Section 4
Description of Systems
EO 2402
Summer 2013

Continuous-time Systems

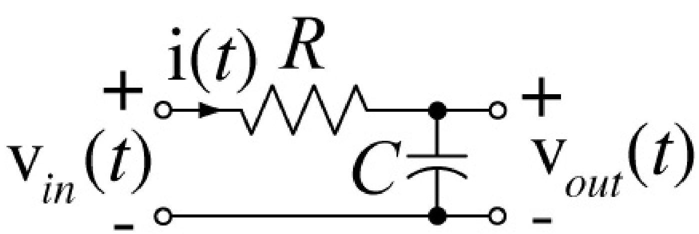
Discrete-time Systems

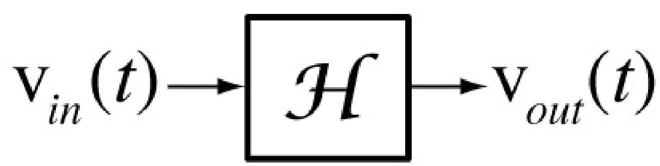
Part A:
Continuous-Time Systems

❖ System Representation

- System is a mathematical model of a process

- Useful to { Analyze / Model / Represent } Actual devices

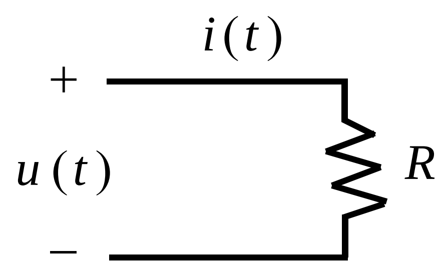




❖ System Representation: Static / Dynamic

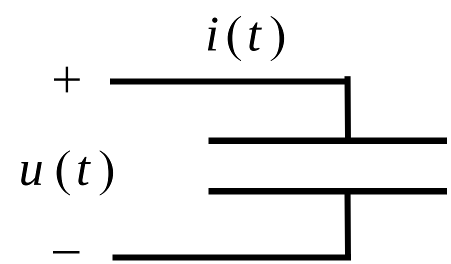
STATIC

- Can not store energy
- Can not remember its states
- System output depends on input at present time only
- Input/output equation represented by algebraic equations

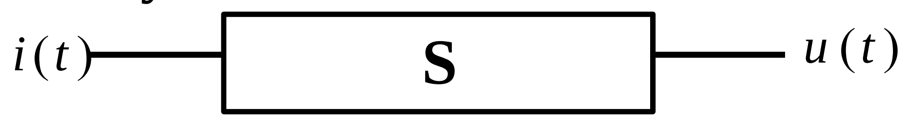


DYNAMIC

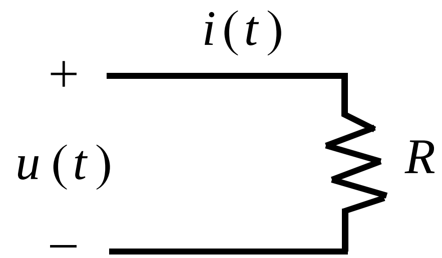
- Can share energy
- Can remember its states
- System output depends on inputs other than present time only
- Input/output equation represented by differential/difference equation



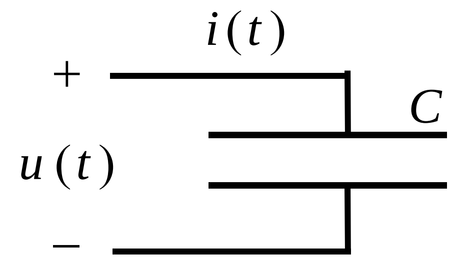
- **Circuits excited by current** $i(t)$



System Representation: Static / Dynamic, cont’



$$u(t) = R\,i(t)$$



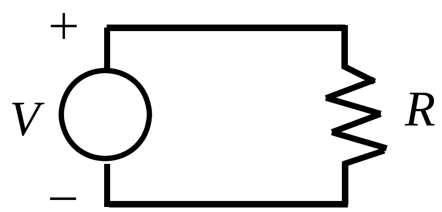
$$u(t) = \frac{1}{C}\int_{-\infty}^{+} i(t)\,dt$$

❖ System Classification: Lumped / Distributed

LUMPED PARAMETER SYSTEM

- Disturbance at any point in system propagates to every point instantaneously

➡ We can ignore the time it takes for signals to propagate around the system



DISTRIBUTED PARAMETER SYSTEM

- Disturbance at any point in system takes some time to propagate to other points of the system

System Classification: Lumped / Distributed, cont'

Makes Sense When

LUMPED PARAMETER SYSTEM

- System physical dimensions are small as compared to system operating wavelength
- **Application:** Microwave engineering rule-of-thumb
- To be considered lumped: no feature of the system can exceed 1/10 of wavelength at the maximum operating frequency.

