I. Signals & Sinusoids

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Signals

- What’s a signal?
  - It’s a function of time, $x(t)$
  - in the mathematical sense
Sinusoidal signal

\[ A \cos(\omega t + \varphi) \]

- **FREQUENCY**
  - Radians/sec
  - Hertz (cycles/sec)
  \[ \omega = (2\pi)f \]

- **PERIOD** (in sec)
  \[ T = \frac{1}{f} = \frac{2\pi}{\omega} \]

- **AMPLITUDE**
  - Magnitude
  \[ |A| \]

- **PHASE**
  \[ \varphi \]
- The cosine signal is periodic

\[ x(t) = A \cos(t + \varphi + 2\pi) = A \cos(t + \varphi) \]

- Period is \(2\pi\)
- Thus adding any multiple of \(2\pi\) leaves \(x(t)\) unchanged

\[ x(t) = A \cos(\omega t + \varphi) \]

What is the period for \(x(t)\)?
Sinusoidal signals

\[ A \cos(2\pi t) \]

\[ Asin(2\pi t) \]
Plotting a cosine function

Generic form \[ A \cos(\omega_0 t + \varphi) \]

Example: \[ 5 \cos(0.3\pi t) \]

\[
\begin{align*}
A &= \\
\omega &= \\
\varphi &= 
\end{align*}
\]
\[ 5 \cos(t + \pi / 3) \]

\[ A = \]
\[ \omega = \]
\[ \varphi = \]
\[-2\sin(4\pi t + \pi / 7)\]

\[A = \]
\[\omega = \]
\[\phi = \]
Caution: phase is ambiguous

\[ A \cos(\omega t + \varphi + 2\pi) = A \cos(\omega t + \varphi) \]
Speech example

- Spoken word: “bat” (BAT.WAV)

Nearly **Periodic** in Vowel “a”

- Waveform $x(t)$ is NOT a Sinusoid
- However, theory tells us
  - $x(t)$ is composed of a sum of sinusoids at different frequencies
  - → analysis done via FOURIER ANALYSIS-
    → Break $x(t)$ into its sinusoidal components
- Called the FREQUENCY SPECTRUM
Signal Operation: Time shift

\[ s(t) = \begin{cases} 
2t, & 0 \leq t \leq 1/2 \\
(4 - 2t)/3, & 1/2 \leq t \leq 2 \\
0, & \text{otherwise}
\end{cases} \]

\[ s_1(t) = s(t - 2) \]
Generic Time Shift

\[ s_1(t) = s(t - T) \]
Signal Operation: Time scaling

\[ s_1(t) = s(at) \]
Signal Operation: Time reversal

\[ s_1(t) = s(-t) \]
Combining time shifting & scaling

\[ s_1(t) = s(at - b), \ a \& b > 0 \]

\[ = s(a(t - b/a)) \]

1. Scale with a
2. Shift right by b/a
Basic Trigonometric equalities

- \( \sin^2(\theta) + \cos^2(\theta) = 1 \)
- \( \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \)
- \( \sin(2\theta) = 2 \sin(\theta)\cos(\theta) \)
- \( \sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \)
- \( \cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \)

Complex Numbers

- To solve: \( z^2 = -1 \)
  - \( z = j \)
  - Math and Physics
  - use \( z = i \)
- Complex number:
  - \( z = x + jy \)
Plot Complex Numbers

-5 + j0

j3

2 + j5

4 - j3

Imaginary Axis

Real Axis

y

x
Combine Complex Numbers
Euler’s FORMULA

- **Complex Exponential**
  - Real part is cosine
  - Imaginary part is sine
  - Magnitude is one

\[ e^{j\theta} = \cos(\theta) + j \sin(\theta) \]

\[ re^{j\theta} = r \cos(\theta) + jr \sin(\theta) \]
Using Euler’s FORMULA

- Allows to write sin/cos in terms of Complex Exponentials

- Recall

\[ e^{j\theta} = \cos(\theta) + j \sin(\theta) \]

\[ e^{j\omega t} = \cos(\omega t) + j \sin(\omega t) \]

\[ e^{-j\omega t} = \cos(\omega t) - j \sin(\omega t) \]

