Example

- An important topic in computer science is graphs and trees. These are useful, for example, in data structures.

Note:
“tree” --- a connected graph with no circuits
“binary” --- at most two edges occur at each node
“ordered” --- order is counted
These count as 2 not 1.

$r$ is the number of leaves

Let $T(x)$ be the ordinary generating function for $t_r$, the number of ordered binary trees with $r$ leaves

$$T(x) = t_0 + t_1 x + t_2 x^2 + t_3 x^3 + \ldots$$
We assume \( t_0 = 0 \). That is, there are no trees with 0 nodes. We form an ordered binary tree as follows:

\[
0 = T^2(x) - T(x) + x
\]

\[
T(x) = \frac{1 \pm \sqrt{1 - 4x}}{2} = \frac{1 \pm 1}{2}(1 - 4x)^{1/2}
\]

\[
= 1 + \sum_{r=1}^{\infty} \frac{(1/2)(1/2)(-1/2)(-1/2)\ldots (-4)^r x^r}{r!12(-1/2)^{r-1}}
\]

\[
= 1 - \sum_{r=1}^{\infty} \frac{1}{2r^2} \cdot 1 \cdot 3 \cdot \ldots (2r-3) \cdot 4^r x^r
\]

Note: The numerator is 1 when \( r = 1 \), not -1.

To form a tree, choose one tree for \( L \) and one for \( R \). Each choice is expressed as \( T(x) \), giving \( T^2(x) \). +x occurs because the single tree with one leaf is not counted in \( T^2(x) \).

Rewrite (1) and solve for \( T(x) \).

Apply the binomial theorem

\[
(1 + x)^n = 1 + \sum_{r=1}^{\infty} \binom{n}{r} x^r
\]

to get

\[
(1 - 4x)^{1/2} = 1 + \sum_{r=1}^{\infty} \binom{1/2}{r} 4^r x^r
\]
Of the two solutions for \( T(x) \), choose the negative one, so that

\[
T(x) = \frac{1}{2} \frac{1}{2} (1 - 4x)^{\frac{1}{2}}
\]

Thus, the number of ordered binary trees with \( r \) leaf nodes is

\[
t_r = \frac{1}{r} \binom{2r}{r} - 1 \quad r \geq 1
\]

\[
t_0 = 0
\]

We have

\[
\begin{array}{cccccccc}
\text{r} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
t_r & 1 & 1 & 2 & 5 & 14 & 42 & 132
\end{array}
\]

There are called “Catalan” numbers after Eugene Charles Catalan, a mathematician from Belgium. In 1838, he solved the following problem:

How many ways are there to place parentheses around \( r \) letters so that inside each pair there are two terms?

Example:

<table>
<thead>
<tr>
<th>Ways to place parentheses</th>
<th>Number of ways</th>
</tr>
</thead>
<tbody>
<tr>
<td>((ab))</td>
<td>1</td>
</tr>
<tr>
<td>(((ab)(cd)))</td>
<td>2</td>
</tr>
<tr>
<td>(((ab)(cd)),((ab)c)(ab))</td>
<td>5</td>
</tr>
</tbody>
</table>

Catalan numbers also count ways to divide polygons into triangles and ways a rook can go from the lower left corner of a half chessboard to the upper right corner.
Generating Function Example - Ordered Binary Trees - J. T. Butler

Number of ways to divide a polygon with $r$ sides into triangles

1 2 5

Number of ways to go from the lower left corner of a half chessboard to the upper right corner

1 2

5