Resistor, capacitor up to 100GHZ

DISTRIBUTED PARAMETER SYSTEM

- System has non perfect conductor wires

OR

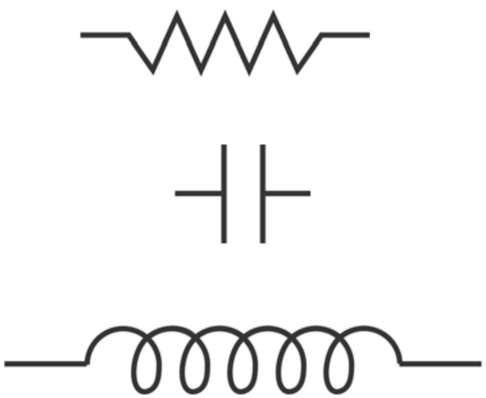
- Is used at high frequencies where wavelength approaches physical dimension of system.

❖ System Classification: Active / Passive

PASSIVE

- Unable to deliver energy to outside world

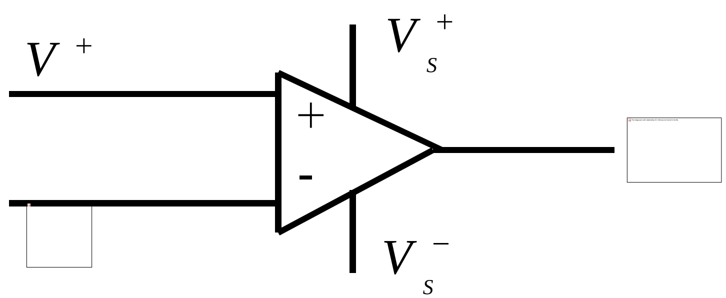
- **EXAMPLES:**



ACTIVE

- Can deliver energy to outside world

- **EXAMPLE**: (op amp)



❖ System Classification: With Memory / Without Memory

WITHOUT MEMORY

- Output at any time depends only output at same time

$$y(t) = R\,x(t)$$

WITH MEMORY

- Output at any time may depend on previous, present, and/or past values of the input.

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$$

❖ System Classification: Causal / Non Causal

CAUSAL

- Output at time to depends only on present and/or past value of input.

$$y(t) = x(t) + x(t-2)$$

NON CAUSAL

- Output at time t_o depends on future values of input

$$y(t) = x(t+1)$$

NOTE: Memoryless $\Rightarrow$ causal

$\not\Leftarrow$

❖ System Classification: Stable

- Definition: A system is said to be stable if any bounded input *x(t)* leads to a bounded output *y(t)*

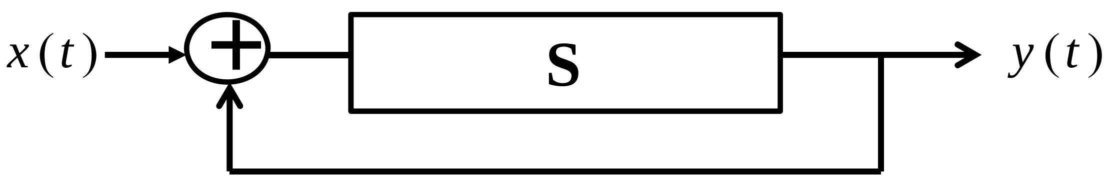
EXAMPLE:

$$y_1(t) = x(t+1)$$

$$y_2(t) = tx(t)$$

❖ System Classification: Feedback

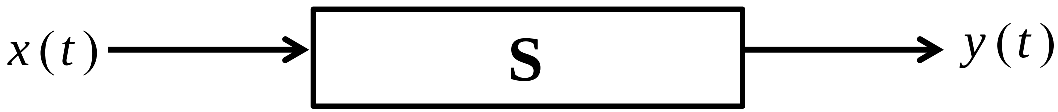
- **Definition:** A system in which output signal is fed-back and added to system



