multiprocessor system diagnosis
with three-valued test outcomes*

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1. Introduction

The traditional application of multiple-valued logic to digital hardware has been at the circuits level. In this chapter, multiple-valued logic concepts are applied at the systems levels, and specifically to the diagnosis of faulty processors in a multiprocessor system. System diagnosis is the process of determining faulty processors in a set of interconnected processors given the outcomes of tests performed by processors on other processors. If the number of faulty processors exceeds a certain threshold, called the diagnosability, the incorrect information supplied by such processors can make it impossible to correctly identify all faulty processors. A problem of significant importance, therefore, is to determine the diagnosability of a system from a description of that system.

In binary system diagnosis, there are two values for the test outcome, 0 (pass) and 1 (fail). Considered here are two types of ternary system diagnosis, where the third value represents either missing test results (2) or incorrect test results due to intermittent faults (0'). It is shown that the diagnosability of a system with respect to either type of three-valued test outcome can be determined from its two-valued behavior provided no two units test each other. However, for multiprocessor systems where two processors test each other, such a strong statement cannot be made. The significance of this result is that for systems where no two units test each other, diagnosability with either missing or incorrect information, can be determined from its two-valued diagnosability. This, in turn, can be determined from easily calculated system parameters. However, when there exists a pair of processors which test each other, the determination of diagnosability is not quite so simple.

Diagnosability of multiprocessor systems with two-valued test outcomes has been well-studied. Preparata, Metze, and Chien [7] showed that, in such systems, if the number of permanently faulty processors exceeds a value \( t \), it may be impossible to determine uniquely all faulty processors, whereas if the number of faulty processors is \( t \) or less, correct identification is always possible. Systems of this type are said to be \( t \)-diagnosable.

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The diagnosis problem is complicated if certain test outcomes are missing, as is shown by Butler [1]. A system is \( t_2/r \)-diagnosable if all permanently faulty processors can be uniquely identified, providing there are \( t' \) or fewer faulty processors and \( r \) or fewer missing test outcomes. The subscript 2 corresponds to the third outcome possible in this model; it represents the fact that a pass or fail outcome is missing. Such outcomes occur because either the testing or tested processors are busy and the test cannot be completed before the beginning of the diagnosis. Such test outcomes also are possible when faults prevent the proper transmission of test results.

Diagnosis is also complicated if certain processors are intermittently faulty, as shown by Mailela and Mason [5, 6]. In this model, a faulty processor \( u_t \) may pass a test by a good processor \( u_t \) because \( u_t \) was, in fact, fault-free at the time the test was applied. A system is \( t_2/r \)-diagnosable if all faulty processors can be uniquely determined provided there are \( t' \) or fewer faulty processors and \( r \) or fewer pass test outcomes from tests of faulty processors by fault-free processors. Such outcomes are called incorrect 0's and are denoted as 0'. When the diagnosis is performed, any 0' is interpreted as a 0, i.e. the primes are transparent to the diagnosis algorithm. In the Preparata, Metze, and Chien model of permanently faulty processors such test results are always 1 (fail) and diagnosis is easier as a result.

Preparata, Metze, and Chien [7] showed that \( t \)-diagnosable systems must satisfy the following two conditions:

1. Each processor is tested by at least \( t \) other processors, and
2. \( n \geq 2t + 1 \),

where \( n \) is the total number of processors. Hakimi and Amin [3] showed these conditions are also sufficient in systems where no two processors test each other. They also showed a set of necessary and sufficient conditions for a general system to be \( t \)-diagnosable. This consists of the two conditions above as well as a third condition which is generally difficult to check. Thus, systems in which no two processors test each other have a special place in the theory of diagnosability.

2. Notation and Introductory Comments

A system is a weighted, directed graph where the set of nodes \( \{u_0, u_1, \ldots, u_n\} \) correspond to processors and the arcs between nodes correspond to tests. \( u_t \) tests \( u_t \) if there exists a directed arc from \( u_t \) to \( u_t \). The test outcome is 0 or 1 if the test result is pass or fail, respectively, and is 2 if the test outcome is missing. Test outcomes are produced according to Fig. 1. \( t \)-diagnosability in Fig. 1(a) is shown as \( (t' + r) \)-diagnosability for reasons which will be clear later. For this case, a fault-free processor will always produce a correct test outcome. That is, the test by a fault-free processor will be 0 (pass) if the tested processor is fault-free and 1 (fail) if it is faulty. A fault-free processor will produce either a 0 or 1 outcome on any processor it tests regardless of whether it is faulty or fault-free.

