right\}$ where $e$ is the minimum error obtained for the optimum filter vector.

(Proof given in Appendix B)

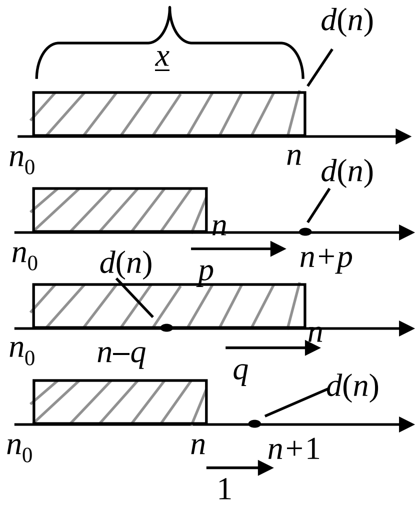
$$P = 2$$

$$\hat{d}(n) = h_0 x(n) + h_1 x(n-1)$$






## Typical Wiener Filtering Problems

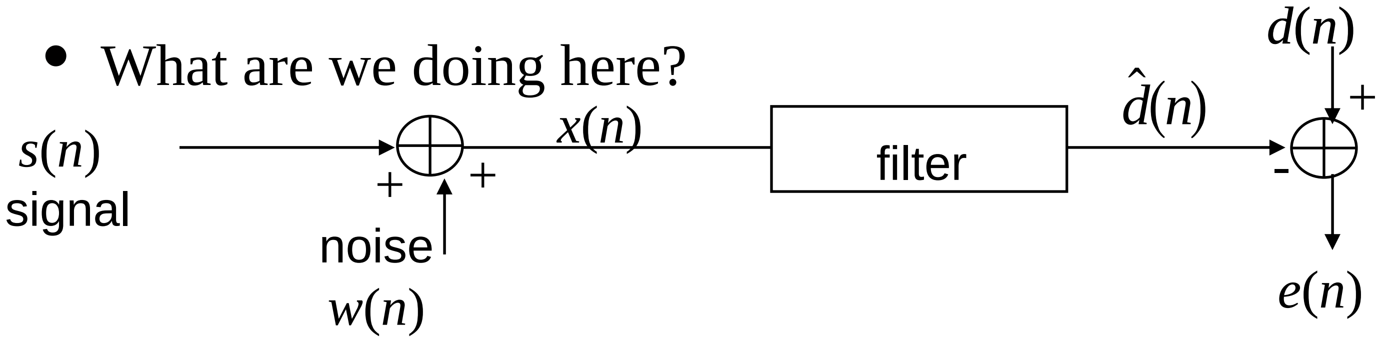



| **Problem** | **Form of Observations** | **Desired Signal** |
|---|---|---|
| Filtering of signal in noise | $x(n) = s(n)+w(n)$ | $d(n) = s(n)$ |
| Prediction of signal in noise | $x(n) = s(n)+w(n)$ | $d(n) = s(n+p)$; $p > 0$ |
| Smoothing of signal in noise | $x(n) = s(n)+w(n)$ | $d(n) = s(n-q)$; $q > 0$ |
| Linear prediction | $x(n) = s(n-1)$ | $d(n) = s(n)$ |

# ❖ FIR Wiener Filtering Concepts

- Filter criterion used: minimization of mean square error between $d(n)$ and $\hat{d}(n)$.

- What are we doing here?




We want to design a filter (in the generic sense can be: filter, smoother, predictor) so that:

$$\hat{d}(n) = \sum_{k=0}^{P-1} h^*(k)x(n-k)$$

How $d(n)$ is defined specifies the operation done:

- filtering: $d(n)=s(n)$
- predicting: $d(n)=s(n+p)$
- smoothing: $d(n)=s(n-p)$

## ❖ How to find $h_k$?

Minimize the MSE: $E\left\{\left|d(n)-\hat{d}(n)\right|^2\right\}$

$$\sum_{k=0}^{P-1} h_k^* x(n-k) = \underline{h}^H \underline{x}$$

$$\underline{h} = \left[h_0, \qquad h_{P-1}\right]^T, \quad \underline{x} = \left[x(n), \qquad x(n-P+1)\right]^T$$

**Wiener filter is a linear filter** $\Rightarrow$ orthogonality principle applies

$$\Rightarrow E\left\{x(n-i)e^*(n)\right\} = 0, \quad \forall i = 0,\ldots,P-1$$

$$E\left\{x(n-i)\left[d(n)-\sum_{k=0}^{P-1} h_k^* x(n-k)\right]^*\right\} = 0, \quad \forall i = 0,\ldots,P-1$$

$$\Rightarrow r_{xd}(-i) - \sum_{k=0}^{P-1} h_k^* R_x(k-i) = 0, \quad \forall i = 0,\ldots,P-1$$

$$r_{dx}^{\,*}(i)=\sum_{k=0}^{P-1} h_k R_x(k-i), \quad \forall i=0,\dots,P-1$$

Matrix form:

$$\begin{array}{c} i=0 \Rightarrow \\ i=1 \Rightarrow \\ \\ \\ \end{array} \begin{bmatrix} r_{dx}^{*}(0) \\ r_{dx}^{*}(1) \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} R_x(0) & R_x(1) & \cdots & R_x(P-1) \\ R_x(-1) & R_x(0) & \cdots & R_x(P-2) \\ \vdots & & & \\ R_x(-P+1) & & & R_x(0) \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{P-1} \end{bmatrix}$$

Note: different notation than in [Therrien, section 7.3]!

❖ **Minimum MSE** (MMSE)

obtained when $\underline{h}$ is obtained from solving WH equations.

For best $\underline{h}$ obtained:

$$\sigma_{e_{\min}}^2 = E\left\{\left|e_{\min}\right|^2\right\} = E\left\{(d(n)-\hat{d}(n))e_{\min}^*\right\}$$

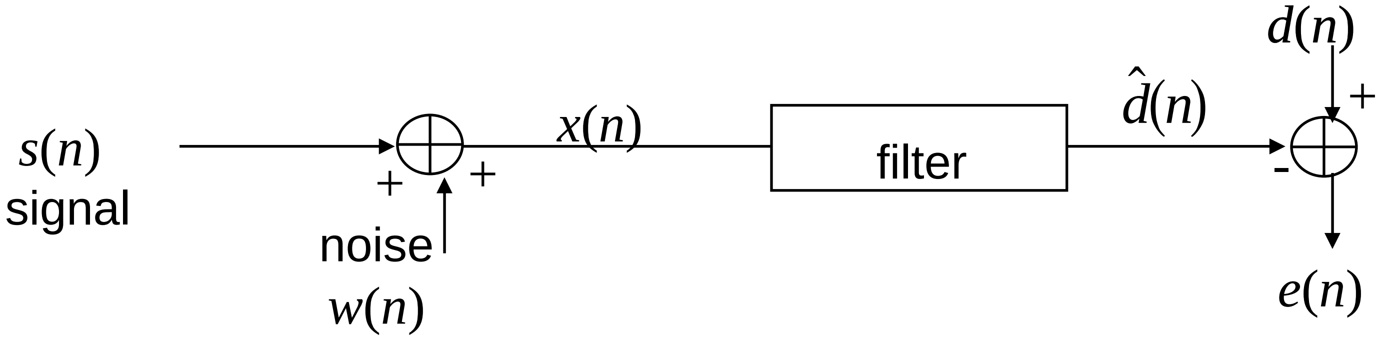
$$= E\left\{d\left(n\right)e_{\min}{}^*(n)\right\}$$

$$= E\left\{d\left(n\right)\left(d\left(n\right)-\sum_{k=0}^{P-1}h_{\mathrm{k}}^* \, x\left(n-k\right)\right)^*\right\}$$

$$= R_d\left(0\right)-\sum_{k=0}^{P-1}h_{\mathrm{k}} \, r_{dx}\left(k\right)$$

$$\boxed{\sigma_{e_{\min}}^2 = R_d\left(0\right)-\underline{h}^T \, \underline{r}_{dx}}$$

# Summary: FIR Wiener Filter Equations




- FIR Wiener filter is a FIR filter such that:

$$\hat{d}(n) = \sum_{k=0}^{P-1} h_k^* x(n-k)$$

where $\sigma_e^2 = E\left[\left|d(n) - \hat{d}(n)\right|^2\right]$ is minimum.

- How $d(n)$ is defined **specifies** the specific type of Wiener filter designed:

  filtering:

  smoothing:

  predicting:

- $\rightarrow$ W-H eqs:
- MMSE:

$$\begin{cases} R_x \underline{h} = \underline{r}_{dx}^* \Rightarrow \underline{h}_{opt} = R_x^{-1} \underline{r}_{dx}^* \\ \sigma_{e_{\min}}^2 = R_d(0) - \underline{h}_{opt}^T \underline{r}_{dx} = R_d(0) - \underline{r}_{dx}^T \underline{h}_{opt} \end{cases}$$

## ❖ One-step ahead Wiener predictor

- tracking of moving series
- forecasting of system behavior
- data compression
- telephone transmission

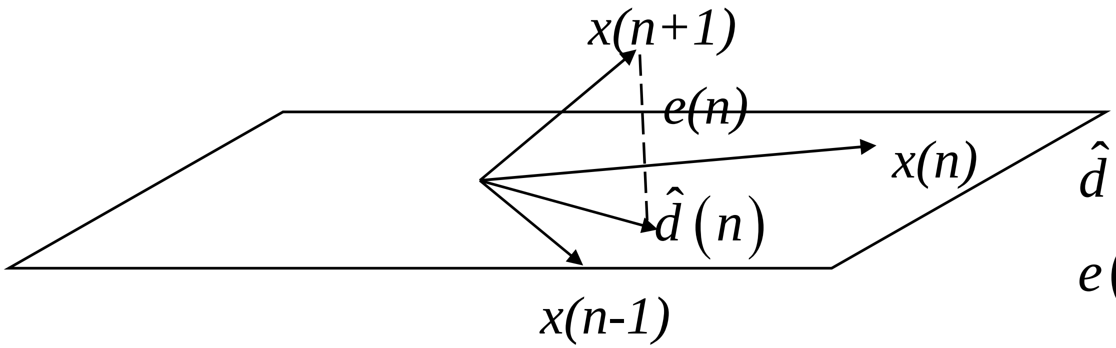
- <u>W-H equations</u>

$$\underline{h}_{\text{opt}} = R_x^{-1}\underline{r}_{dx}^{*}$$

$$\text{where} \quad \hat{d}(n) = \sum_{\ell=0}^{P-1} h_l^{*} x(n-\ell)$$

$$d(n) = \ ?$$

❖ **Wiener predictor geometric interpretation:** Assume a 1-step ahead predictor of length 2 (no additive noise)




$$\hat{d}(n) =$$

$$e(n) =$$

*e(n)* is the error between true value *x(n+1)* and predicted value for *x(n+1)* based on predictor inputs *x(n)* and *x(n-1)*