❖ Continuous-Time Systems Building Blocks

Impulse response to system S

- **Ideal Delay**



- Mathematical Definition

$$y(t) = x(t - T_d)$$

- System impulse response

$$x(t) = \delta(t) \Rightarrow y(t) = h(t)$$
$$=$$

- **Integrator**

$$x(t) \longrightarrow \boxed{\mathbf{S}} \longrightarrow y(t)$$

- Mathematical Definition

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

- System impulse response

$$x(t) = \delta(t)$$

$$\Rightarrow \; h(t) =$$

EXAMPLE: Integrator Output

- Assume: $x(t) = e^{-0.8t}u(t)$
- Find $y(t) = S\left\{x(t)\right\}$

- **Differentiator**

$$x(t) \longrightarrow \boxed{\quad S \quad} \longrightarrow y(t)$$

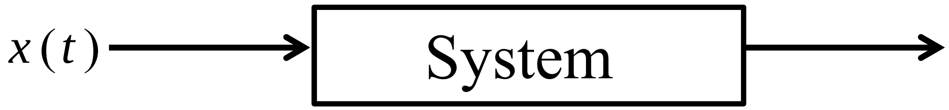
- Mathematical Definition:

$$y(t) = \frac{dx(t)}{dt}$$

EXAMPLE: Differentiator Output

$$x(t) = e^{-0.8t}u(t) \qquad \text{Find:} \quad y(t) = S\left\{x(t)\right\}$$

❖ Linearity

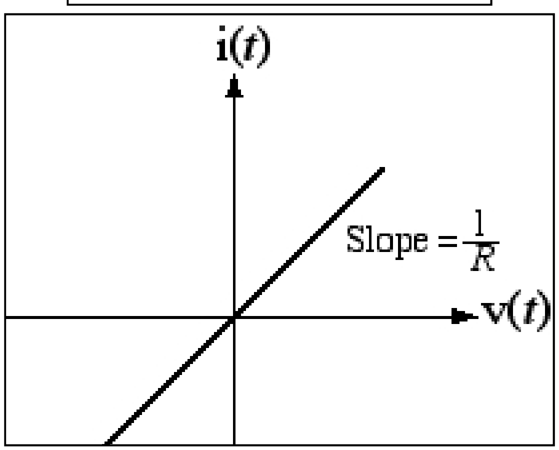


- **Definition**: A system is linear iff for inputs $x_1(t)$ and $x_2(t)$ and any constants α and β we have

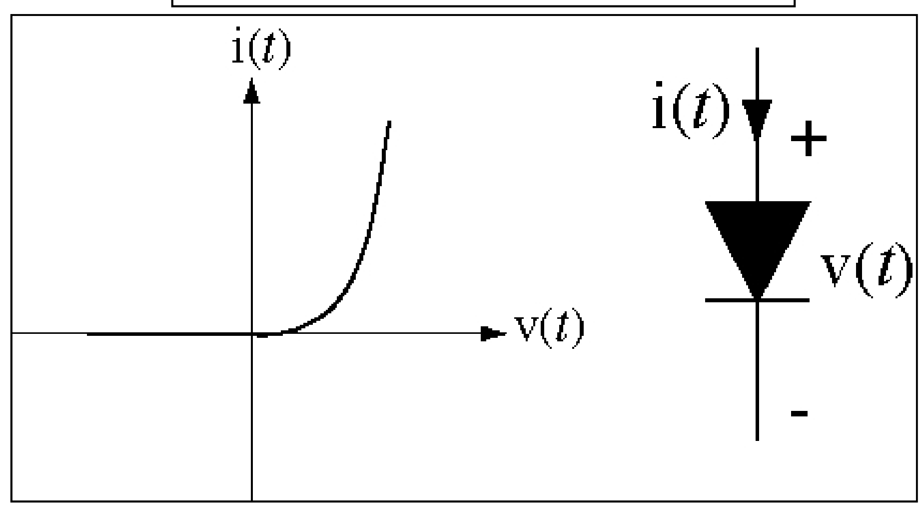
$$S\left\{\alpha\, x_1(t) + \beta\, x_2(t)\right\} = \alpha\, S\left\{x_1(t)\right\} + \beta\, S\left\{x_2(t)\right\}$$

Many real systems are non-linear because the relationship between excitation amplitude and response amplitude is non-linear

V-I Diagram for a Linear Resistor



V-I Diagram for a Diode



Linearity, cont'

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow$$

EXAMPLES:

$$y_1(t) = 2x(t)$$

$$y_2(t) = 2x(t) + 1$$

$$y_3(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)\,d\tau$$

$$y_4(t) = \left|x(t)\right|$$

$$y_5(t) = x^2(t)$$

$$y_6(t) = \cos(x(t)), \qquad x(t)\ finite$$

❖ Invertibility

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

Definition: A system is invertible if knowledge of the output *y(t)* allows to uniquely determine the input *x(t)*.