- Solve for cosine and sine
Evaluate the complex numbers

\[ z_1 = e^{-j\pi/2}; \quad z_2 = 2e^{j\pi/3} \]
\[ z_3 = 2e^{j3\pi/2}; \quad z_4 = 6e^{-j\pi/7} \]
\[ z_5 = \sum_{k=0}^{4} e^{-j\pi k/2} \]
POLAR FORM for \( \exp(j\theta) \)

- Vector Form
  - Length = 1
  - Angle = \( \theta \)

POLAR < --> RECTANGULAR transformation

- Relate \((x,y)\) to \((r,\theta)\)

\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta \\
r^2 &= x^2 + y^2 \\
\theta &= \tan^{-1}\left(\frac{y}{x}\right)
\end{align*}
\]
Real Part Example

\[ A \cos(\omega t + \varphi) = \Re \left\{ A e^{j\varphi} e^{j\omega t} \right\} \]

Evaluate \[ x(t) = \Re \left\{ -3 j e^{j\omega t} \right\} \]
**Imaginary Part Example**

\[ A \sin(\alpha t + \varphi) = \text{Im}\left\{ Ae^{j\varphi} e^{j\omega t} \right\} \]

Evaluate \[ x(t) = \text{Im}\left\{ -3je^{j\omega t} \right\} \]
**Example:** A sinusoidal signal $x(t)$ is defined by:

$$x(t) = \text{Re}\left\{ (1 + j)e^{j\pi t} \right\}$$

A maximum value of $x(t)$ is be located at:
(a) $t = 0$ sec.
(b) $t = 1/4$ sec.
(c) $t = 1$ sec.
(d) $t = 7/4$ sec.
(e) none of the above
Example: rewrite the cosine function using complex exponentials

\[ A \cos (7t) = \]
Example: Determine the amplitude $A$ and phase $\varphi$ of the sinusoid that is the sum of the following three sinusoids:

$$x(t) = \cos(\pi + \pi/2) + \cos(\pi t + \pi/4) + \cos(\pi t + 3\pi/4)$$

\[a) \quad A = 0 \quad \theta = 0\]
\[b) \quad A = 1 \quad \theta = \pi/2\]
\[c) \quad A = 1 + \sqrt{2} \quad \theta = 0\]
\[d) \quad A = 1 + \sqrt{2} \quad \theta = \pi/2\]
\[e) \quad A = 3 \quad \theta = \pi/2\]
How to combine different tones

- Most signals may be represented as Sinusoids with DIFFERENT Frequencies
  - SYNTHESIZE by Adding Sinusoids

\[ x(t) = \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k) + A_0 \]

- Two-sided SPECTRUM Representation
  - Graphical Form shows amplitude at DIFFERENT Frequencies (derived using complex exponentials)
\[ x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k) \]

\[ x(t) = X_0 + \sum_{k=1}^{N} \Re \{ X_k e^{j2\pi f_k t} \} \]

\[ X_k = A_k e^{j\varphi_k} \]

**Frequency** = \( f_k \)

\[ \Re \{ z \} = \frac{1}{2} z + \frac{1}{2} z^* \]
$$x(t) = 4e^{-j\pi/2} + 7e^{j\pi/3} + 7e^{-j\pi/3} + 4e^{j\pi/2}$$

**f (in Hz)**

-250  -100  0  100  250
**Example**: rewrite the cosine function using complex exponentials and plot the two-sided spectrum

\[ A \sin(7t) = \]
Speech Signal: “BAT”

- Nearly **Periodic** in the Vowel Region
- Period is (Approximately)
  \[ T = 0.0065 \text{ sec} \]
Periodic Signals

- Repeat every T secs
  - Definition: \( x(t) = x(t + T) \)

- Example: \( x(t) = \cos^2(3t) \)
  \[ T = \ldots \]

- Speech can be “quasi-periodic”

- Period of Complex Exponential
  \[ x(t) = e^{j\omega t} \]
  \[ x(t + T) = x(t) \, ? \]
Harmonic Signal Spectrum

\[ x(t) = \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k) + A_0 \]

Periodic signal can only have: \( f_k = k f_0 \)

Fundamental frequency
T = 1/f_0 fundamental period

\[ x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k) \]

\[ X_k = A_k e^{j\varphi_k} \]

\[ x(t) = X_0 + \sum_{k=1}^{N} X_k \]
Example: Harmonic Signal (3 Frequencies)

What is the fundamental frequency ($f_0$)?

