3.5.2c Let \( \mathbf{u}_1 = (0,1)^T \) and \( \mathbf{u}_2 = (1,0)^T \). Then \([\mathbf{u}_1, \mathbf{u}_2]\) is an ordered basis for \( \mathbb{R}^2 \). Find the transition matrix corresponding to the change of basis from the standard basis to \([\mathbf{u}_1, \mathbf{u}_2]\).

**Solution:** Let \( U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} \). Then \( U \) is the transition matrix corresponding to the change of basis from \([\mathbf{u}_1, \mathbf{u}_2]\) to the standard basis. It follows that \( U^T \) is the matrix that we're after. Performing the computation, we find that \( U^T = U \), i.e., \( U \) is its own inverse.

3.5.4 Let \( E = [(5,3)^T, (3,2)^T] \), and let \( \mathbf{x} = (1,1)^T, \mathbf{y} = (1,-1)^T \), and \( \mathbf{z} = (10,7)^T \). Find the values of \([\mathbf{x}]_E, [\mathbf{y}]_E, \) and \([\mathbf{z}]_E\).

**Solution:** From our discussion in class, we know that the transition matrix from the ordered basis \( E \) to the standard basis is just

\[
T = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}.
\]

Since we want to go the other way, the transition matrix we want is

\[
W = T^{-1} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}.
\]

It is now an easy step to determine \([\mathbf{x}]_E = (-1,2)^T\), \([\mathbf{y}]_E = (5,-8)^T\), and \([\mathbf{z}]_E = (-1,5)^T\).

3.5.5 Let \( \mathbf{u}_1 = (1,1,1)^T, \mathbf{u}_2 = (1,2,2)^T, \) and \( \mathbf{u}_3 = (2,3,4)^T \). Let \( e_i \) denote the \( i \)th standard basis vector. In part (a) of this problem, we are to find the transition matrix corresponding to the change of basis from \([e_1, e_2, e_3]\) to \([\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]\). We know that the transition matrix corresponding to the change of basis from \([\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]\) to \([e_1, e_2 e_3]\) is \( U = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \). It follows that the matrix in which we are interested in is \( U^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \).

In part (b), we are to compute the coordinates of various vectors with respect to \([\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]\). This involves nothing more than matrix multiplication, and the solution is not included here.

3.5.6 We are given vectors \( \mathbf{v}_1 = (4,6,7)^T, \mathbf{v}_2 = (0,1,1)^T, \) and \( \mathbf{v}_3 = (0,1,2)^T \), along with the set \{\( \mathbf{u}_i \)\} of vectors from the preceding exercise. The task is to find the transition matrix from \([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]\) to \([\mathbf{u}_1, \mathbf{u}_2 u_3]\). This is just \( U^{-1}V \), where \( U^{-1} \) is the matrix found in the preceding problem and \( V \) is the matrix whose \( i \)th column is \( \mathbf{v}_i \). We are then given the vector \( \mathbf{x} = 2\mathbf{v}_1 + 3\mathbf{v}_2 - 4\mathbf{v}_3 \), and asked to find its coordinates with respect to \([\mathbf{u}_1, \mathbf{u}_2 u_3]\). This is, once again, a matter of computing a product, in this case \( U^{-1}V[\mathbf{x}]_V \), where \([\mathbf{x}]_V = (2,3,-4)^T \).
3.5.7 Given \( v_1 = (1, 2)^T, \ v_2 = (2, 3)^T, \) and \( S = \begin{bmatrix} 3 & 5 \\ 1 & -2 \end{bmatrix}, \) find vectors \( w_1 \) and \( w_2 \) such that \( S \) is the transition matrix from \( [w_1, w_2] \) to \( [v_1, v_2] \).

**Solution:** Let \( W = \begin{bmatrix} w_1 & w_2 \end{bmatrix} \) and \( V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \). It follows that \( S = V^{-1}W \). Then
\[
W = VS = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 9 & 4 \end{bmatrix}.
\]

3.5.8 This is closely related to the previous problem. We are given \( v_1 = (2, 6)^T, \ v_2 = (1, 4)^T, \) and \( S = \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}. \) This time we want to find \( u_1 \) and \( u_2 \) such that \( S \) is the transition matrix from \( [v_1, v_2] \) to \( [u_1, u_2] \). We know that \( S = U^{-1}V \). Multiplying from the left by \( U \) and from the right by \( S^{-1} \), we have \( U = VS^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix}. \) So \( u_1 = (0, -1)^T \) and \( u_2 = (1, 5)^T \).

3.5.9 Let \([x, 1]\) and \([2x - 1, 2x + 1]\) be ordered bases for \( P_2 \). We want to find two transition matrices, one for changing basis from \([2x - 1, 2x + 1]\) to \([x, 1]\) and the other for changing basis from \([x, 1]\) to \([2x - 1, 2x + 1]\).

**Solution:** It is up to us to choose what the ‘standard’ basis is for \( P_2 \). We can choose this basis to be \([x, 1]\), with associated matrix \( U = I_2 \). Relative to this standard, the matrix associated with the other basis is \( V = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}. \) Now the transition matrix for changing basis from \([2x - 1, 2x + 1]\) to \([x, 1]\) is \( V, \) while that for the inverse change is \( V^{-1} = \frac{1}{4} \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}. \)