Recurrence Relations - J. T. Butler

Recurrence Relations

Example: Fibonacci Sequence

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>a_n</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
</tr>
</tbody>
</table>

This originated in 1202 by Leonardo Fibonacci. He was educated in North Africa, where his father held a diplomatic post. He published Liber abaci in 1202, which introduced the Hindu-Arabic placed-valued decimal system and Arabic numerals in Europe.

Born: Pisa, Italy

The French mathematician, Edouard Lucas, used Fibonacci numbers to prove $2^{127} - 1$ is prime. This 39-digit number was the largest known until 1952 when a computer was used to find five higher primes, the largest being $2^{2281} - 1$. The Lucas sequence is

$$1, 2, 3, 5, 8, 13, ...$$

We have

$$a_n = a_{n-1} + a_{n-2}.$$

This is a recurrence relation. It can be solved in a way similar to differential equations.

Assume a solution of the form

$$a_n = A \alpha^n.$$

We have

$$0 = A \alpha^n - A \alpha^{n-1} - A \alpha^{n-2}.$$

or

$$0 = \alpha^2 - \alpha - 1.$$

This is called the characteristic equation. Solving for $\alpha$ yields

$$\alpha = \frac{1 \pm \sqrt{1 + 4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

Applying the boundary conditions gives

$$a_0 = 1 = A_1 + A_2 \frac{\sqrt{5}}{2},$$

$$a_1 = 1 = A_1 \frac{\sqrt{5}}{2} + A_2 \frac{\sqrt{5}}{2}.$$

and we have

$$a_n = A_1 \left( \frac{\sqrt{5}}{2} \right)^n + A_2 \left( \frac{-1}{2} \right)^n.$$
Solving for $A_1$ and $A_2$ yields

$$A_1 = \frac{1}{\sqrt{5}} \left( \frac{\sqrt{5} + 1}{2} \right) \quad A_2 = -\frac{1}{\sqrt{5}} \left( \frac{\sqrt{5} - 1}{2} \right)$$

Thus

$$a_n = \frac{1}{\sqrt{5}} \left( \frac{\sqrt{5} + 1}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{\sqrt{5} - 1}{2} \right)^n$$

or

$$a_n = 0.724 (1.62)^n + 0.276 (-0.62)^n$$

Note that the second term approaches 0 as $n$ approaches $\infty$.

**Properties of Fibonacci Numbers**

1. **Fibonacci Numbers in Anatomy**
   
   A study showed that the average ratio of $H/B$ for children in England was 1.62. For children in India (where height is smaller), this ratio was also 1.62.

2. **Fibonacci Numbers in Sunflower Heads**
   
   In sunflowers, the seeds of the head form two spirals. The number of seeds in each spiral is a Fibonacci number.

3. **Fibonacci Numbers in Flower Petals**
   
   - Daisy - 13 petals
   - Aarons - 5 petals
   - Hibiscus - 5 petals

4. **Fibonacci Numbers in Fruits**
   
   - Pineapple - 8-13-21 rows
   - Orange - 8 sections
5. Fibonacci Numbers in the Solar System

<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>70</td>
<td>1</td>
<td>51</td>
</tr>
<tr>
<td>Venus</td>
<td>189</td>
<td>2</td>
<td>102</td>
</tr>
<tr>
<td>Earth</td>
<td>152</td>
<td>3</td>
<td>154</td>
</tr>
<tr>
<td>Mars</td>
<td>249</td>
<td>8</td>
<td>256</td>
</tr>
<tr>
<td>Aster.</td>
<td>8</td>
<td>13</td>
<td>666</td>
</tr>
<tr>
<td>Jupiter</td>
<td>816</td>
<td>21</td>
<td>1075</td>
</tr>
<tr>
<td>Saturn</td>
<td>1504</td>
<td>34</td>
<td>1744</td>
</tr>
<tr>
<td>Uranus</td>
<td>3002</td>
<td>55</td>
<td>2817</td>
</tr>
<tr>
<td>Neptune</td>
<td>4537</td>
<td>89</td>
<td>4558</td>
</tr>
<tr>
<td>Pluto</td>
<td>7375</td>
<td>144</td>
<td>7375</td>
</tr>
</tbody>
</table>

For example, \( \frac{51.2}{7375/144} = 0.0724 \) (1.62)^2


Similar relationships are seen in the positions of the moons of the planets Jupiter and Saturn.

