Rule of Sum and Product

Rule of sum:
If one event can occur in \( m \) ways and another event can occur in \( n \) ways, there are \( m + n \) ways in which exactly one event can occur.

Example

- Suppose there are 5 red balls and 10 green balls. Then, there are \( 5 + 10 = 15 \) ways to choose a ball of any color.

Rule of Product:
If one event can occur in \( m \) ways and another event can occur in \( n \) ways, then there are \( m \cdot n \) ways the two events can occur together.

Example

- Suppose there are 5 red balls and 10 green balls. Then, there are \( 5 \cdot 10 = 50 \) ways to choose a red ball AND a green ball.

Example

The address register in your PC has 24 bits. How many locations can your PC access?

Each bit can be chosen independently in 2 ways. By the rule of product, there are \( 2 \cdot 2 \cdot ... \cdot 2 = 2^{24} = 16,777,216 \) or 16 megabytes.

Permutations

How many ways \( P(n,n) \) are there to arrange \( n \) of \( n \) objects?

By the rule of product
\[
P(n,n) = n \times (n-1) \times (n-2) \times ... \times 2 \times 1
\]

\( n \) ways to choose 1st object
\( n-1 \) ways to choose 2nd object
\( n-2 \) ways to choose 3rd object
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\[ n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1 = n! \]

“\( n \) factorial”

**Also,**

\[ P(n, r) = n \times (n-1) \times (n-2) \times \ldots \times (n-r+1) \]

\[ = \frac{n!}{(n-r)!} \]

This is called *permutation.*

Also, \( P(n, r) = n \times (n-1) \times (n-2) \times \ldots \times (n-r+1) \) can be viewed as a distribution of balls to cells

\[ r \text{ positions (cells)} \]

\[ \begin{array}{c}
\vdots \\
\end{array} \]

\[ n \text{ ways to fill the 1st position} \\
\begin{array}{c}
\vdots \\
\end{array} \]

\[ n-r+1 \text{ ways to fill the rth position} \]

**Notation:**

\( P(n, r) \) is the number of permutations of \( r \)

objects from a set of \( n \) distinct objects

Alternative notation:

\[ P^n_r = n! P_r \left( = P(n, r) \right) \]

**Remember**

Given a set \( S \) of \( n \) distinct objects, a *permutation* is an arrangement or ordering of objects from \( S \).

**Example**

How many permutations are there of two elements of \( S=\{a,b,c,d\} \)?

There are 12: \( ab, ba, ac, ca, ad, da, bc, cb, bd, db, cd, \) and \( dc \).

\[ P(4, 2) = 4 \times 3 = 12. \]

**Example**

How many ways are there to arrange the three letters in \( ALL \)?

If all letters were distinct, like \( AL_1L_2 \), there be \( 3! = 6 \). But two of letters are identical.

\[ \begin{array}{ll}
ALL & \leftrightarrow AL_1L_2 \\
LAL & \leftrightarrow L_1AL_2 \\
LLA & \leftrightarrow L_2L_1A \\
\end{array} \]

If \( a \neq b \neq c \neq d \) then \( 4! = 24 \) ways.
We observe that:
2!(the number of arrangements of ALL) = number of arrangements of AL₁L₂. That is, the number of arrangements of ALL
\[ \frac{3!}{2!} = 3. \]

In general,
If there are \( n_i \) objects of the first type, \( n_2 \) objects of the second type, \( \ldots \), \( n_r \) objects of the \( r \)-th type, such that the total number \( n \) of objects \( n = n_1 + n_2 + \ldots + n_r \), then the number of arrangements of the \( n \) objects is
\[ \frac{n!}{n_1! n_2! \ldots n_r!}. \]

Example
If six people, A, B, C, D, E, and F are seated about a round table, how many circular arrangements are possible?

We can arbitrarily choose one person, say A, to be at the top. Then, we can choose the order of the remaining ones in 5! ways. Thus, the number of circular permutations on 6 objects is 5!.

Suppose that A, B, and C are females, and D, E, and F are males. We now want to arrange them so the sexes alternate. How many ways are there to place the people?

Choose A, a female to be in top position. There are 3! ways to arrange the males and 2! ways to arrange the remaining females or 3! \times 2! = 12 ways.
Combinations

How many ways are there to arrange 5 red balls and 10 green balls?

There are 15 objects, 5 of the first kind and 10 of the second. Thus, there are 
\[
\frac{15!}{5!10!}
\]
arrangements.

Suppose we want the number of ways to choose 5 objects from 15 distinct objects. Notice, there is a one-to-one correspondence between an arrangement of 5 red balls and 10 green balls and a selection of 5 objects from 15. Let the red balls signify which object is selected.

Let \( C(n, r) \) be the number of ways to select \( r \) objects from \( n \). Then, we saw

\[
C(15,5) = \frac{15!}{5!10!}
\]
In general, \( C(n, r) = \frac{n!}{r!(n-r)!} \).

Alternative words:

\( C(n, r) \) is the number of ways to choose \( r \) objects from \( n \) distinct objects.

\( C(n, r) \iff \text{“} n \text{ choose } r \text{”} \)

Alternative notation:

\( C_r^n = \binom{n}{r} = C(n, r) \)

Notes:

1. \( C_r^n = \frac{n!}{r!(n-r)!} \)
2. When one chooses an object, it cannot be chosen again. This is choice without repetition.

Note

Given a set \( S \) of \( n \) distinct objects, a combination is a subset \( S' \) of \( S \).

