Dollar Paradox

Consider the following game:
Mr. Smith flips a coin. If it is heads, he pays you $1.00. If it is tails, he flips again. If it is heads, he pays you $2.00. If it is tails, he flips again. If it is heads, he pays you $4.00, etc.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>You receive</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>$1.00</td>
</tr>
<tr>
<td>TH</td>
<td>$2.00</td>
</tr>
<tr>
<td>TTH</td>
<td>$4.00</td>
</tr>
<tr>
<td>TTTH</td>
<td>$8.00</td>
</tr>
</tbody>
</table>

However, you must pay to play this game. **What should you be willing to pay?**

Of course, if it costs $1.00, you should play; you will receive back at least $1.00. However, should you pay $10,000, for example?

Yes, you should be willing to pay any amount. Why? You can expect to win
\[
\frac{1}{2} \times 1.00 + \frac{1}{4} \times 2.00 + \frac{1}{8} \times 4.00 + \ldots,
\]
which, even with inflation, is a large amount of money.

Combinations with Repetition

**Question:**
How many ways can one choose \( r \) objects from \( n \) distinct objects when repetition is allowed? That is, one can choose any object 0, 1, 2, ... etc. times.

**Example:**
If \( r = 2 \) and \( n = 3 \), then this number is 6. Let \( S = \{a,b,c\} \). Then, one can choose \{a,b\}, \{a,c\}, \{b,c\}, \{a,a\}, \{b,b\}, and \{c,c\}. If repetition is not allowed, this number is 3 - \{a,b\}, \{a,c\}, and \{b,c\}. 


Answer:
Each object can be chosen 0, 1, 2, ... times. A choice of \( r \) objects can be represented as follows. Arrange \( n + r - 1 \) objects in a row, and choose \( n-1 \) as dividing lines.

The \( r \) remaining objects represent the choice with repetition. For example, for \( r = 2 \) and \( n = 3 \), we have

\[
\begin{align*}
0 \phi 0 \phi & \Leftrightarrow ab \\
0 \phi 0 & \Leftrightarrow ac \\
\phi 0 & \Leftrightarrow bc
\end{align*}
\]

\( \phi \) are the dividing lines. Choose them as shown above.

Example:
Out of a large number of 1¢, 5¢, 10¢, and 25¢ coins, in how many ways can six coins be chosen?

This is the same as selecting six coins from 1¢, 5¢, 10¢, and 25¢ with repetition or

\[
\binom{4 + 6 - 1}{6} = \binom{9}{6} = 84.
\]

Example:
When three distinct dice are rolled, the number of outcomes is \( 6 \cdot 6 \cdot 6 = 216 \).

Here, 1 2 1 ≠ 2 1 1

How many outcomes are there if the 3 dice are indistinguishable (we do not specify that one outcome occurred on the first roll, second roll, and third roll?)

This is the same as selecting 3 numbers from 6 with repetition or

\[
\binom{6 + 3 - 1}{3} = \binom{8}{3} = 56.
\]
**Question:**
How many ways are there to go from the southwest corner of a chessboard to the northeast corner by making only north and east moves?

![Chessboard diagram](Diagram)

**Answer:**
To go from A to B, make 7 north and 7 east moves. Once you choose the north moves, the east moves are determined. The north moves can be chosen in

\[ C(14,7) = \frac{14!}{7! \cdot 7!} = 3432 \]

ways.

**Question:**
How many of these 3432 ways consist of four east move segments separated by three north move segments? A segment is one, two, etc. moves in the same direction.

**Answer:**
The number of ways of choosing a path of four east move segments is the same as placing seven indistinguishable balls in four boxes with no box left empty.

This is the same as distributing three balls in four boxes with repetition. There are

\[ C(4+3-1,3) = \frac{6!}{3! \cdot 3!} = 20 \]

ways to do this.

Similarly, the number of ways to choose three north segments is

\[ C(3+4-1,4) = \frac{6!}{4! \cdot 2!} = 15 \]

Therefore, the answer is

\[ 20 \cdot 15 = 300. \]
Example

How many solutions are there to the equation
\[ x_1 + x_2 + x_3 + x_4 = 7, \text{ where } x_i \geq 0 \text{ for } 1 \leq i \leq 4? \]

Imagine there are four cells representing \( x_1, x_2, x_3, \) and \( x_4. \) Consider some distribution of 7 balls to these cells. Interpret the number of balls in a cell as the value of \( x_i. \) We can conclude that the number of solutions is the number of ways to distribute 7 balls to 4 cells, where each cell can have 0, 1, 2, ..., 7 balls. This is just the number of ways to select 7 objects from 4 with repetition or \( C(4 + 7 - 1, 4 - 1) = 120. \)

In general, the following are identical.

a) The number of integer solutions of the equation
\[ x_1 + x_2 + \ldots + x_n = r, \text{ where } x_i \geq 0 \text{ and } 1 \leq i \leq n. \]
b) The number of ways to select \( r \) objects from \( n \) distinct objects with repetition.
c) The number of ways to distribute \( r \) nondistinct objects to \( n \) distinct cells.

Example:

How many integer solutions are there to the equation
\[ x_1 + x_2 + \ldots + x_6 = 10, \text{ where } x_i \geq 0 \text{ for } 1 \leq i \leq 6? \] How many such solutions are there to \( x_1 + x_2 + \ldots + x_6 < 10? \)

From our previous discussion, there are \( C(6 + 10 - 1, 6 - 1) = 3003 \) solutions. We can view the nonnegative solutions of \( x_1 + x_2 + \ldots + x_6 < 10 \) as equivalent to the solutions of \( x_1 + x_2 + \ldots + x_6 + x_7 = 9, \) where \( x_i \geq 0, \) for \( 1 \leq i \leq 7. \) In effect, \( x_7 \) is a "slack" variable. The number of solutions to this equation is \( C(7 + 9 - 1, 7 - 1) = 5005. \)

Example

- Some problems can be viewed either as a permutation or as a selection. Consider
- Four volleyball teams A, B, C, and D, each with 9 players, must be made from 36 players. In how many ways can this be done?
Selection: To form Team A, select 9 players from the 36 players in \( \binom{36}{9} \) ways. To form Team B, select 9 players from the remaining 27 players in \( \binom{27}{9} \) ways. Similarly, form Teams C and D in \( \binom{18}{9} \) ways. By the rule of product, the four teams can be formed in

\[
\frac{36!}{9!9!9!9!} \approx 2.145 \times 10^{19}
\]

Permutation: View the 36 players as arranged in a line. We ask "How many ways can we arrange 9 A’s, 9 B’s, 9 C’s, and 9 D’s?, for each arrangement corresponds to an assignment to the four teams. This is the problem of counting permutations of non-distinct objects.

\[
\frac{36!}{9!9!9!9!} \approx 2.145 \times 10^{19}
\]