EXAMPLES:

$$y_1(t) = \sin(x(t))$$

$$y_2(t) = |x(t)|$$

❖ Time-Invariance

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

Definition: A system is time-invariant if a shift in the input *x(t)* produces the same shift in the output *y(t)*

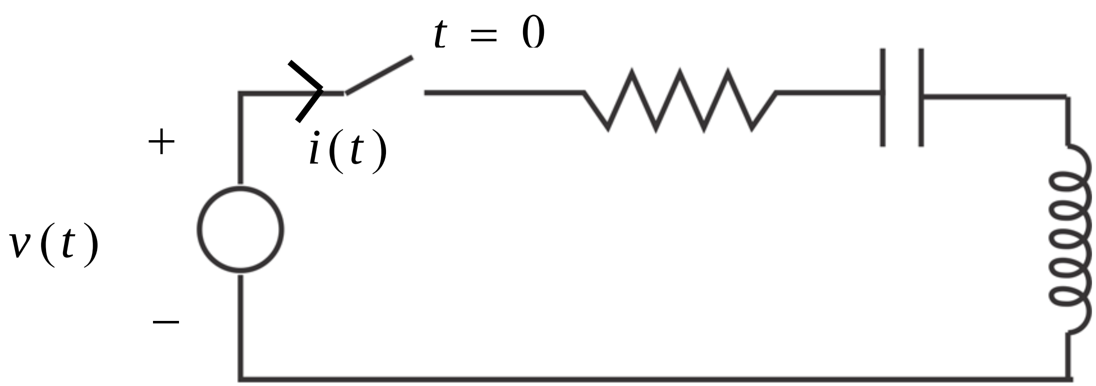
LTI System Definition: When the system is linear AND time-invariant it is called **LTI**

EXAMPLES:

$$y_1(t) = 2x(t)$$
$$y_2(t) = x(2t)$$
$$y_3(t) = x(t - D)$$

❖ Representation of Systems by Differential Equations

- **Introduction**



Assume: No initial energy stored in inductor or capacitor (at t=0, current in inductor and voltage in capacitor are equal to 0)

$$v(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_0^t i(t)dt$$

Input: *i(t)*
Output: *v(t)*

$$\frac{dv(t)}{dt} = R\frac{di(t)}{dt} + L\frac{d^2 i(t)}{dt^2} + \frac{1}{C}i(t)$$

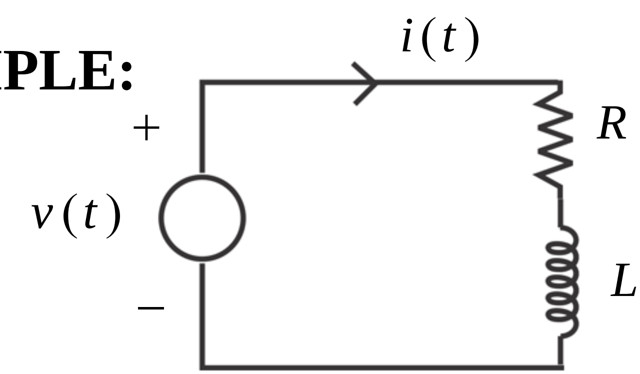
Representation of Systems by Differential Equations, cont’

- Most continuous-time dynamic systems represented by ODEs
- Order of differential equation = number of elements which can store energy

$$a_0 y(t) + a_1 \frac{dy(t)}{dt} + \ldots + a_N \frac{d^N y(t)}{dt^N} = b_0 x(t) + b_1 \frac{dx(t)}{dt} + \ldots b_M \frac{d_M x(t)}{dt^M}$$

$$t \geq 0$$

EXAMPLE:



$$\begin{cases} v(t) = R\,i(t) + L \dfrac{di(t)}{dt} \\ i(0) = I_o \end{cases}$$

Representation of Systems by Differential Equations, cont'

$$y(t) + a_1 \frac{dy(t)}{dt} + \ldots + a_N \frac{d^N y(t)}{dt^N} = b_0 x(t) + b_1 \frac{dx(t)}{dt} + \ldots b_M \frac{d^M x(t)}{dt^M}$$

With

- N initial conditions $\left.\frac{d^k y(t)}{dt^k}\right|_{t=0} \quad for \quad k = 0, \ldots, N-1;$
- $x(t) = 0 \quad t < 0$

