Example of Asymptotic Approximations - Structured Flowcharts - J. T. Butler

Enumeration of Structured Flowcharts
A structured flowchart consists of a set of decision nodes interconnected by paths.

Assume that interchanging $F_1$ and $F_2$ in an IF-THEN-ELSE flowchart leaves the flow-chart unchanged. Let $ITE(x)$, $DW(x)$, and $F(x)$ be the ordinary generating functions for the number of IF-THEN-ELSE, DO-WHILE, and SEQUENCE flowcharts, respectively. A structured flowchart is a SEQUENCE of $p$ IF-THEN-ELSE and DO-WHILE flowcharts where $p \geq 0$.

By hand calculation, we have $ITE(x) = x + 2x^2 + \ldots$, $DW(x) = x + 2x^2 + \ldots$, and $F(x) = 1 + 2x + 8x^2 + \ldots$. The flowcharts with $j$ nodes, for $0 \leq j \leq 2$, are shown on the next slide.

We can calculate $ITE(x)$ directly as follows:

$$ITE(x) = \frac{x}{2} F(x) F(x) + \frac{x}{2} F(x^2) \quad (1)$$
Consider the left term in (1) first. The $x$ represents the contribution to the node count by $N$, the topmost node, while the two factors $F(x)F(x)$ represent the contributions to the node count by sub-flowcharts $F_1$ and $F_2$. The factor $1/2$ appears because of double counting which occurs in $xF(x)F(x)$. That is, a flowchart with distinct choices for $F_1$ and $F_2$ is considered the same as the flowchart obtained by interchanging $F_1$ and $F_2$. Both flowcharts are included in $xF(x)F(x)$.

However, double counting does not occur if we happen to choose $F_1$ and $F_2$ the same. Such charts contribute only one-half of what they should in $xF(x)F(x)/2$. Thus, we must add the remainder of their contribution to the node count. This contribution is expressed by $xF(x^2)/2$, the right term of (1).

Similarly, for DO-WHILE flowcharts we

$$DW(x) = xF(x)$$

(2)

All structured flowcharts consist of a SEQUENCE of no, one, two, … IF-THEN-ELSE and DO-WHILE flowcharts. Thus, $F(x)$ is

$$F(x) = 1 + [ITE(x) + DW(x)] + [ITE(x) + DW(x)] + \ldots$$

(3)

The constant term of (3) corresponds to the flowchart of no nodes. The $[ITE(x) + DW(x)]$ term is the contribution of a SEQUENCE of one IF-THEN-ELSE or DO-WHILE flowchart. The $[ITE(x) + DW(x)]^2$ term corresponds to a SEQUENCE of two such flowcharts, etc. We can write (3) as follows

$$F(x) = \frac{1}{1 - (ITE(x) + DW(x))}$$

(4)

Substituting (1) and (2) into (4) and rearranging yields

$$F(x) = \frac{xF^3(x)}{2} + xF^2(x) + \frac{xF(x)F(x^2)}{2} + 1$$

(5)
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We seek $f_n$, the coefficient of $x^n$ in $F(x)$. This is accomplished by equating coefficients on each side of (5). For example,

$$f_0 = 1,$$
$$f_1 = \frac{f_0}{2} + f_0^3 + \frac{f_1}{2} = 2,$$

and

$$f_2 = \frac{3f_0f_1}{2} + 2f_0f_1 + \frac{f_2}{2} = 8$$

We have

$$F(x) = 1 + 2x + 8x^2 + 43x^3 + 258x^4 + \ldots$$

The calculation of these coefficients was done by computer using polynomial manipulation programs. That is, substituting $[0,0,0,\ldots]$ for $F(x)$ on the left produces $[1,0,0,\ldots]$ for $F(x)$ on the left. Substituting $[1,0,0,\ldots]$ for $F(x)$ on the right produces $[1,2,0,\ldots]$ on the left, etc. Substituting $[1,2,0,\ldots]$ for $F(x)$ on the right produces $[1,2,8,0,\ldots]$ on the left, etc.

Enumeration of Structured Flowcharts by Node Type and Height

I. Counting flowcharts by node type

Note that each node in a structured flowchart can be classified as an IF-THEN-ELSE type or DO-WHILE type depending on how it is used. Let $F(x,y)$ be an ordinary generating function for such flowcharts, where $y$ accounts for IF-THEN-ELSE nodes.

A typical term in $F(x,y)$ is $a_{np}x^n y^p$, where $a_{np}$ is the number of structured flowcharts with $n$ nodes $p$ of which are IF-THEN-ELSE nodes. We have

$$ITE(x,y) = \frac{xyF^2(x,y)}{2} + \frac{xyF(x^2,y^2)}{2}$$

$$DW(x,y) = xF(x,y)$$

$$F(x,y) = \frac{1}{1 - ITE(x,y) - DW(x,y)}$$

Substituting (4) and (5) into (6) and rearranging yields

$$F(x,y) = \frac{xyF^3(x,y)}{2} + \frac{xyF(x,y)}{2} + \frac{xyF(x,y)F(x^2,y^2)}{2} + 1$$

(9)
Equating coefficients on each side of (7), we get
\[ a_0 = 1 \quad a_{2n} = 2a_na_{n0} = 2 \]
\[ a_{2n} = a_{n0} = 1 \quad a_{21} = \frac{3a_n\alpha_{n1} + 2a_n\alpha_{n1}}{2} + a_n\alpha_{n1} = 4 \]
\[ a_{n1} = \frac{a_n^3}{2} + \frac{a_n^2}{2} = 1 \quad a_{n2} = \frac{3a_n\alpha_{n1} + a_n\alpha_{n1}}{2} = 2 \]