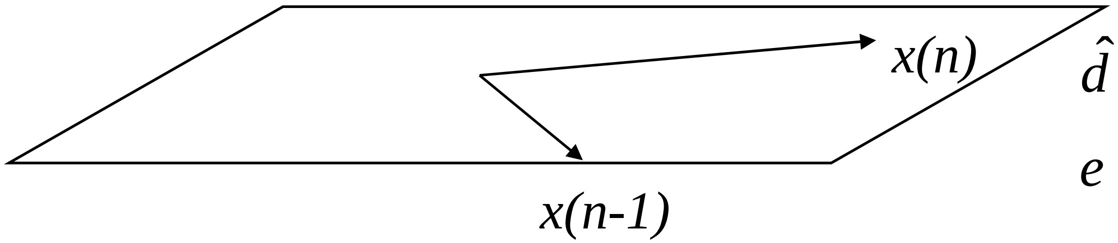
→ represents the new information in *x(n+1)* which is not already contained in *x(n)* or *x(n-1)*

→ *e(n)* is called the **innovation process** corresponding to *x*(*n*)

# Geometric interpretation, cont'

Assume *x(n+1)* only has **NO** new information (i.e., information in *x(n+1)* is that already contained in *x(n)* and *x(n-1)*. Filter of length 2.

Plot $x(n+1), \hat{d}\left(n\right), e\left(n\right)$



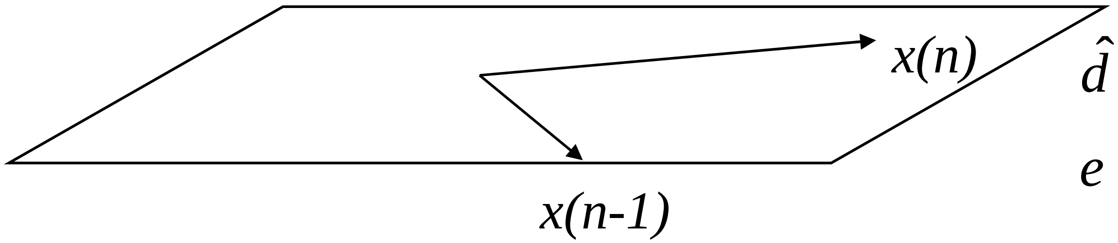

$\hat{d}\left(n\right) =$

$e\left(n\right) =$

# Geometric interpretation, cont'

Assume *x(n+1)* only has new information (i.e., information in *x(n+1)* is that **NOT** already contained in *x(n)* and *x(n-1)*. Filter of length 2.

Plot $x(n+1), \hat{d}(n), e(n)$

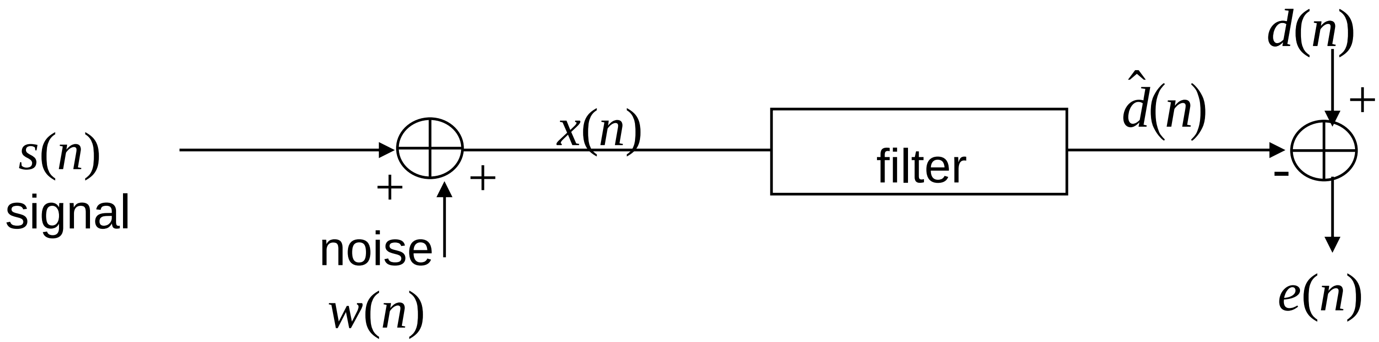



$\hat{d}(n) =$

$e(n) =$

## ❖ Example 1: Wiener filter (filter case: $d(n) = s(n)$ & white noise)

Assume $x(n)$ is defined by




$s(n)$, $w(n)$ uncorrelated

$w(n)$ white noise, zero mean $\qquad R_w(n) = 2\delta(n)$

$s(n)$ $\qquad R_s(n) = 2\,(0.8)^{|n|}$

| Filter length | Filter coefficients | MMSE |
|---|---|---|
| 2 | [0.405, 0.238] | 0.81 |
| 3 | [0.382, 0.2, 0.118] | 0.76 |
| 4 | [0.377, 0.191, 0.01, 0.06] | 0.7537 |
| 5 | [0.375, 0.188, 0.095, 0.049, 0.029] | 0.7509 |
| 6 | [0.3751, 0.1877, 0.0941, 0.0476, 0.0249, 0.0146] | 0.7502 |
| 7 | [0.3750, 0.1875, 0.0938, 0.0471, 0.0238, 0.0125, 0.0073] | 0.7501 |
| 8 | [0.3750, 0.1875, 0.038, 0.049, 0.0235, 0.0119, 0.0062, 0.0037 | 0.75 |

## ❖ Example 2: Application to Wiener filter (filter case: $d(n) = s(n)$ & colored noise)

$s(n)$, $w(n)$ uncorrelated, and zero-mean

$w(n)$ noise with $R_w(n) = 2\,(0.5)^{|n|}$

$s(n)$ signal with $R_s(n) = 2\,(0.8)^{|n|}$

| Filter length | Filter coefficients | MMSE |
|---|---|---|
| 2 | [0.4156, 0.1299] | 0.961 |
| 3 | [0.4122, 0.0750, 0.0878] | 0.9432 |
| 4 | [0.4106, 0.0737, 0.0508 0.0595] | 0.9351 |
| 5 | [0.4099, 0.0730, 0.0499, 0.0344, 0.0403] | 0.9314 |
| 6 | [0.4095, 0.0728, 0.0495, 0.0338, 0.0233, 0.0273] | 0.9297 |
| 7 | [0.4094, 0.0726, 0.0493, 0.0335, 0.0229, 0.0158, 0.0185] | 0.9289 |
| 8 | [0.4093, 0.0726, 0.0492, 0.0334, 0.0227, 0.0155, 0.0107, 0.0125] | 0.9285 |

## ❖ Example 3: 1-step ahead predictor

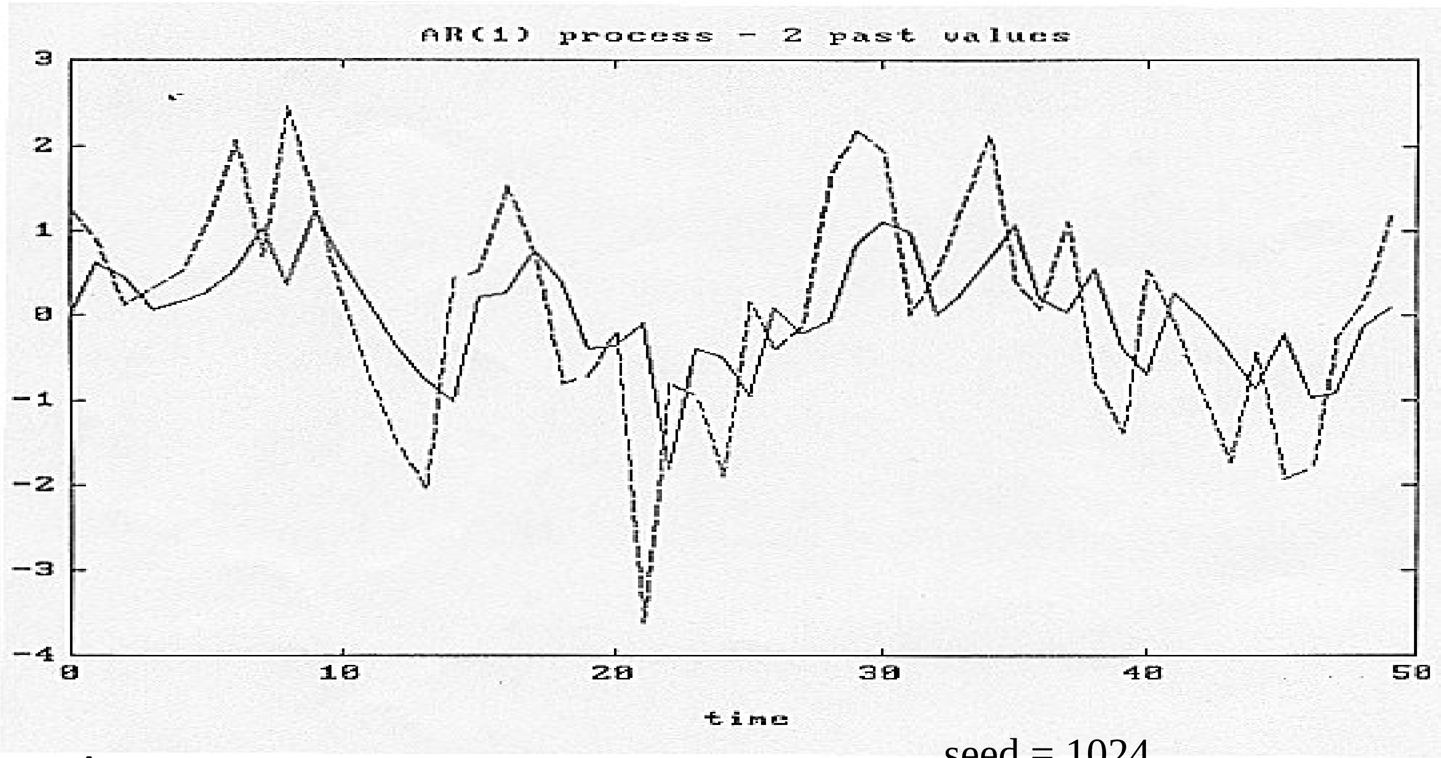
- - - - - - - AR (1) process

——— Predictor

$a = 0.5$

RP $x(n)$ defined as $x(n) = x(n-1) + v(n)$ $\quad |a| < 1$

$v(n)$ is white noise. 1-step predictor of length 2.