Leads to general solution: $y(t) = y_p(t) + y_h(t)$

Particular solution due to input *x(t)*

Homogeneous solution due only to initial conditions when input signal *x(t)*= 0

Note: When initial conditions = 0, *y(t)* only depends on the input *x(t)* → system is LTI
When initial conditions ≠ 0, when checking for linearity, only input gets changed while initial conditions remain the same → system is nonlinear

❖ How to Solve Differential Equations / Part 1: Homogeneous Equation (homogeneous solution)

Assume $x(t) = 0$

$$\Rightarrow \quad y(t) + a_1 \frac{dy(t)}{dt} + \ldots + a_N \frac{d^N y(t)}{dt^N} = 0$$

$$\text{with} \quad i.c \quad y(0), \quad \frac{d}{dt} y(t)\bigg|_{t=0}, \ldots, \frac{d^{N-1}}{dt^{N-1}} y(t)\bigg|_{t=0}$$

1. Form characteristic equation $a_0 + a_1 s + \ldots + a_N s^N = 0$

2. Solve for roots s_i of equation: $i = 1, \ldots, N$

   poles, eigenvalues of system

3. For distinct roots

   Solution of homogeneous equation is given by:

$$y_h(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + \ldots + c_N e^{s_N t}$$

Homogeneous Equation, cont'

Example: Assume $a_0 y(t) + a_1 \frac{dy(t)}{dt} + \ldots + a_2 \frac{d^2 y(t)}{dt^2} = 0$

$\rightarrow$ Characteristic Equation: $a_0 + a_1 s + a_2 s^2 = 0$

Special cases: * Multiple roots: 2 equal and real roots: $s_1 = s_2 = s$

$$y_h(t) = c_1 e^{st} + c_2 t e^{st}$$

* Complex conjugate roots: $s_1 = a + jb,\ s_2 = s_1^*$

$$y_h(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} = e^{at}\left(a_1 \cos bt + a_2 \sin bt\right)$$

where $a_1 = c_1 + c_2 \quad a_2 = (c_1 - c_2)\, j$

❖ How to Solve Differential Equations/ Part 2: Non Homogeneous Equation (particular solution)

$$y(t) + a_1 \frac{d}{dt} y(t) + \ldots + a_N \frac{d^N y(t)}{dt^N} = b_0 x(t) + \ldots + b_M \frac{d^M x(t)}{dt^M}$$

- Expression depends on specific input *x(t)*

input $x(t)$	$y_p(t)$
A	B
Ae^{at}	Be^{at}
$A\cos(at+\theta)$	$B_1\cos(at) + B_2\sin(at)$
At^n	$\sum_{k=0}^{n} B_k t^k$

Constraints: if any term of $y_p(t)$ appears in $y_h(t)$, the specific common term in $y_p(t)$ gets multiplied by the smallest integral power of t large enough so that none of the resulting terms in $y_p(t)$ appear in $y_h(t)$

Any unknown constant identified by replacing expression in differential equation

How to Solve Differential Equations, cont'

$$y(t) + a_1 \frac{d}{dt} y(t) + \ldots + a_N \frac{d^N y(t)}{dt^N} = b_0 x(t) + \ldots + b_M \frac{d^M x(t)}{dt^M}$$

Overall procedure to solve the differential equation:

1) Compute homogeneous solution $y_h(t)$ [assume RHS=0]

2) Identify particular solution $y_p(t)$ due to specific input $x(t)$. Identify unknown constants (if any) by replacing $y_p(t)$ in differential equation

3) Form overall solution as $y(t)=y_h(t)+ y_p(t)$

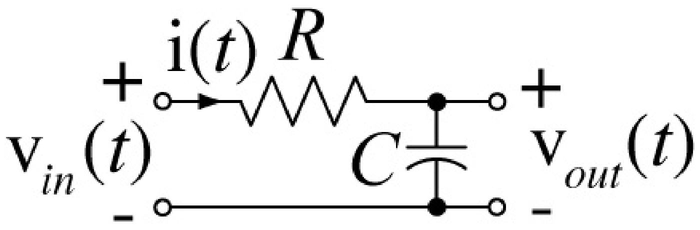
4) Identify unknown constants present in $y(t)$ by using initial conditions.