Let \( f_n \approx g_n \) mean \( \lim_{n \to \infty} \frac{f_n}{g_n} = 1 \)

Thus,

\[
\frac{\alpha_n}{1 + \sqrt{5} / 2} = 0.724 (1.62)\]

Note:

\[
\frac{\alpha_{n+1}}{\alpha_n} = 1 + \sqrt{5} / 2 = 1.62
\]

1.62 is the Golden Ratio or Divine Proportion. It is the ratio of the lengths of two sides of a rectangle that has the most “pleasing” appearance. It is commonly represented by \( \phi \) (phi).

The Fibonacci Square

Cut the 8 x 8 square below into four pieces as shown.

Area = 8 x 8 = 64

Rearrange the pieces to form a 5 x 13 rectangle, as shown.

Area = 5 x 13 = 65. The area has been increased by 1.
How to become rich!!

Do this with gold!

**Identities**

Fibonacci numbers satisfy many identities.

**Theorem**:  
\[ a_n a_{n-2} - a^2_{n-1} = (-1)^n \]  
(1)

*This is due to J. D. Cassini* Histoire Acad. Roy. Paris 1 (1680), 201.

**Proof**: By induction

a) (1) is satisfied for \(a_0 = a_1 = 1\) and \(a_2 = 2\).

b) Assume (1) holds. We have

- \(a_{n+1} - a_n - a_{n-1} = (-1)^{n+1}\)
- \(a_{n+1} a_{n-1} - a_n a_{n-2} = (-1)^{n+1}\)
- \(a_{n+1} a_{n-1} - a_n (a_{n-1} + a_{n-2}) = (-1)^{n+1}\)
- \(a_{n+1} a_{n-1} - a^2_n = (-1)^{n+1}\)

(1) with \(n\) replaced by \(n+1\)

Q.E.D.

**Fibonacci Sequence**

\[
\begin{array}{ccccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
 a_n & 1 & 1 & 2 & 3 & 5 & 8 & 13 \\
\end{array}
\]

\[ a_n = a_{n-1} + a_{n-2} \]

\[ a_n a_{n-2} = a^2_{n-1} + (-1)^n \]

\[ 13 \times 5 = 8^2 + (-1)^6 \]

Note that the diagonals sum to Fibonacci numbers. Thus,

\[ a_n = \sum_{k=0}^{n} \binom{n-k}{k} \]
Problem:
In a disk system, the appearance of a pulse corresponds to a 1 and no pulse to a 0. However, because of instabilities, the reading of data can become improperly synchronized if too many consecutive 0’s occur. Derive an expression for \( b_n \), the number of \( n \) bit binary numbers that contain no pair of 0’s. We will use these as data.

Solution:
We can form an \( n+1 \) bit sequence by appending a 0 or 1 to an \( n \) bit sequence. A 1 can be appended to an \( n \) bit sequence that ends in either a 0 or 1. However, a 0 can only be appended to an \( n \) bit sequence that ends in a 1 only.

Let \( b^1_n, b^0_n \) be the number of \( n \) bit binary numbers with no 00 that end in 0(1).