seed = 1024

$$\hat{x}(n) = +a_1 x(n-1) + a_2 x(n-2)$$

$$a = \begin{bmatrix} +0.5 \\ 0 \end{bmatrix}$$

## ❖ Link between Predictor behavior & input signal behavior

***1) Case 1****:* $s(n)$ = process with correlation

$$R_s(k) = \delta(k) + 0.5\delta(k-1) + 0.5\delta(k+1)$$

Investigate performances of N-step predictor as a function of changes N

***2) Case 2****:* $s(n)$ = process with correlation

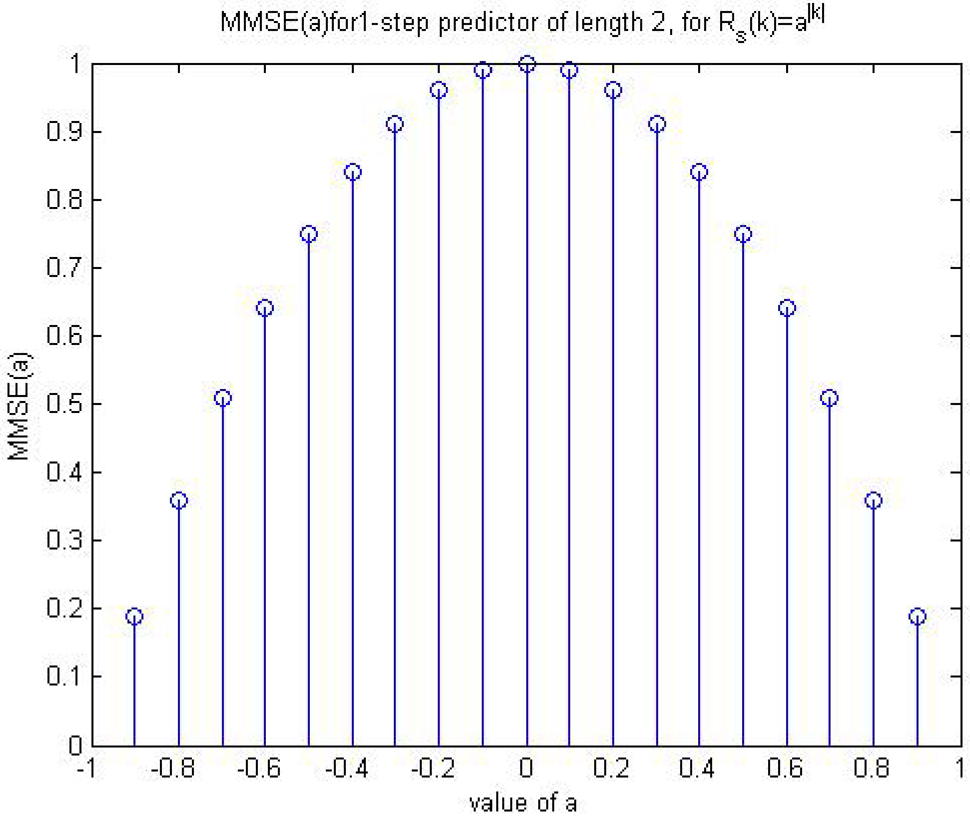
$$R_s(k) = a^{|k|}, |a| < 1$$

Investigate performances of predictor as a function of changes in *a*

***1) Case 1****:* $s(n)$ = wss process with

$$R_s(k) = \delta(k) + 0.5\delta(k-1) + 0.5\delta(k+1)$$

## *2) Case 2:* $s(n)$ = wss process with $R_s(k) = a^{|k|}, |a| < 1$




```
% EC3410 -  MPF
% Compute FIR filter coefficients for
% a 1-step ahead predictor of length 2
% for correlation sequence of type R(k)=a^{|k|}
% Compute and plot resulting MMSE value
A=[-0.9:0.1:0.9];
for k0=1:length(A)
   a=A(k0);
for k=1:3
   rs(k)=a^(k-1);
end
Rs=toeplitz(rs(1:2));
rdx=[rs(2);rs(3)];
h(:,k0)=Rs\rdx;
mmse(k0)=rs(1)-h(:,k0)'*rdx;
end
stem(A,mmse)
xlabel('value of a')
ylabel('MMSE(a)')
title('MMSE(a)for1-step predictor of length 2, …
        for R_s(k)=a^{|k|}')
```

❖ **Example 4:**

$$R_s(n) = 2(0.8)^{|n|}$$

$$R_w(n) = 2\delta(n)$$

$s(n)$ = process with

$w(n)$ = white noise, zero mean

$s(n)$, $w(n)$ uncorrelated

Design the 1-step ahead predictor of length 2.
Compute MMSE.

| 1- step ahead Predictor | | | Filter |
|---|---|---|---|
| **Length** | **Coefficients** | **MMSE** | **Filter MMSE** |
| 2 | [0.3238, 0.1905] | 1.2381 | 0.81 |
| 3 | [0.3059, 01.6, 0.0941] | 1.2094 | 0.76 |
| 4 | [0.3015, 0.1525, 0.0798, 0.0469] | 1.2023 | 0.7537 |
| 5 | [0.3004, 0.1506, 0.0762, 0.0762, 0.04, 0.0234] | 1.2006 | 0.7509 |
| 6 | [0.3001, 0.1502, 0.0753, 0.0381, 0.0199, 0.0199] | 1.2001 | 0.7502 |
| 7 | [0.3, 0.15, 0.0751, 0.0376, 0.0190, 0.001, 0.0059] | 1.2 | 0.7501 |
| 8 | [0.3, 0.15, 0.075, 0.0375, 0.0188, 0.0095, 0.0050, 0.003] | 1.2 | 0.75 |

| N-step ahead Predictor | | | Filter |
|---|---|---|---|
| **Length** | **1-step ahead MMSE** | **2-step ahead MMSE** | **Filter MMSE** |
| 2 | 1.2381 | 1.5124 | 0.81 |
| 3 | 1.2094 | 1.494 | 0.76 |
| 4 | 1.2023 | 1.4895 | 0.7537 |
| 5 | 1.2006 | 1.4884 | 0.7509 |
| 6 | 1.2001 | 1.4881 | 0.7502 |
| 7 | 1.2 | 1.4880 | 0.7501 |
| 8 | 1.2 | 1.4880 | 0.75 |

## ❖ **Example 5:**

$$R_s(n) = 2(0.8)^{|n|}$$

$$R_w(n) = 2(0.5)^{|n|}$$

$s(n)$ = process with

$w(n)$ = wss noise, zero mean

$s(n)$, $w(n)$ uncorrelated

- Design the 1-step ahead predictor of length 2
- Design 1-step back smoother of length 2

| 1-step ahead predictor (Col. Noise) | | |
|---|---|---|
| **Length** | **Coefficients** | **MMSE** |
| 2 | [0.3325, 0.1039] | 1.3351 |
| 3 | [0.3297, 0.06, 0.0703] | 1.3237 |
| 4 | [0.3285, 0.0589, 0.0406, 0.0476] | 1.3185 |
| 5 | [0.3279, 0.0584, 0.04, 0.0275, 0.0322] | 1.3161 |
| 6 | [0.3276, 0.0582, 0.0396, 0.0270, 0.0186, 0.0218] | 1.315 |
| 7 | [0.3275, 0.0581, 0.0394, 0.0268, 0.0183, 0.0126, 0.0148] | 1.3145 |
| 8 | [0.3275, 0.0581, 0.0394, 0.0267, 0.018, 0.0124, 0.0085, 0.0100] | 1.3142 |

| Length | MMSE (Col. Noise) | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | N-step ahead Predictor | | | | N-step back Smoother | | | | Filter |
| | 1-step | 2-step | 3-step | 4-step | 1-step | 2-step | 3-step | 4-step | |
| 2 | 1.3351 | 1.5744 | 1.7276 | 1.8257 | 0.961 | 1.3351 | 1.5744 | 1.7276 | 0.961 |
| 3 | 1.3237 | 1.5672 | 1.7230 | 1.8227 | 0.925 | 0.9432 | 1.3237 | 1.5672 | 0.9432 |
| 4 | 1.3185 | 1.5638 | 1.7208 | 1.8213 | 0.9085 | 0.9085 | 0.9351 | 1.3185 | 0.9351 |
| 5 | 1.3161 | 1.5623 | 1.7199 | 1.8207 | 0.9009 | 0.8926 | 0.9009 | 0.9314 | 0.9314 |
| 6 | 1.315 | 1.5616 | 1.7194 | 1.8204 | 0.8975 | 0.8853 | 0.8853 | 0.8975 | 0.9297 |
| 7 | 1.3145 | 1.5613 | 1.7192 | 1.8203 | 0.8959 | 0.8819 | 0.8781 | 0.8819 | 0.9289 |
| 8 | 1.3142 | 1.5611 | 1.7191 | 1.8202 | 0.8952 | 0.8804 | 0.8748 | 0.8748 | 0.9285 |