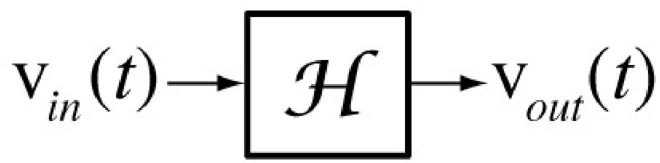
❖ How to Solve Differential Equations - Examples

$$\frac{d^2 y(t)}{dt^2} + \frac{dy(t)}{dt} = 2t; \quad y(0) = \frac{d}{dt} y(t)\big|_{t=0} = 0$$

$$\frac{d^3 y(t)}{dt^2} - y(t) = \sin(t); \quad y(0) = \frac{d}{dt} y(t)\big|_{t=0} = \frac{d^2 y(t)}{dt^2}\big|_{t=0} = 0$$

❖ How to Solve Differential Equations – Circuit Example 1



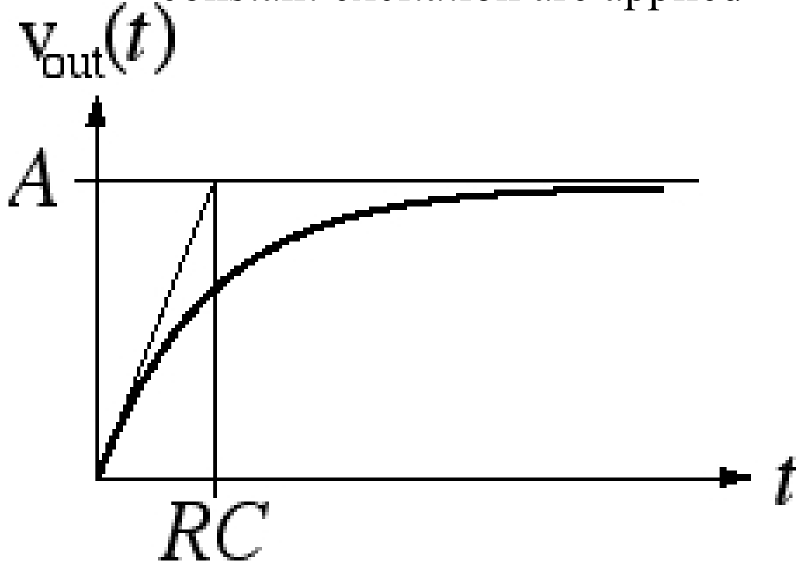


Assume R=1Ω, C=1F

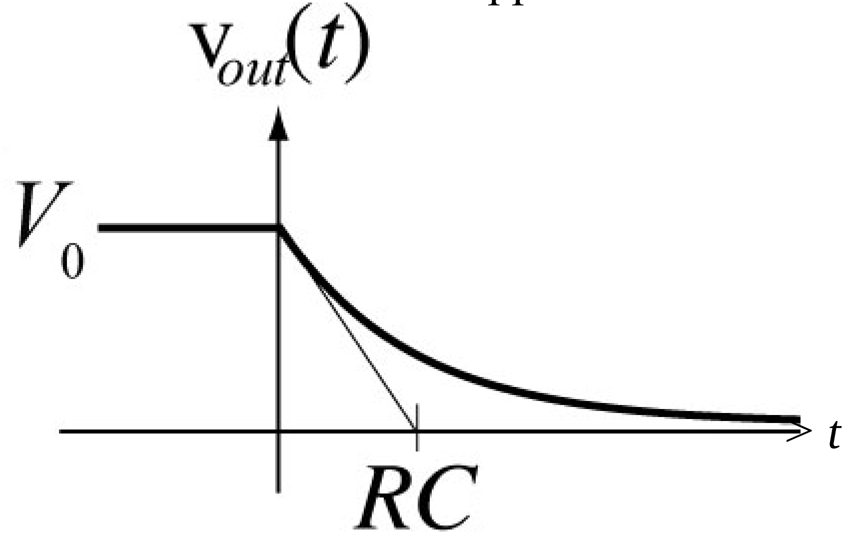
Input voltage $v_{in}(t)=Au(t)$

Initial voltage $v_{out}(0)=V_0$

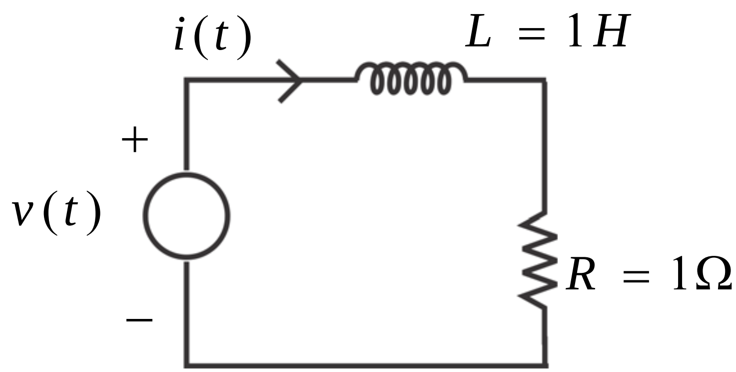
When no initial voltage and constant excitation are applied