What are the harmonics ($k f_0$)?
Example: Irrational Spectrum

We need a SPECIAL RELATIONSHIP to get a PERIODIC SIGNAL

Signal 1

Signal 2

Are Signal 1 and Signal 2 periodic?
NON-Harmonic Signal

Sum of Cosine Waves with Nonharmonic Frequencies
Frequency analysis

- This is a much HARDER problem
- Given a recording of a song, have the computer write the music
- Can a machine extract frequencies?
  - Yes, if we COMPUTE the spectrum for $x(t)$ during short intervals
- How to compute the frequency information using MATLAB $\texttt{spectrogram.m}$
  - **ANALYSIS** program
    - Takes $x(t)$ as input &
    - Produces spectrum values $X_k$
    - Breaks $x(t)$ into SHORT TIME SEGMENTS
      - Then uses a fast implementation of the Fourier Transform)
Plotting spectrograms using MATLAB

The MATLAB function `spectrogram.m` plots the spectrogram

\[ S = \text{SPECTROGRAM}(X, \text{WL}, \text{NOVERLAP}, \text{NFFT}, \text{Fs}) \]

WL: divides data \( X \) into segments of length equal to WL, and then windows each segment with a rectangular window of length WL.
NOVERLAP: number of samples each segment of \( X \) overlaps. NOVERLAP must be an integer smaller than WL
NFFT: specifies the number of frequency points used to calculate the Fourier transforms.
Fs: sampling frequency
Spectrogram Example

- Two **Constant** Frequencies

\[ x(t) = \cos(2\pi(660)t) \sin(2\pi(12)t) \]
What is the fundamental frequency?
Example: Stepped frequencies

- C-major SCALE: successive sinusoids
  - Frequency is constant for each note

Note: ARTIFACTS at frequency transitions
Time-Varying Frequency signals

- Frequency can change continuously
- Examples:
  1. FREQUENCY MODULATION (FM)
     
     \[ x(t) = \cos\left(2\pi f_c t + \nu(t)\right) \]

  2. CHIRP SIGNALS
     
     Linear Frequency Modulation (LFM)
     
     \[ x(t) = A\cos(\alpha t^2 + 2\pi f_0 t + \phi) \]
     
     Phase change is quadratic
     
     Frequency changes LINEARLY vs. time
Instantaneous Frequency (IF) definition

- Definition

\[
x(t) = A \cos(\psi(t))
\]

\[
\Rightarrow \omega_i(t) = \frac{d}{dt} \psi(t)
\]

- Example: Sinusoidal signal

\[
x(t) = A \cos(2\pi f_0 t + \varphi)
\]

\[
\psi(t) =
\]

\[
\Rightarrow \omega_i(t) = \frac{d}{dt} \psi(t) =
\]
Example:

\[ x(t) = 2 \cos(2\pi (30t^2 - 30t + 0.1)) \]

Derive the expression for the instantaneous frequency
Chirp waveform
Linear Chirp Spectrogram
How to determine the equation for a specific linear chirp

Assume \( x(t) \) sweeps from \( f_1 = 5000 \text{Hz} \) at \( T_1 = 0 \text{s} \) to \( f_2 = 1000 \text{Hz} \) at \( T_2 = 2 \text{s} \), find its equation and plot its spectrogram.
Other types of CHIRPS

- $\psi(t)$ can be anything:

$$x(t) = A \cos(\alpha \cos(\beta t) + \varphi)$$

$$\Rightarrow \omega_i(t) = \frac{d}{dt} \psi(t) =$$

Example: Sinewave frequency modulation (FM)

Additional examples in the CD-ROM Demos in Chapter 3