# Comments

## ❖ Example 6:

$s(n)$ and $w(n)$ defined as before

with $w(n)$ zero mean,

and $s(n)$ and $w(n)$ uncorrelated

$$R_s(n) = 2(0.8)^{|n|}$$

$$R_w(n) = 2(0.5)^{|n|}$$

Design the 3-step ahead predictor of length 2, and associated MMSE

## ❖ Wiener Filters and Error Surfaces

Recall $\underline{h}_{\text{opt}}$ computed from

$$\boxed{R_x \, \underline{h}_{\text{opt}} = \underline{r}^*_{dx}}$$

$$\sigma_e^2 = E\left\{\left|d(n) - \underline{h}^H \underline{x}\right|^2\right\} = E\left\{\left(d(n) - \underline{h}^H \underline{x}\right)\left(d(n) - \underline{h}^H \underline{x}\right)^*\right\}$$

$$= R_d(0) + \underline{h}^H E\left\{\underline{x}\underline{x}^H\right\}\underline{h} - 2\,\text{Real}\left(\underline{h}^T \underline{r}_{dx}\right)$$

↳ for real signals $d(n)$, $x(n)$

$$\boxed{\sigma_e^2 = R_d(0) + \underline{h}^T R_x \underline{h} - 2\underline{h}^T \underline{r}_{dx}}$$

using the fact that

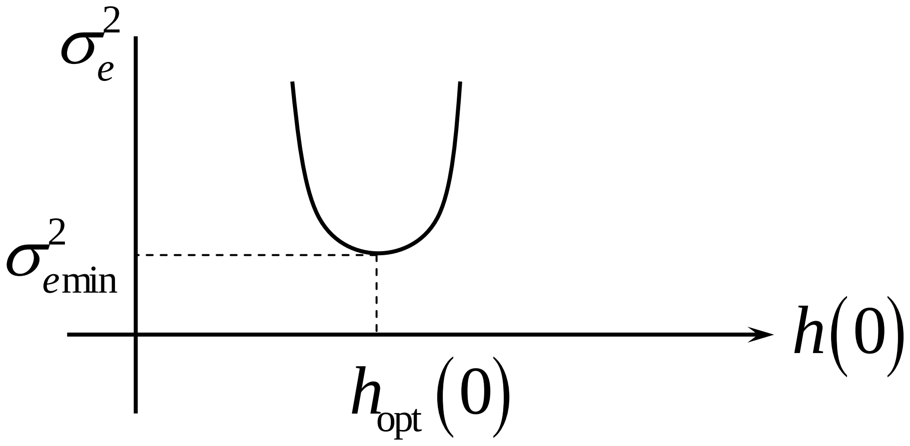
$$\left(\underline{h}^H \underline{x}\right)^* = \underline{x}^H \underline{h}$$

$$d(n)\,\underline{h}^H \underline{x} = \underline{h}^H d(n)\,\underline{x}$$

$$\boxed{\sigma_e^2 = R_d(0) + \underline{h}^T R_x \underline{h} - 2\underline{h}^T \underline{r}_{dx}}$$

- <u>For filter length $P = 1$</u> $\underline{h} = h_0$ $\underline{x} = x(n)$

$$\longrightarrow \sigma_e^2 = R_d(0) + h(0)^2 R_x(0) - 2h(0) r_{dx}(0)$$



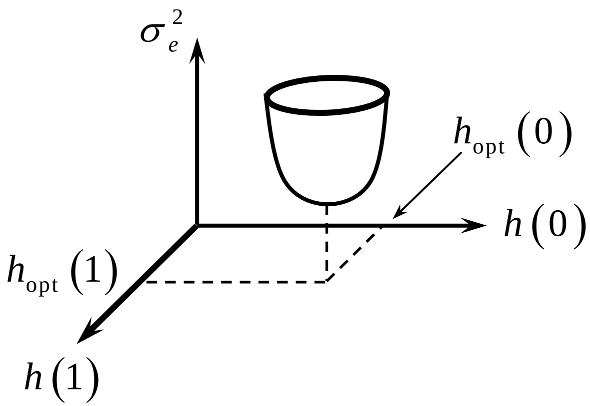
$$\boxed{\sigma_e^2 = R_d(0) + \underline{h}^T R_x \underline{h} - 2\underline{h}^T \underline{r}_{dx}}$$

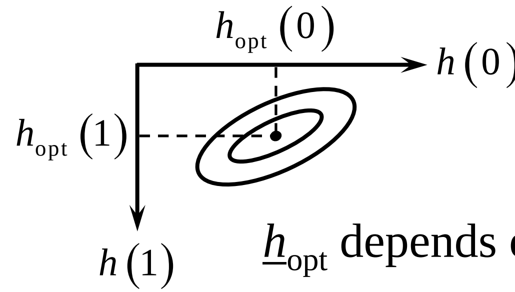
- For filter length $P = 2$

$$\underline{h} = \left[h(0), h(1)\right]^T; \;\; \underline{x} = \left[x(n), x(n-1)\right]^T$$

$$\begin{aligned}
\sigma_e^2 &= R_d(0) + \underline{h}^T R_x \underline{h} - 2\underline{h}^T \underline{r}_{dx} \\
&= R_d(0) + \left[h(0), h(1)\right] \begin{bmatrix} R_x(0) & R_x(1) \\ R_x(1) & R_x(0) \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \end{bmatrix} \\
&\qquad - 2\left[h(0), h(1)\right] \begin{bmatrix} r_{dx}(0) \\ r_{dx}(1) \end{bmatrix}
\end{aligned}$$

$$\Rightarrow \;\; \sigma_e^2 = A_0 h(0)^2 + A_1 h(1)^2 + A_2 h(1) + A_3 h(0) + A_4 h(0) h(1) + R_d(0)$$




$\underline{h}_{\text{opt}}$ depends on $\underline{r}_{dx}$ and $R_x$

$$\sigma_e^2 = R_d(0) + \underline{h}^T R_x \underline{h} - 2\underline{h}^T \underline{r}_{dx}$$

shape of $\sigma_e^2$ depends on $R_x$ information: $\lambda(R_x)$ & eigenvectors

moves $\sigma_e^2$ up and down

$$\frac{\partial \sigma_e}{\partial \underline{h}} = 2R_x\underline{h} - 2\underline{r}_{dx}$$

$$\downarrow$$

$$R_x \underline{h} = \underline{r}_{dx}$$

$R_x$: specifies shape of $\sigma_e^2(\underline{h})$

$\underline{r}_{dx}$: specifies where the bowl is in the 3-d plane but doesn't change the shape of the bowl

$R_d(0)$: moves bowl up and down in 3-d plane but doesn't change shape or location of bowl

## ❑Correlation matrix Eigenvalue Spread Impact on Error Surface Shape

$\longrightarrow$ see plots

Eigenvector Direction for 2 × 2 Toeplitz Correlation Matrix

$$R_x = \begin{bmatrix} R_x(0) & R_x(1) \\ R_x(1) & R_x(0) \end{bmatrix} \rightarrow \begin{matrix} \text{normalize} \\ \text{correlation} \end{matrix} \quad \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}$$

eigenvalues of $R_x$

$$(1-\lambda)^2 - a^2 = 0 \quad \Rightarrow \quad \lambda = \begin{cases} 1-a \\ 1+a \end{cases}$$

eigenvectors

$$\begin{pmatrix} 1-\lambda & a \\ a & 1-\lambda \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix} = 0$$

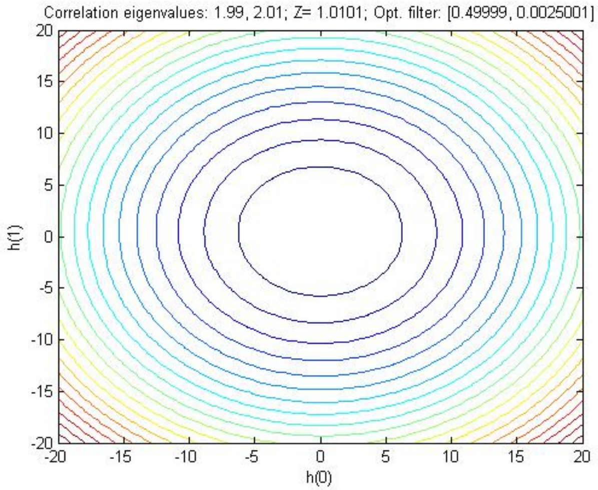
$$\Rightarrow \quad (1-\lambda)u_{11} + a u_{12} = 0$$

$$\lambda_1 = 1-a \quad \Rightarrow \quad (\cancel{1} - \cancel{1} + a)u_{11} + a u_{12} = 0$$

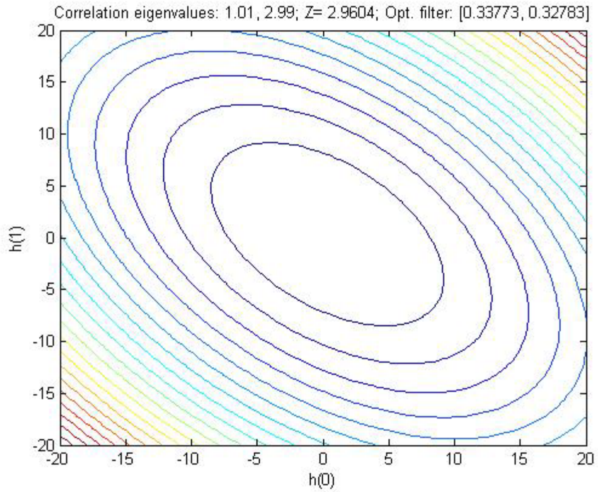
$$\Rightarrow \quad u_{11} = -u_{12} \quad \Rightarrow \underline{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 1+a \quad \Rightarrow \quad \underline{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