Fig. 1(b), corresponding to \( t_2/r \)-diagnosability, shows that a 2 (missing) test outcome can occur in any test situation. Except for this, Fig. 1(b) is the same as Fig. 1(a).
Fig. 1(c) shows that diagnosability under the assumption of intermittent faults is identical to the case of permanent faults except when a fault-free processor tests a faulty processor. When this happens, an incorrect 0, 0', can occur.

As an example of these ideas, consider the system $S_t$ in Fig. 2. In Fig. 2(a) a missing test outcome occurs on the test of faulty unit $u_0$ by fault-free processor $u_t$. An examination of the remaining test outcomes will show they are consistent with Fig. 1(b). Fig. 2(b) shows a possible set of test outcomes for the case where an intermittent fault occurs. The outcomes are consistent with Fig. 1(c).

During normal diagnosis it is, of course, not known a priori that $u_0$ is faulty and the other units are fault-free. However, if we assume there can be no more than one
faulty processor, it is possible to determine that \( u_0 \) is indeed faulty in Fig. 2(a). This follows from the two following observations. In this three processor system,

1. the single faulty processor should fail both tests applied to it and,
2. the two fault-free processors should pass the tests applied to each other.

The four tests covered in the two conditions are distinct and so, even if one is missing, there is still enough information to uniquely identify the faulty processor. In fact, the missing test outcome of the test \( u_0 \) by \( u_1 \) should be 1 (fail). System \( S_1 \) allows the detection of a single faulty processor even when one test outcome is missing. It follows that \( S_1 \) is 1/1-diagnosable.

Consider now the case where a single intermittent fault can occur as in Fig. 2(b). During diagnosis, the 0' test outcome of the test of \( u_0 \) by \( u_1 \) is seen as a normal 0 (pass) test outcome. In this context, there would be no difference in the set of test outcomes if \( u_2 \) was intermittently faulty and \( u_0 \) was fault-free. Thus, there is an ambiguity. Consequently, correct identification of the faulty processor is not possible and it follows that \( S_1 \) is not 1/1-diagnosable.

3. Diagnosis in general systems

The results of this section are summarized in Fig. 3, where the three types of diagnosability are represented as circles. Implication is indicated by a double arrow. For example, the arrow on the right indicates that \( \tau' + \tau \)-diagnosability in any system \( S \) implies that \( S \) is also \( \tau \)-diagnosable. The absence of an arrow indicates the implication does not exist for some choice of \( \tau' \) and \( \tau \). For example, the absence of an arrow from the top circle to the bottom left circle indicates there exists a
Fig. 3. Summary of results for general systems.

$t'/\tau$-diagnosable system which is not $t_0/\tau$-diagnosable. System $S_1$ of Fig. 2 is, in fact, an instance of this case. Thus, we have

**Observation 1** [2]. There exists a system $(S_i)$ which is $t'/\tau$-diagnosable but not $t_0/\tau$-diagnosable, for some $t', \tau \geq 1$.

Fig. 2 can also be used to show that $t'/\tau$-diagnosability in a system does not imply $(t' + \tau)$-diagnosability. Consider the case where all test outcomes in $S_1$ are 1. Such a set of test outcomes can be produced by any arrangement of two permanently faulty processors. Thus, $S_1$ is not 2-diagnosable, and so

**Observation 2** [1]. There exists a system $(S_i)$ which is $t'/\tau$-diagnosable but not $(t' + \tau)$-diagnosable, for some $t', \tau \geq 1$.

Consider now the question of whether $(t' + \tau)$-diagnosability implies either $t'/\tau$- or $t_0/\tau$-diagnosability. The following theorem is an affirmative answer to the first part.