For \( n \geq 2 \)
\[
\begin{align*}
 b^1_{n+1} &= b^1_n + b^0_n \quad (1) \text{ ends in a 1} \\
 b^0_{n+1} &= b^1_n \quad (2) \text{ ends in a 0}
\end{align*}
\]

From (2)
\[
 b^0_n = b^1_{n-1} \quad (3)
\]

Substitute (3) into (1)

\[
b^1_{n+1} = b^1_n + b^0_n \]
This looks like the Fibonacci recurrence. In fact,
\[
\begin{align*}
 b^1_1 &= 1 \quad (1) \\
 b^1_2 &= 2 \quad (01, 11)
\end{align*}
\]

Choose \( b^1_0 = 1 \).
Thus,
\[
\begin{align*}
 b^1_n &= a_n \\
 b^0_n &= b^1_{n-1} = a_{n-1}
\end{align*}
\]

and
\[
b_n = b^1_n + b^0_n = a_n + a_{n-1} = a_{n+1} .
\]

\[
\begin{array}{cccccccc}
 n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
 b_n & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 55 \\
 2^n & 2 & 4 & 8 & 16 & 32 & 64 & 128 & 256
\end{array}
\]

Note:
\[
\lim_{n \to \infty} \frac{b_n}{2^n} = \frac{0.724(1.62)^{n+1}}{2^n} = 0 .
\]
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**Solution by Generating Function**

We have

\[ a_n = a_{n-1} + a_{n-2}. \]

Multiply both sides by \( x^n \). This yields

\[ a_n x^n = a_{n-1} x^n + a_{n-2} x^n. \]

This represents an arbitrarily large number of equalities, like so

\[ a_2 x^2 = a_1 x + a_0 x, \]
\[ a_3 x^3 = a_2 x^2 + a_1 x + a_0 x, \]
\[ a_4 x^4 = a_3 x^3 + a_2 x^2 + a_1 x + a_0 x, \]

etc.

Summing over all equations yields

\[ \sum_{n=2}^{\infty} a_n x^n = x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2}. \]

With

\[ A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots, \]

\[ A(x) - a_1 x - a_0 = x [A(x) - a_0] + x^2 A(x). \]

Solving yields

\[ A(x) = \frac{1}{1 - x - x^2} = 1 + x + 2 x^2 + 3 x^3 + 5 x^4 + 8 x^5 + \ldots. \]

These first few terms can be extracted using a polynomial manipulation package like MACSYMA.

\[ A(x) = \frac{1}{1 - x - x^2} = \frac{1}{x + x - 1}. \]

Expanding in partial fractions, yields

\[ A(x) = \frac{1}{\sqrt{5}} + \frac{-1}{\sqrt{5} - 1}. \]

From the last expression, we can write the \( n \)th term in the power series expansion of \( A(x) \) as

\[ a_n = \frac{1}{\sqrt{5}} \left( \frac{\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5} - 1} \left( \frac{1}{2} \sqrt{5} \right)^n. \]

which agrees, as it should, with the expression as derived from the recurrence relation solution.
To find known sequences\(^*\), you can send an E-mail message to sequences@research.att.com, in which the first line is, for example, "lookup 1 1 2 3 5 8 13". Sometime later, you will receive an E-mail showing the result of the search. Also visit the web page http://netlib.att.com/netlib/att/math/sloane/doc/eistop.html.


An interesting application of recurrence relations is the Fibonacci Number System

An alternative to the binary number system is the Fibonacci number system.

Recall that in the standard binary number system, an integer \( N \) is the sum of powers of 2. That is,

\[
N = b_{m-1}2^{m-1} + b_{m-2}2^{m-2} + \ldots + b_12^1 + b_02^0.
\]

In the Fibonacci number system, an integer \( N \) is the sum of Fibonacci numbers. That is,

\[
N = b_{m-1}a_m + b_{m-2}a_{m-1} + \ldots + b_1a_2 + b_0a_1,
\]

where \( a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 5, a_5 = 8, \ldots \).

Example:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>( b_2 )</th>
<th>( b_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>100</td>
<td>011</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>101</td>
<td>110</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>1000</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

A disadvantage of the Fibonacci Number System is that certain integers have more than one representative. For example, \( 100 = 011 = 3 \). However, if we restrict the binary tuples to having no pair of adjacent 1’s, then each integer has a unique representative. Such representations are useful in encoding applications. For example, due to physical limitations of CD-ROMs, data with adjacent 1’s cannot be reliably stored, and a Fibonacci-like number system is used\(^1\).