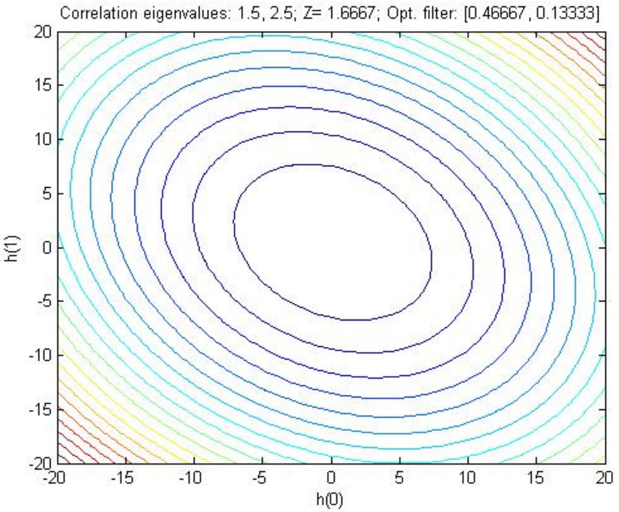
# ❖ Error surface shape and eigenvalue ratios




*a=0.1*

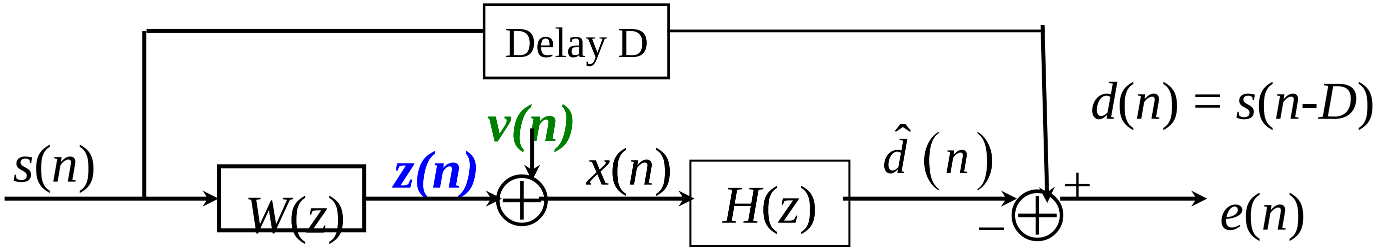



*a=0.99*




*a=0.5*

# ❖ Application to Channel Equalization




Goal: Implement the equalization filter H(z) as a stable causal FIR filter

- Goal: Recover *s(n)* by estimating channel distortion (applications in communications, control, etc.)
- Information Available:

$$x(n) = z(n) + v(n)$$

*channel output* — *additive noise due to sensors*

$$d(n) = s(n-D)$$

*s(n):* original data samples

- Assumptions: 1) $v(n)$ is stationary, zero-mean, uncorrelated with $s(n)$.

  2) $v(n) = 0$ & D=0

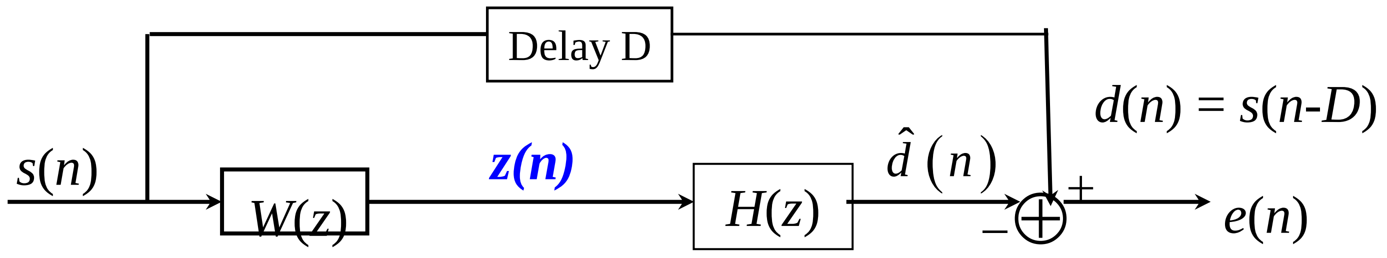
$$\text{Assume: } W(z) = 0.2798 + z^{-1} + 0.2798z^{-2}$$

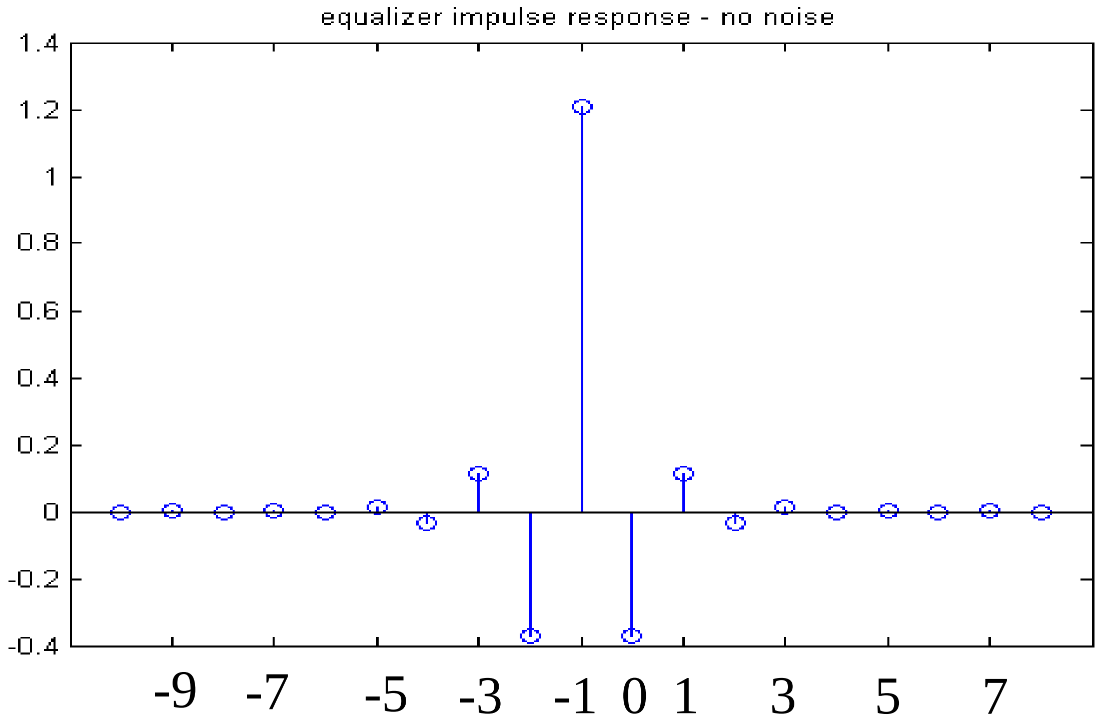
**Questions:**

1) Assume $v(n) = 0$ & D=0. Identify the type of filter (FIR/IIR) needed to cancel channel distortions. Identify resulting $H(z)$

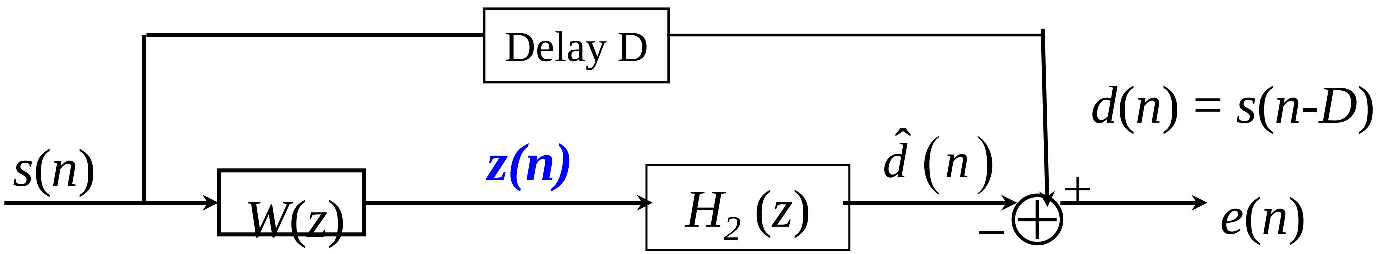
2) Identify whether the equalization filter is causal and stable.