\( \text{combination} \iff \text{selection} \)

Given a set \( S \) of \( n \) distinct objects, a permutation is an arrangement of elements.

\( \text{permutation} \iff \text{arrangement} \)
**Question:**
Let \( S = \{a,b,c,d\} \). Then \( \{a,b\} \) and \( \{c,d\} \) are combinations. How many combinations of size 2 are there of \( S \)?

**Answer:**
\( \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \) and \( \{c,d\} \) are all of the combinations of \( S \) of size 2. There are 6. There are 12 permutations, two for each combination above.

**Rule of sum:**
\[
C(n, r) = C(n-1, r) + C(n-1, r-1)
\]
Choose \( r \) objects from \( n-1 \) objects not including a special object.
Choose the special object and \( r-1 \) other objects from \( n-1 \) objects not including the special object.

**Pascal’s Triangle**

<table>
<thead>
<tr>
<th></th>
<th>Sum</th>
<th>Diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1 1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>1 2 1</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>1 3 3 1</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>1 4 6 4 1</td>
<td>16</td>
<td>0</td>
</tr>
<tr>
<td>1 5 10 10 5 1</td>
<td>32</td>
<td>0</td>
</tr>
<tr>
<td>1 6 15 20 15 6 1</td>
<td>64</td>
<td>0</td>
</tr>
</tbody>
</table>

**How to build Pascal’s triangle**
1. Write the 1’s
   
   \[
   \begin{array}{ccccccc}
   & & & & & & 1 \\
   & & & & 1 & & 1 \\
   & & & 1 & & 1 & \\
   & 1 & & 1 & & & \\
   \end{array}
   \]

2. Fill in the remaining numbers by adding pairs
   
   \[
   \begin{array}{ccccccc}
   & & & & & & 1 \\
   & & & & 1 & & 1 \\
   & & & 1 & & 1 & \\
   1 & & 1 & & & & \ldots \\
   \end{array}
   \]

**Brief history**
This was first published in 1653 by Pascal in one of the first publications on probability.
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**Note:**

Pascal’s Rule is used to derive the entries of this table. That is, \( C(0,0) = 1 \)

and

\[
C(n, r) = C(n - 1, r) + C(n - 1, r - 1).
\]

**Note:**

\( C(n, r) \) is the number of ways to choose \( r \) objects from \( n \) distinct objects.

\( C(n, r) \) is called a “binomial coefficient”.

Binomial coefficients satisfy thousands of identities.

**Naming the Ace Paradox**
A Deck of Cards is a set of 52 cards
Each card as a value
2 3 4 5 6 7 8 9 10 J Q K A
where J = Jack, Q = Queen
K = King, and A = Ace
and a suit
Hearts Diamonds Spades Clubs

EXAMPLE:
There are four aces
ace of hearts
ace of diamonds
ace of spades
ace of clubs
Similarly, there are four kings, four queens, etc.

Definition:
“To deal 52 cards to 4 people” means to randomly give 13 cards to 4 people. Usually, a person knows only those cards given to him/her.

Definition:
A hand is a set of 13 cards given to one person.

Question: Naming the ace
52 cards are distributed to 4 people. One person says “I have an ace.” What is the probability he/she has another ace?

Answer: $5359/14498 < \frac{1}{2}$.
However, suppose he/she says, “I have an ace of spades.” What is the probability he/she has another ace?

Answer: $11686/20825 > \frac{1}{2}!$
Small Deck

Shown below is a much smaller deck. In this case, we give two cards each to two people.

Below are the six different hands with two cards

1  
2  
3  

There are 6 ways a person can have a hand of two cards, as shown. If someone says, “I have an ace”, the probability he/she has another ace is 1/5 (there are 5 hands with at least one ace, only one of which has another ace). If he/she says, “I have an ace of spades,” then the probability he/she has another ace is 1/3.

Children Paradox

Mr. Smith has two children. At least one is a boy. What is the probability that the other is a boy also?

There are four equally likely outcomes

<table>
<thead>
<tr>
<th>Older</th>
<th>Younger</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boy</td>
<td>Boy</td>
</tr>
<tr>
<td>Boy</td>
<td>Girl</td>
</tr>
<tr>
<td>Girl</td>
<td>Boy</td>
</tr>
<tr>
<td>Girl</td>
<td>Girl (impossible)</td>
</tr>
</tbody>
</table>

Since only one of three possible outcomes consists of two boys, the probability is 1/3!

Playing Keno

Question: Keno is a game in which you choose 6 distinct numbers from 1 to 80. The casino chooses 20 distinct numbers from 1 to 80. You enter a choice of numbers for $0.60 and the casino pays you an amount of money depending on how many of the 6 numbers you have chosen matches the numbers chosen by the casino.
For example, you win $1,000 if all six numbers you have chosen were also chosen by the casino. What is the probability $P(k)$ that exactly $k$ of the numbers you have chosen were chosen by the casino, for $0 \leq k \leq 6$?

**Answer:**

Number of ways $k$ of 80 numbers can match

Number of ways casino can choose remaining numbers

Number of ways player can choose remaining numbers

\[
P(k) = \frac{\binom{6}{k} \binom{70}{8-k}}{\binom{80}{8}}
\]

Total number of ways casino can choose numbers

Total number of ways player can choose numbers

This table shows the exact probability, as calculated from (1). Also shown is the probability as extracted during a class experiment in which 40 students chose 6 numbers and the instructor chose 20.

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**How to Become Rich by Playing Keno**

You cannot. Spend your money somewhere else.