**Theorem:** Any integer \( n \) can be expressed uniquely as

\[
n = \sum_{i=1}^{m} b_i a_i ,
\]

where \( b_i = 0 \) or \( 1 \), \( b_i b_{i+1} = 0 \), \( b_m = 1 \), \( a_i \) is the \( i \)th Fibonacci number, and \( a_m \) is the largest Fibonacci number just less than or equal to \( n \).

**Proof:** By induction.

The statement holds for \( m = 3 \), e.g., \( 1 = 1 \), \( 2 = 10 \), \( 3 = 100 \), and \( 4 = 101 \).

Assume that it holds for all \( m' < m \), and consider an \( n \) such that \( a_m \) is the largest Fibonacci number \( \leq n \). Let \( b_m = 1 \), and consider \( b_{m-1} \), which is determined by \( n-b_m \).

If \( n-a_m \geq a_{m-1} \), then \( n \geq a_m + a_{m-1} = a_{m+1} \), and it follows that \( a_m \) is not the largest Fibonacci number greater than or equal to \( n \). Therefore, \( n - a_m < a_{m-1} \), and it follows that \( b_{m-1} = 0 \).

By the inductive assumption, \( n - a_m \) has a unique representation of the form described in the problem statement. Since \( b_m b_{m-1} = 10 \), so does \( n \).

Q.E.D.

**Solving recurrence relations**

This is similar to the process of solving differential equations.

**Example:** Solve the difference equation

\[
a_n + 2a_{n-1} = n + 3 ,
\]

subject to the initial condition

\[
a_0 = 3 .
\]

First, solve for the homogeneous solution

\[
a^{(h)}_n + 2a^{(h)}_{n-1} = 0
\]

\[
a^{(h)}_n = -2a^{(h)}_{n-1} .
\]

Assume a solution of the form

\[
a^{(h)}_n = A \alpha^n .
\]

Substituting yields

\[
A \alpha^n = -2A \alpha^{n-1}
\]

\[
\alpha = -2 .
\]

Thus,

\[
a^{(h)}_n = A (-2)^n .
\]

Assume a particular solution of the form

\[
a^{(p)}_n = B n + D .
\]

Substituting this yields

\[
\]
which yields

\[ 3Bn + 3D - 2B = n + 3. \]

Comparing coefficients yields

\[ 3B = 1 \quad \text{and} \quad 3D - 2B = 3. \]

That is,

\[ B = \frac{1}{3} \quad \text{and} \quad D = \frac{11}{9}. \]

and so

\[ a_n^{(p)} = n + \frac{11}{9}. \]

The total solution is the sum of the homogeneous and particular solution.

\[ a_n = a_n^{(h)} + a_n^{(p)} = A (-2)^n + \frac{n}{3} + \frac{11}{9}. \]

Applying the boundary condition \( a_0 = 3 \) yields \( A = 16/9 \) and so

\[ a_n = \frac{16}{9} (-2)^n + \frac{n}{3} + \frac{11}{9}. \]

When the particular and homogeneous solutions are not linearly independent

This is similar to the process of solving differential equations under similar conditions.

**Example:** Solve the recurrence relation

\[ a_n = 2a_{n-1} + 2^n - 1 \]

subject to the initial condition \( a_0 = 0 \).

First, solve for the homogeneous solution

\[ a_n^{(h)} - 2a_{n-1}^{(h)} = 0 \]
\[ a_n^{(h)} = 2a_{n-1}^{(h)}, \]

which has a solution of the form

\[ a_n^{(h)} = A \cdot 2^n. \]

Assume a particular solution of the form

\[ a_n^{(p)} = B2^n + D. \]

Substituting this yields

\[ B2^n + D = 2B2^{n-1} + D + 2^n - 1. \]

Note that we cannot solve for \( B \); it drops out of both sides of the equation.