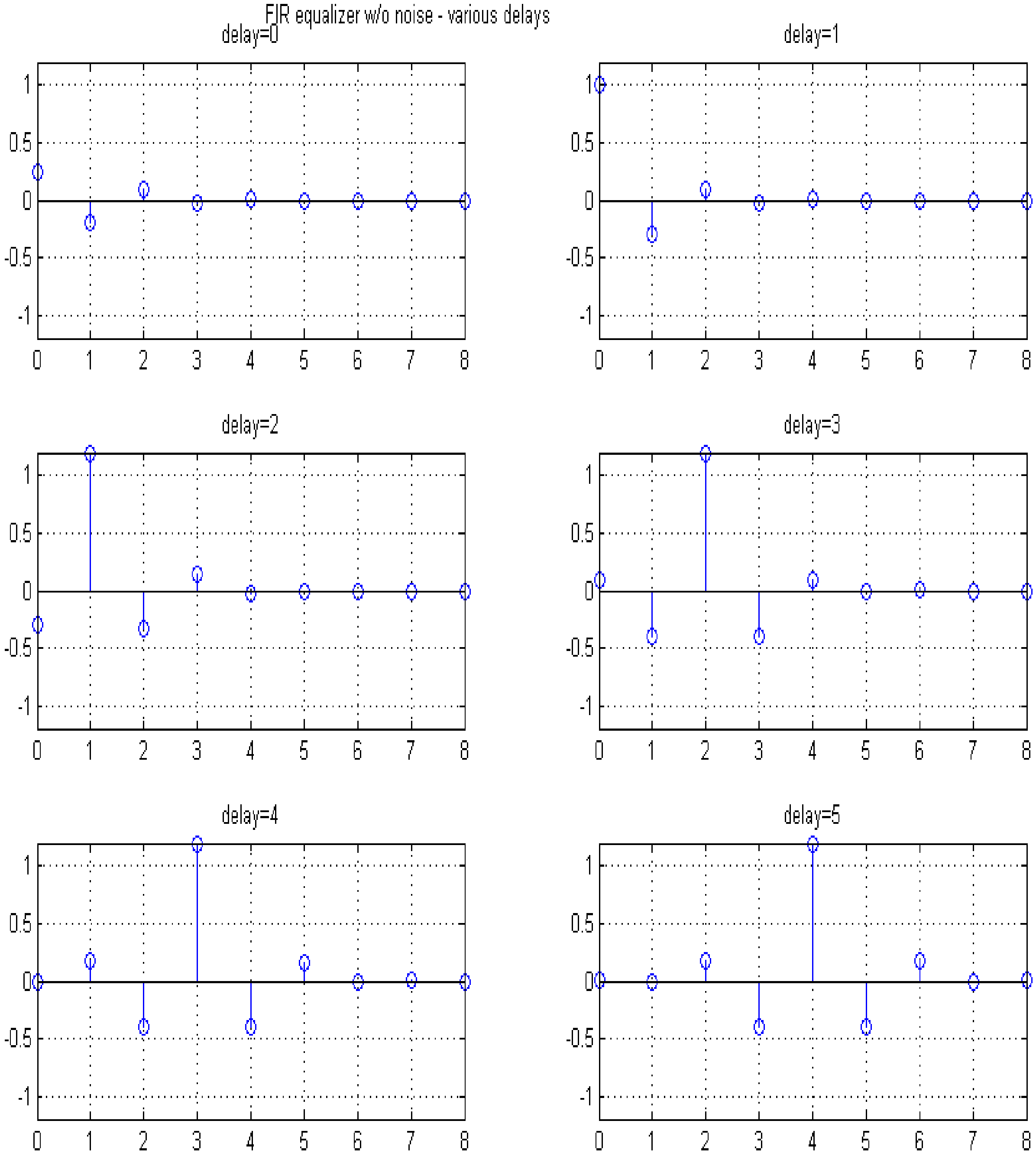
3) Assume $v(n) = 0$ & D≠0. Identify resulting $H_2(z)$ in terms of $H(z)$.

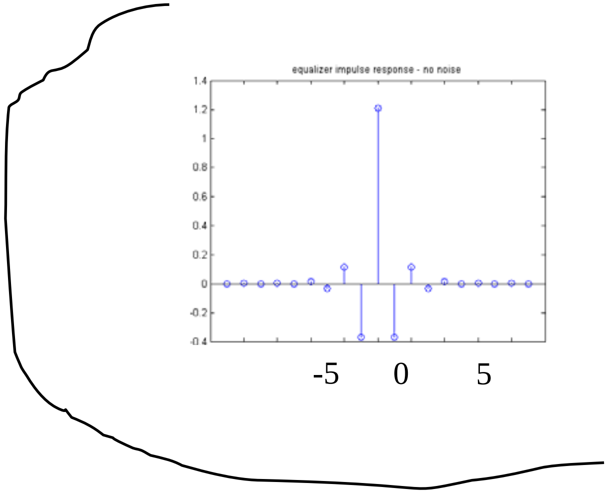



equalizer impulse response - no noise
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-9
-7
-5
-3
-1
0
1
3
5
7

Assume D≠0

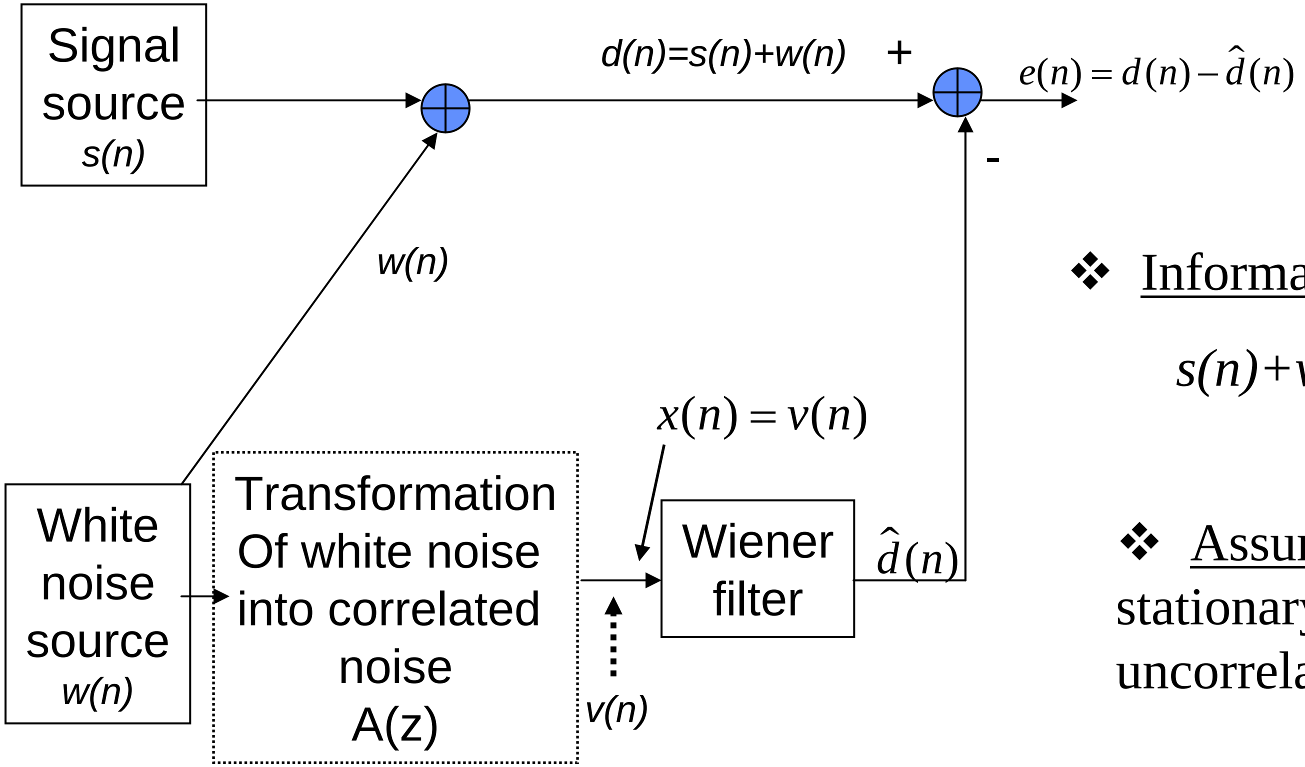



FIR equalizer w/o noise - various delays
delay=0
delay=1
delay=2
delay=3
delay=4
delay=5


equalizer impulse response - no noise
-5
0
5

# ❖ Application to Noise Cancellation

❑ Goal: Recover *s(n)* by compensating for the noise distortion while having only access to the related noise distortion signal *v(n)* (applications in communications, control, etc.)




❖ Information Available:

*s(n)+w(n)* & *v(n)*

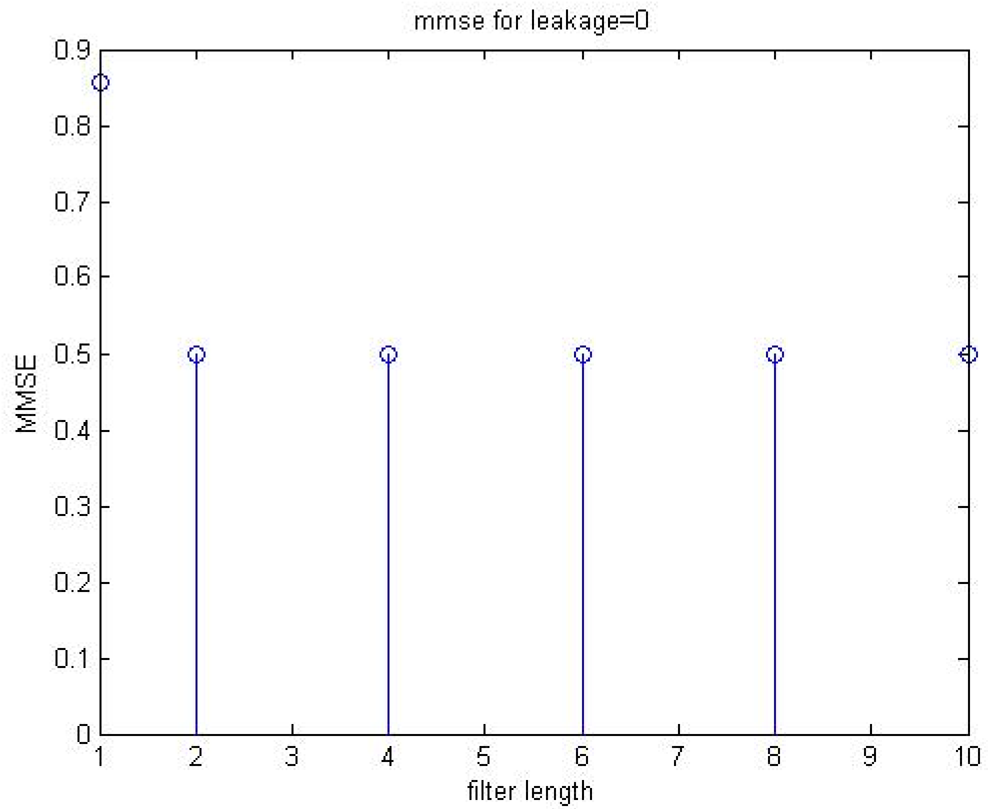
❖ Assumption: $w(n)$ is stationary, zero-mean, uncorrelated with $s(n)$.