When initial voltage V_0 and no excitation are applied



❖ How to Solve Differential Equations – Circuit Example 2



Initial current in induction: I_o

$v(t)=Bu(t)$

Find the expression for the current $i(t)$ when

$a)\ \mathrm{B}=1 \qquad I_0 = 1$

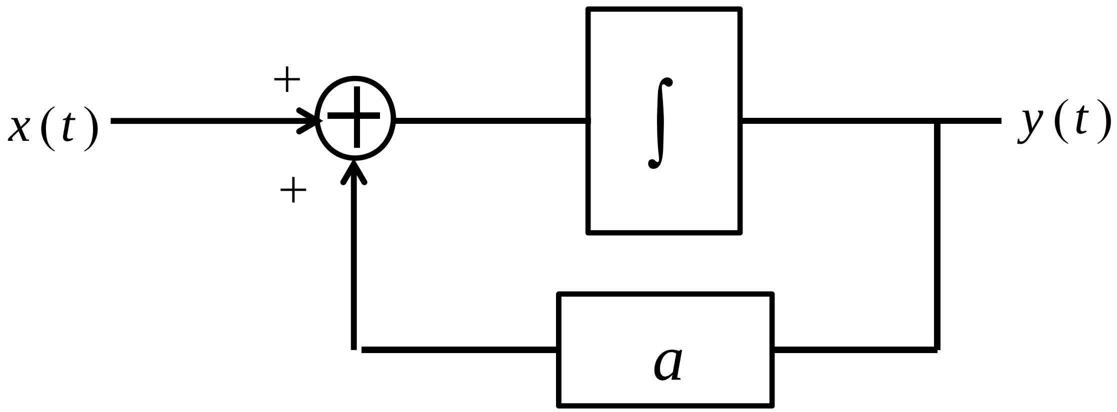
$b)\ \mathrm{B}=2 \qquad I_0 = 0$

❖ Example

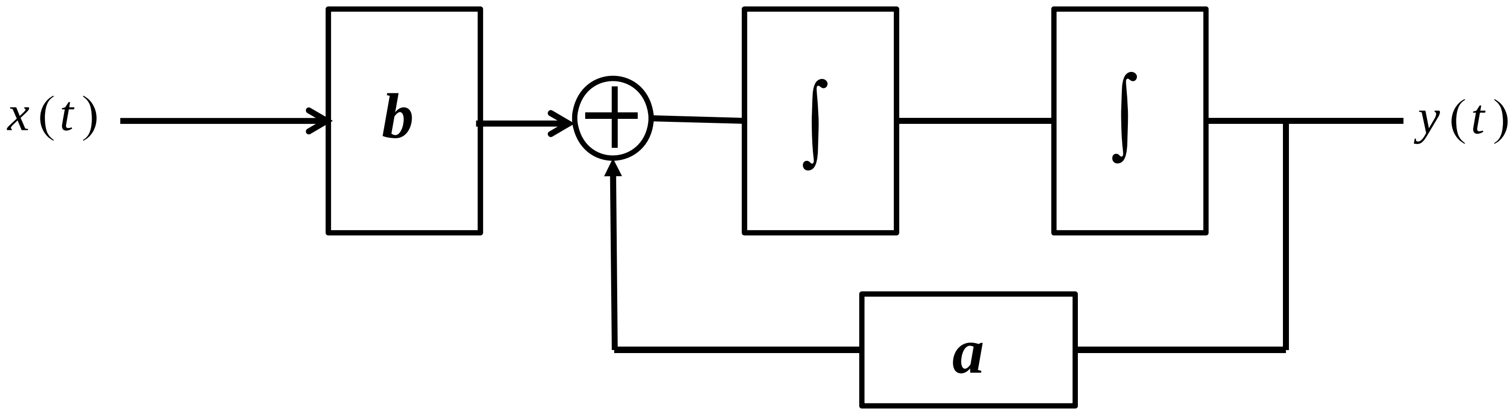
Compute the impulse response of the causal system described by:

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

❖ Representing Systems: From Block Diagram to Differential Equations



Representing Systems, cont'



❖ Representing Systems: From Differential Equations to Block Diagrams

1) $\frac{d}{dt} y(t) - a y(t) = x(t)$ 2) $\frac{d^2 y(t)}{dt} - a y(t) = b x(t)$

Part B:
Discrete-Time Systems

❖ Basic Discrete-Time System Characteristics

- Events occur at sample points in time but not between them.