Assume now a particular solution of the form

\[ a_n^{(p)} = Bn2^n + D. \]

Substituting this yields

\[ [Bn2^n + D] = 2\cdot [B(n-1)2^{n-1} + D] + 2^n - 1. \]

Equating coefficients yields the following.

\[ Bn2^n = 2Bn2^{n-1} - 2B \cdot 2^{n-1} + 2^n \quad \text{and} \quad D = 2D - 1. \]
Therefore, $B = 1$ and $D = 1$, and we have $a_n = a(n) + d(n) = 2^n + n2^n + 1$. Solving for $A$ yields $a_0 = 0 = A1 + 0 + 1$, and $A = -1$. Therefore, $a_n = (n-1)2^n + 1$.

**Domain and range transformation**

Applies to recurrence relations that are not linear.

Domain $n = 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6$

Range $a_n = 1 \ 1 \ 2 \ 3 \ 5 \ 8 \ 13 \ ...$

**Range Transformation**

Consider $a_n = 3a_{n-1}$ for $n \geq 1$ and $a_0 = 1$. Let $b_n = \log_2 a_n$. Thus, $b_n = \log_2 a_n = 2\log_2 a_{n-1} + \log_2 3 = 2b_{n-1} + \log_2 3$ with $b_0 = 0$.

Solving $b_n = 2b_{n-1} + \log_2 3$ using methods described earlier yields $b_n = (2^n - 1)\log_2 3$ or $a_n = 2^{(2^n - 1)\log_2 3} = 3^{2^n - 1}$.

**Domain Transformation**

The merge sort divides a set of $n$ numbers in half, sorts each half, then merges each half together using at most $n-1$ comparisons. If $n$ is a power of 2, an upper bound $t_n$ on the number of comparisons is $t_n = 2t_{n/2} + n - 1$ for $n \geq 2$

$t_1 = 0$.

Let $n = 2^k$ and $a_k = t_n$. Thus, $a_k = 2a_{k-1} + 2^k - 1$ for $k \geq 1$

$a_0 = 0$. Solving this yields $a_k = -2^k + k2^k + 1 = (k-1)2^k + 1$.

From this, we obtain $t_n = (\log_2 n - 1)\log_2 n + 1$. 

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10
Distinct Recurrence Relations Generate the Same Sequence


The number of “p-equivalent” classes of cascade functions $d_n$ on $n$ variables is given by the recurrence relation

$$d_n = \sum_{r=2}^{n-1} rd_{n-r-1} + n + 1 \quad n \geq 3,$$  \hspace{1cm} (1)

where $d_0 = 0$, $d_1 = 1$ and $d_2 = 3$. Rearranging yields

$$2d_n = \sum_{r=1}^{n} rd_{n-r-1} + 1 \quad n \geq 1.$$  \hspace{1cm} (2)

The range of (2) is noted to now apply to $n = 1$ and $n = 2$.

Letting $i = r - 1$ yields,

$$2d_n = \sum_{i=0}^{n-1} (i+1)d_{n-i} + 1.$$  \hspace{1cm} (3)

and substituting $o_i = i + 1$ yields

$$2d_n = \sum_{i=0}^{n-1} o_id_{n-i} + 1.$$  \hspace{1cm} (4)

Multiplying (4) by $x^n$ and summing yields

$$2 \sum_{n=1}^{\infty} d_n x^n = \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} o_i d_{n-i} x^n + \sum_{n=1}^{\infty} x^n.$$  \hspace{1cm} (5)

Let

$$O(x) = 1 + 2x + 3x^2 + \ldots = \frac{1}{1 - x}.$$  \hspace{1cm} (6)

We have

$$2D(x) = O(x)D(x) + \frac{x}{b - xg}.$$  \hspace{1cm} (7)

Rearranging (6) yields

$$D(x) = \frac{x - x^2}{1 - 4x + 2x^2} = x + 3x^2 + 10x^3 + 34x^4 + \ldots.$$  \hspace{1cm} (8)

Equating coefficients in (8) yields

$$d_n = 4d_{n-1} - 2d_{n-2} \quad n \geq 3.$$  \hspace{1cm} (9)

Comparing (1) and (9) shows a very interesting relation. The recurrence relation expressed in the sum of (1) describes exactly the recurrence relation of (9) without sums!