Similarly, we can count the number of \( n \)-node structured flowcharts where \( q \) of the nodes are of type DO-WHILE. Let formal variable \( z \) account for such nodes, and let \( F(x,z) \) be the corresponding generating function. A typical term in \( F(x,z) \) is \( b_{nq}x^n z^q \), where \( b_{nq} \) is the number of \( n \)-node structured flowcharts with \( q \) DO-WHILE nodes.

\[
F(x, z) = \frac{XF(x, z)}{2} + xzXF(x, z) + \frac{XF(x, z)F(x', z')}{2} + 1
\]

Equating coefficients on each side of (8) we get
\[ h_{n0} = 1 \quad b_{2n} = \frac{3h_{n0}h_{n1} + h_{n0}h_{n1}}{2} = 2 \]
\[ h_{2n} = \frac{h_{n0}^3 + h_{n1}^2}{2} = 1 \quad b_{21} = \frac{3h_{n0}h_{n1} + 2h_{n0}h_{n1} + h_{n0}h_{n1}}{2} = 4 \]
\[ b_{n1} = h_{n0}^3 = 1 \quad b_{n2} = 2h_{n1} \]

II. Counting Flowcharts by Height

A structured flowchart consists of a SEQUENCE of \( h = 0, 1, 2, \) etc. IF-THEN-ELSE and DO-WHILE flowcharts. \( h \) is called the height. Consider a generating function \( F(x,t) \) which accounts for height. A typical term in \( F(x,t) \) is \( c_{nht}x^n t^h \), where \( c_{nht} \) is the number of \( n \)-node flowcharts of height \( h \).

\[
F(x, t) \text{ is } \quad F(x, t) = 1 + \left[ [ITE(x) + DW(x)]^2 \right] + \left[ [ITE(x) + DW(x)]^3 \right] t^2 + \ldots
\]
or
\[
F(x, t) = \frac{1}{1 - 1 ITE(x) - 1 DW(x)} \quad (11)
\]
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Substituting the expressions for $ITE(x)$ and $DW(x)$ shown in (1) and (2), respectively, and rearranging yields

$$F(x,t) = \frac{xt F^2(x) F(x,t)}{2} + \frac{xt F(x^2 F(x,t))}{2} + xt F(x) F(x,t) + 1$$

(12)

Knowing the coefficients in $F(x)$, we can derive the coefficients of $F(x,t)$ in a manner similar to past calculations

$$c_{00} = 1$$
$$c_{11} = \frac{f_1 c_{00}}{2} + \frac{f_0 c_{10}}{2} + f_c = 2$$
$$c_{21} = \frac{2c_{00} f_1 t}{2} + f_c c_{01} = 4$$
$$c_{11} = \frac{f_0 c_{11}}{2} + f_c c_{01} = 4$$

Calculation of the Average Height of $n$-node Structured Flowcharts

From the section “Counting Flowcharts by Height”, the generating function for $n$-node height $h$ flowcharts was

$$F(x,t) = \frac{xt F^2(x) F(x,t)}{2} + \frac{xt F(x^2 F(x,t))}{2} + xt F(x) F(x,t) + 1$$

where $x$ and $t$ account for $n$ and $h$, respectively.

A typical term in $F(x,t)$ is

$$c_{i} t + c_{n_1} t^2 + \ldots + c_{n_m} t^m h$$

where $c_{ni}$ is the number of $n$-node flowcharts of height $i$.

If $F(x,t)$ is differentiated with respect to $t$, this term becomes

$$c_{i} t + c_{n_1} t^2 + \ldots + nc_{n_m} t^{n-1} h$$

Setting $t = 1$ yields

$$c_{i} + 2c_{n_1} + \ldots + nc_{n_m} g$$

The coefficient is a weighted sum in which each $n$-node flowchart with height $i$ contributes $i$.

The average depth of $n$-node flowcharts is just

$$\frac{c_{d1} + 2c_{d2} + \ldots + nc_{dn}}{f_n}$$

where $f_n$ is the number of $n$-node structured flowcharts.
Thus, we proceed by differentiating $F(x,t)$ with respect to $t$ and setting $t$ to 1.

$$F'(x,1) = \frac{\partial F(x,t)}{\partial t}\bigg|_{t=1} = \frac{x F'(x) + F'(x) F'(x,t)}{2} + x F''(x) + x F(x) F'(x,1)$$

We have

$$c_{i_1} = \frac{f_0^1 + f_0^2 + f_0 + f_0^0}{2} + \frac{f_0^1 + f_0^2 + f_0 + f_0^0}{2} + f_0 + f_0 = 2$$

Avg. ht. = $\frac{c_{i_1}}{f_1} = \frac{2}{2} = 1.0$

$$2c_{i_1} = \frac{3f_0^1 f_1 + f_0^2 c_{i_1} + 0 + f_0 f_1 + f_0 c_{i_1} + 2 f_0 f_1 + f_0 c_{i_1}}{2} = 12$$

Avg. ht. = $\frac{c_{i_1} + 2c_{i_1}}{f_1} = \frac{12}{8} = 1.5$

**Large Structured Flowcharts**

- $\mu_{ITE}$ -- fraction of flowcharts of height 1 which are IF-THEN-ELSE.
- $\mu_{DWW}$ -- fraction of flow charts of height 1 which are DO-WHILE.