Assume: $v(n) = av(n-1) + w(n), \quad a = 0.6,$

$w(n)$ white noise with variance $\sigma_w^2$

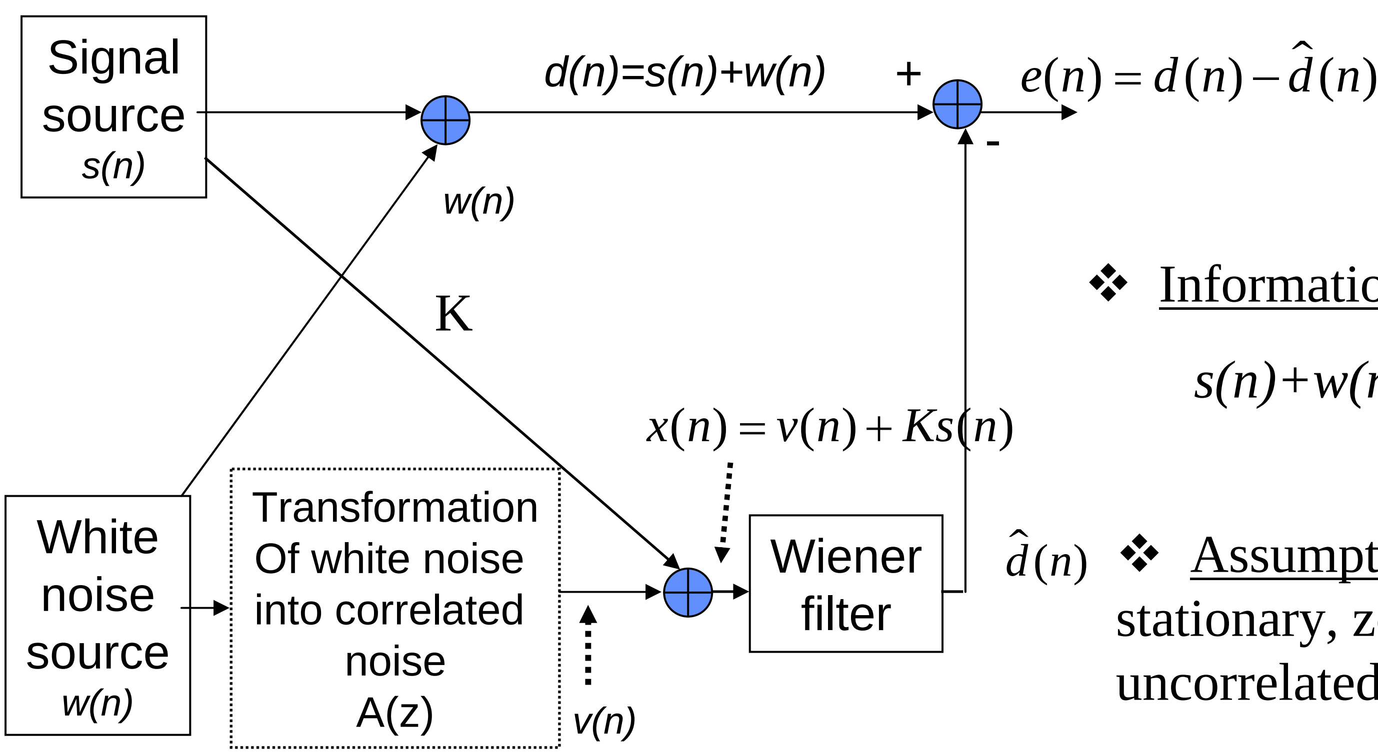
$s(n) = \sin(\omega_0 n + \phi), \quad \phi \sim U[0, 2\pi]$

Compute the FIR Wiener filter of length 2 and evaluate filter performances




# Results Interpretation

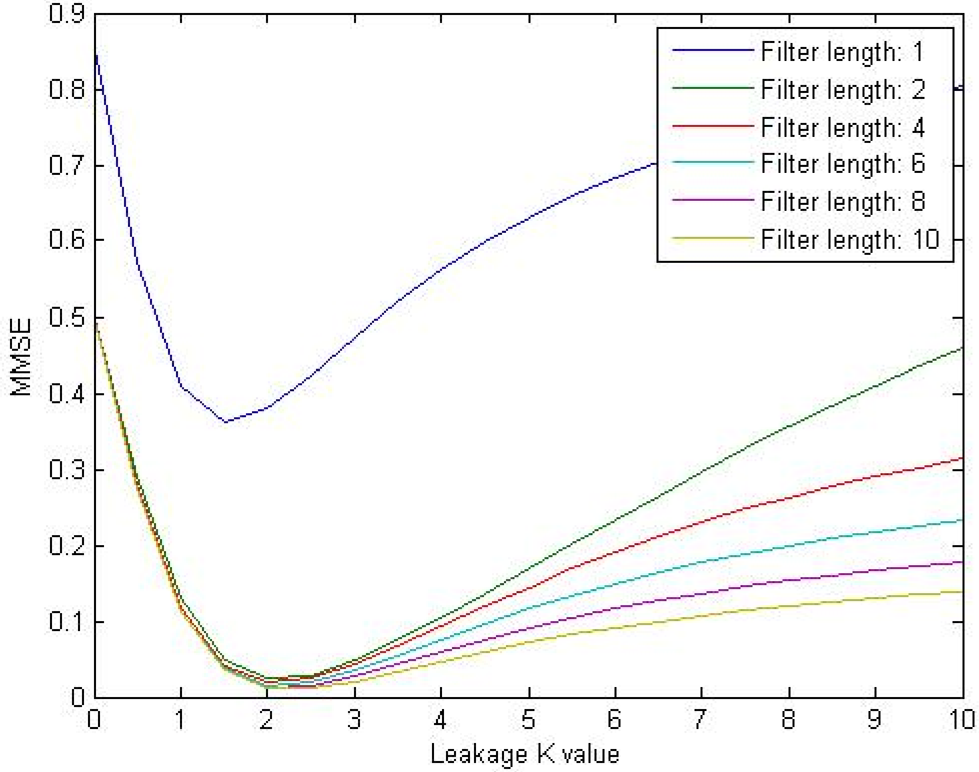
# ❖ Application to Noise Cancellation (with information leakage)




❖ Information Available:

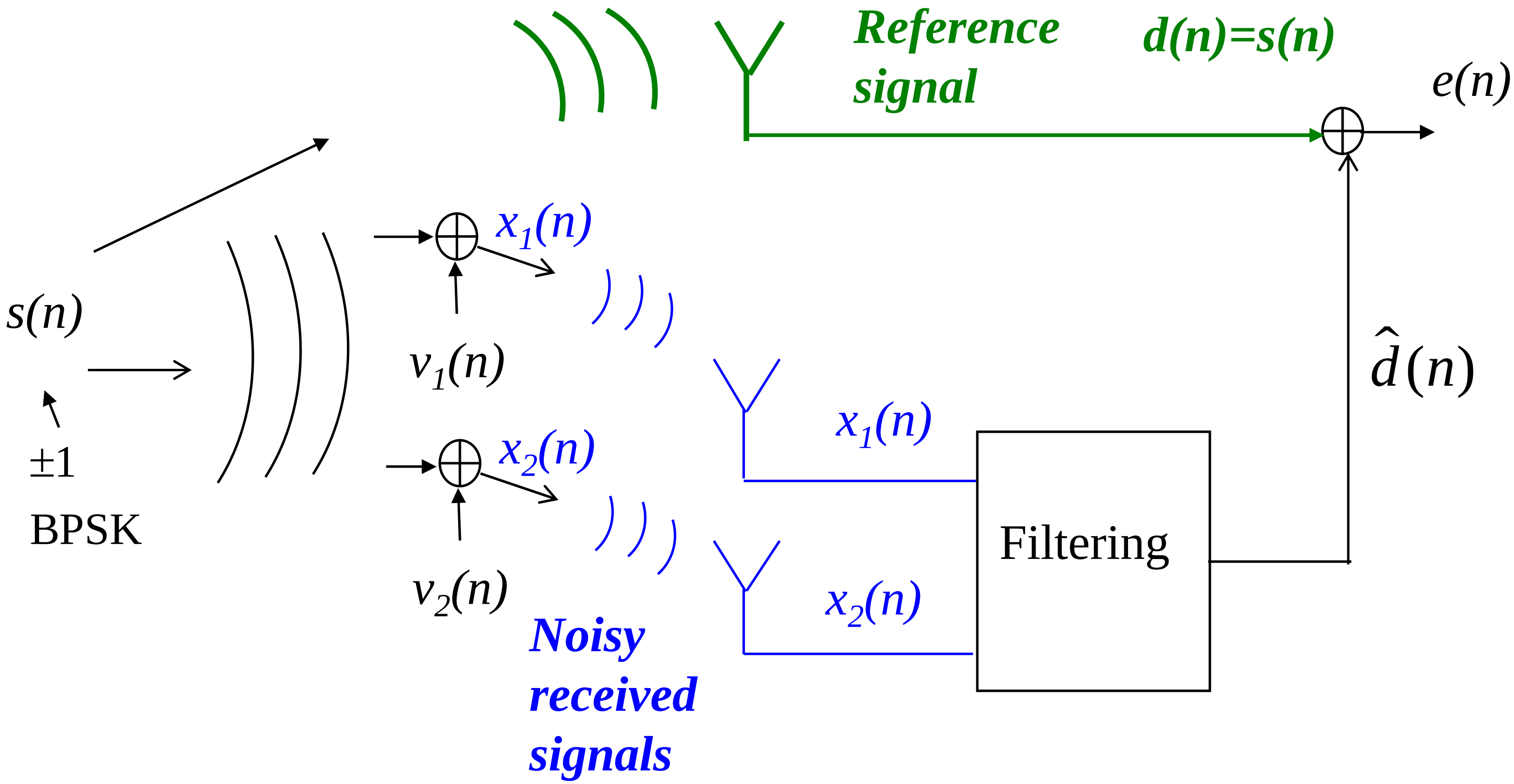
*s(n)+w(n)* & *v(n)+Ks(n)*

❖ Assumption: $w(n)$ is stationary, zero-mean, uncorrelated with $s(n)$.