- Discrete-time example: **digital computer** ➔ Significant events occur at the end of each clock cycle.

- Similar block diagram configurations as for continuous system structures

- Discrete-time systems can be described by **difference** (not differential) **equations**.

$$x[n] = x(nT_s) \longrightarrow \boxed{\text{Discrete System}} \longrightarrow y[n] = y(nT_s)$$

$$y[n] = 0.8y[n-1] + x[n]$$

❖ Solving Difference Equations

$$Example:$$

$$y[n] = 0.8y[n-1] + x[n],$$

$$x[0] = 1,\ x[n] = 0, n > 0$$

- Solving can be done recursively using a computer

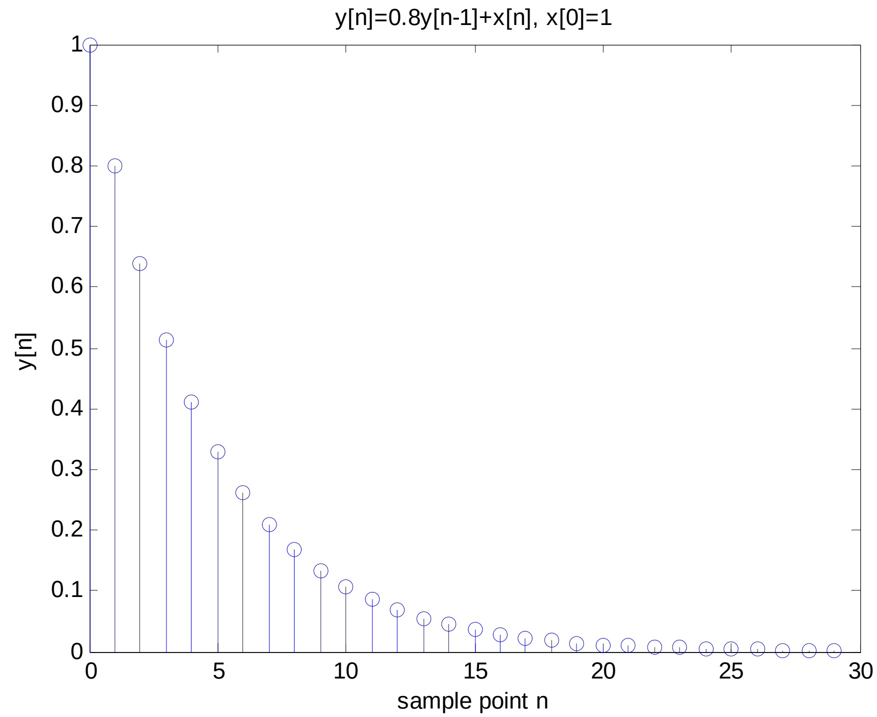
$y[1] = 1;$

for k=2:N

$\quad y[k] = 0.8 * y[k-1];$

end

stem(0:N-1,y)



❖ Discrete-Time System Properties

Mostly derived in the same fashion as for continuous systems

Example: - Discrete system is linear if:

$$S\left\{ax_1[n]+bx_2[n]\right\} = aS\left\{x_1[n]\right\}+bS\left\{x_2[n]\right\}$$

- LTI Discrete system is stable if:

$$\sum_{k=-\infty}^{+\infty} \left|h[h]\right| < \infty$$

Appendix

❖ Differential Equations - Homogeneous Equation case

$$y(t) + a_1 \frac{dy(t)}{dt} + \ldots + a_N \frac{d^N y(t)}{dt^N} = 0$$

$$\text{with } \; i.c \quad y(0), \; \left.\frac{d}{dt} y(t)\right|_{t=0}, \ldots, \left.\frac{d^{N-1}}{dt^{N-1}} y(t)\right|_{t=0}$$

1. Form characteristic equation $a_0 + a_1 s + \ldots + a_N s^N = 0$

2. Solve for roots s_i of equation: $i = 1, \ldots, N$

   poles, eigenvalues of system

3. For distinct roots

   Solution of homogeneous equation is given by:

$$y_h(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + \ldots + c_N e^{s_N t}$$

Why ?

A few words on the characteristics of the solution:

Assume 1st order differential equation:

$$\frac{dy(t)}{dt} + ay(t) = 0$$

$$\Rightarrow \frac{dy(t)}{dt} = -ay(t)$$

y(t) and y'(t) must have same functional form
→which is only possible when $y(t)=e^{Kt}$

More details about differential equations in Web Appendix D [Roberts Textbook]