# Results Interpretation

## ❖ Application to Spatial Filtering



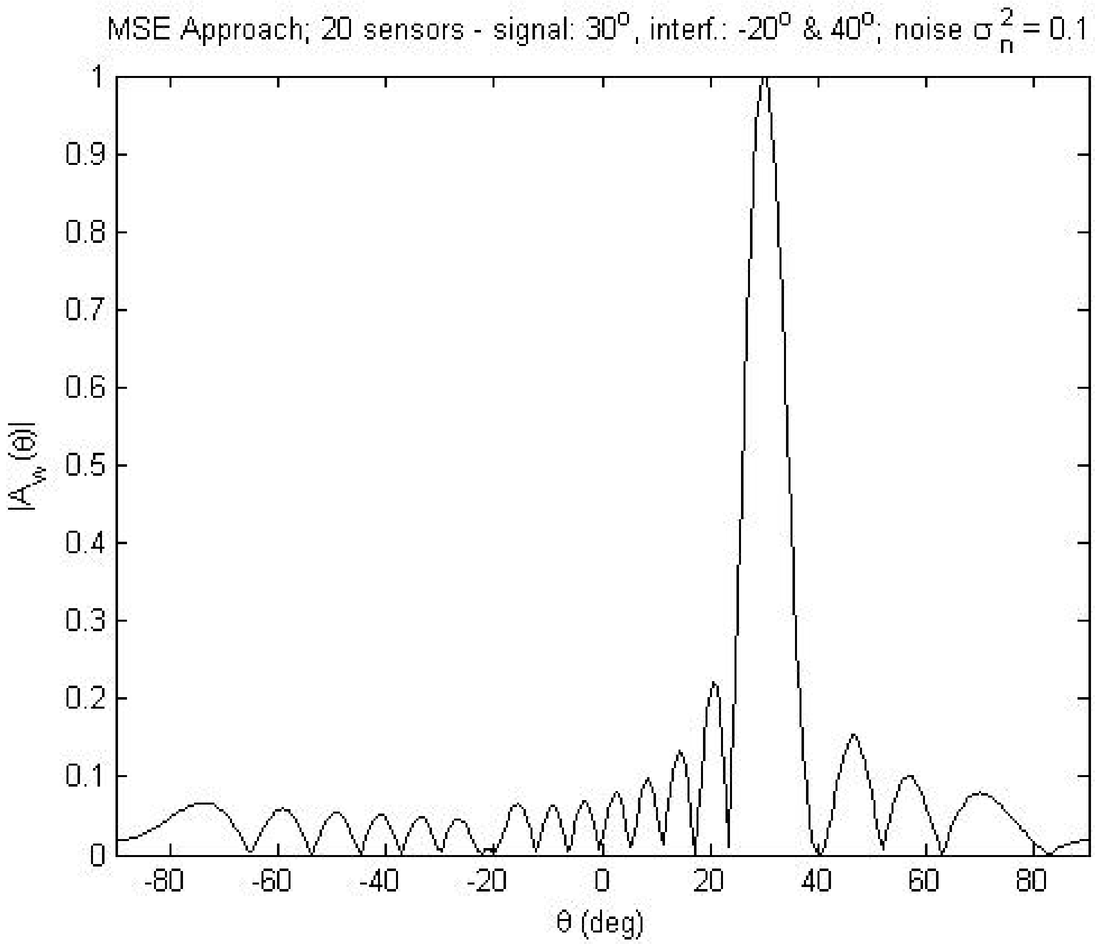

❖ Information Available: *snapshot in time of received signal retrieved at two antennas & reference signal*

❖ Assumption: $v_1(n)$, $v_2(n)$ *zero mean wss white noise RPs independent of each other and of s(n).*

❖ Goal: Denoise received signal

## ❖ Application to Spatial Filtering, cont'

## **Example:** Gain Pattern at filter output



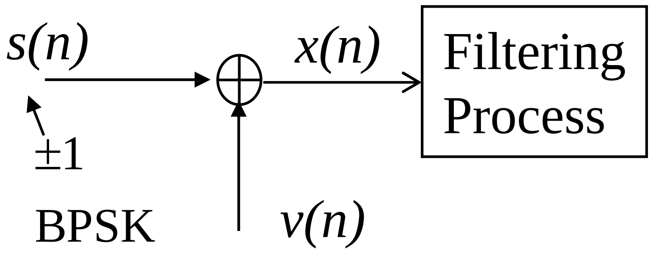

**Example:**
N-element array,
- desired signal at $\theta_0 = 30°$,
- interferences at $\theta_1 = -20°$, $\theta_2 = 40°$
- Noise power: 0.1

Array steering vector

$$A_w(\theta) = \frac{\left|\underline{w}^H \underline{a}(\theta)\right|^2}{\underline{w}^H \underline{w}}$$

# Application to Spatial Filtering, cont'

## Did we gain anything by using multiple receivers?




- <u>Assumption</u>: *v(n) zero mean RP independent of s(n).*
- <u>Goal</u>: Compute filter coefficients, filter output, and MMSE.

# Appendices

**Proof:**

$$\xi = E\left\{\left|s - \widehat{d}\left(\underline{x}\right)\right|^2\right\}$$

$$= E\left\{\underbrace{\left(s - \widehat{d}\left(\underline{x}\right)\right)\left(s - \widehat{d}\left(\underline{x}\right)\right)^T}\right\}$$

Define $L\left(s, \hat{s}\left(\underline{x}\right)\right)$ : loss function

$$\xi = \iint L\left(s, \widehat{d}\left(\underline{x}\right)\right) f\left(s|\underline{x}\right) f\left(\underline{x}\right) d\,\underline{x}^M\, ds, \quad \text{Bayes Rule}$$

$$= \int \underbrace{\left[\int L\left(s, \widehat{d}\left(\underline{x}\right)\right) f\left(s|\underline{x}\right) ds\right]}_{K} \underbrace{f\left(\underline{x}\right)}_{\geq 0} d^M\, \underline{x}$$

$\xi$ is minimized if $K$ is minimized for each value of $\underline{x}$

Problem: find $\widehat{d}\left(\underline{x}\right)$ so that $K$ is minimum

$$\frac{\partial K}{\partial \widehat{d}} = \frac{\partial}{\partial \widehat{d}}\left[\int \left(s - \widehat{d}\left(\underline{x}\right)\right)^2 f\left(s|\underline{x}\right) ds\right]$$

$$= -2\int \left(s - \widehat{d}\left(\underline{x}\right)\right) f\left(s|\underline{x}\right) ds$$

$$\frac{\partial K}{\partial \widehat{d}} = 0 \quad \Rightarrow \quad \int \left(s - \widehat{d}\left(\underline{x}\right)\right) f\left(s|\underline{x}\right) ds = 0$$

$$\Rightarrow \quad \int s\, f\left(s|\underline{x}\right) ds = \int \widehat{d}\left(\underline{x}\right) f\left(s|\underline{x}\right) ds$$

$$\Rightarrow \int s\, f\left(s|\underline{x}\right) ds = \widehat{d}\left(\underline{x}\right) \int f\left(s|\underline{x}\right) ds$$

$$\Rightarrow \int s\, f\left(s|\underline{x}\right) ds = \widehat{d}\left(\underline{x}\right) \times 1$$

$$\Rightarrow E\left[s \mid \underline{x}\right] = \widehat{d}\left(\underline{x}\right)$$

For a minimum we need: $\dfrac{\partial^2 K}{\partial \hat{d}^2} > 0$

Note:

$$\frac{\partial^2 K}{\partial \hat{d}^2} = (-2)\,\partial\left(\int\left(s-\hat{d}\left(\underline{x}\right)\right)f\left(s\middle|\underline{x}\right)ds\right)/\,\partial\hat{d}$$

$$=(-2)\int -f\left(s\middle|\underline{x}\right)ds$$

$$=(2)\times 1 > 0$$

# Appendix B: Proof of corollary of orthogonality principle

Proof:

$$e = s - \underline{h}^H \underline{x} = s - \left( \underline{h} + \underline{\hat{h}} - \underline{\hat{h}} \right)^H \underline{x}$$

where $\underline{\hat{h}}$ is the weight vector defined so that the orthogonality principle holds. Resulting error is called

$$\hat{e} = s - \underline{\hat{h}}^H \underline{x}$$

$$\sigma_e^2 = E\left\{ |e|^2 \right\} = E\left\{ \left( s - \underline{\hat{h}}^H \underline{x} + \left( \underline{\hat{h}} - \underline{h} \right)^H \underline{x} \right) \left( \qquad\qquad \right)^H \right\}$$

$$= E\left\{ \left( \hat{e} + \left( \underline{\hat{h}} - \underline{h} \right)^H \underline{x} \right) \left( \qquad\qquad \right)^H \right\}$$

$$= E\left\{ \left| \hat{e} \right|^2 + \left( \underline{\hat{h}} - \underline{h} \right)^H \underline{x}\, \hat{e}^H + \left| \left( \underline{\hat{h}} - \underline{h} \right)^H \underline{x} \right|^2 + \hat{e}\, \underline{x}^H \left( \underline{\hat{h}} - \underline{h} \right) \right\}$$

$$= E\left\{ \left| \hat{e} \right|^2 \right\} + \left( \underline{\hat{h}} - \underline{h} \right)^H \cancel{E\left\{ \underline{x}\, \hat{e}^H \right\}} + E\left\{ \left| \left( \underline{\hat{h}} - \underline{h} \right)^H \underline{x} \right|^2 \right\} + \cancel{E\left\{ \hat{e}\, \underline{x}^H \right\}} \left( \underline{\hat{h}} - \underline{h} \right)$$

$$= E\left\{ \left| \hat{e} \right|^2 \right\} = E\left\{ (s - \underline{\hat{h}}^H \underline{x}) \hat{e}^H \right\} = E\left\{ s\, \hat{e}^H \right\} - \underline{\hat{h}}^H \cancel{E\left\{ \underline{x}\, \hat{e}^H \right\}}$$

$$= E\left\{ s\, \hat{e}^H \right\} = E\left\{ s\, \hat{e}^* \right\}$$

# References

[1] C.W. Therrien, *Discrete Random Signals and Statistical Signal Processing*.
[2] D. Manolakis, V. Ingle, S. Kogon, *Statistical and Adaptive Signal Processing*, Artech House, 2005.
[3] S. Haykin, *Adaptive Filter Theory*, Prentice Hall 2002.