**Theorem 1** [1]. If $S$ is $t$-diagnosable, then $S$ is $t'/\tau$-diagnosable, where $t' + \tau = t$.

**Proof.** On the contrary, assume there is a system $S$ which is $t$-diagnosable but not $t'/\tau$-diagnosable for $t' + \tau = t$. Since $S$ is not $t'/\tau$-diagnosable, there are two distinct fault patterns $FP_s$ and $FP'_s$ each with no more than $t'$ permanently faulty processors that produce the same set $\sigma_s$ of test outcomes containing $\tau$ or fewer 2's. It will be shown that this implies the existence of two fault patterns $FP_s$ and $FP'_s$ each with no more than $t$ permanently faulty processors, which produce the same set $\sigma_s$ of test outcomes containing no 2 test outcomes. Thus, $S$ is not $t$-diagnosable as assumed.

The faulty processors of $FP_s$ and $FP'_s$ are chosen as follows.

$$FP_s = FP_s \cup \emptyset \quad \text{and} \quad FP'_s = FP'_s \cup \emptyset,$$
where \( \theta \) is the set of processors neither in \( FP_s \) nor \( FP'_s \) which apply tests producing 2 outcomes. Note that \( FP_s \) and \( FP'_s \) each contain no more than \( t \) permanently faulty processors.

\( \sigma_s \) is the same as \( \sigma_a \) with the following exceptions:

1. A 2 produced by a test applied by a processor in both \( FP_s \) and \( FP'_s \) is replaced by either 0 or 1.

2. A 2 produced by a test \( \rho \) applied by a processor in \( \neg FP_s \cap FP'_s \) (\( FP_s \cap \neg FP'_s \)) is replaced by a 0 or 1 depending on whether the processor to which it is applied is in \( FP_s \) (\( FP'_s \)) or not, respectively.

3. A 0 produced by a test of a processor in \( \theta \) by a processor in \( \neg FP_s \cup \neg FP'_s \) is replaced by a 1.

Note that \( \sigma_s \) is a valid syndrome for both \( FP_s \) and \( FP'_s \).

This completes the proof.

It should be noted that the diagnosability of a system with missing test outcomes may be better than that indicated by Theorem 1. Recall that \( S_1 \), the system of Fig. 2, is at most 1-diagnosable. Yet, it is also 1\( \| \)1-diagnosable.

Regarding the second part of the question above, quite a different result is true.

**Observation 3** [2]. There exists a system \( (S_2) \) which is \( t \)-diagnosable but not \( t_0 \| \tau \)-diagnosable, where \( t + \tau = t \).

To see this, consider the 11-processor system \( S_2 \) shown in Fig. 4. \( S_2 \) is a design \( D_4(9) \) (a system of 9 processors shown in [7] to be 4-diagnosable) augmented with two nodes, \( u_0 \) and \( u_{10} \), and the eight directed arcs, as shown.

![Diagram of S_2](image)

*Fig. 4. A system \( S_2 \) which is 4-diagnosable but not 3\( \| \)1-diagnosable.*
Claim 1. $S_2$ is 4-diagnosable.

Assume there are 4 or fewer faulty processors. The claim is justified by considering three cases.

1. Both $u_0$ and $u_{10}$ are fault-free. Thus, all faulty processors must be in $D_4(9)$. Since there are 4 or fewer faulty processors and $D_4(9)$ is 4-diagnosable, they can be uniquely identified.

2. One of $u_0$ and $u_{10}$ is fault-free. If there are 3 faulty processors in $D_4(9)$, they can be uniquely identified. Further, at least one of $u_0$ and $u_{10}$ is tested by a known fault-free processor. If it is faulty, we are done, since by assumption there are 4 or fewer faulty processors. If it is fault-free, its test of the other processor will determine that the other processor is faulty. If there are 2 or fewer faulty processors in $D_4(9)$, then both $u_0$ and $u_{10}$ are tested by known fault-free processors, and we are done.

3. Neither $u_0$ nor $u_{10}$ is fault-free. Thus, there can be no more than two faulty processors in $D_4(9)$ and so both $u_0$ and $u_{10}$ are tested by known fault-free processors.

Claim 2. $S_2$ is not 3/1-diagnosable.

Consider the case where both $u_2$ and $u_3$ are faulty and one of $u_0$ and $u_{10}$ is intermittently faulty. Let the set of test outcomes of the augmented tests be as shown in Fig. 4. If $u_0$ is the faulty processor, the result of the test by fault-free processor $u_0$ is an incorrect 0; otherwise the result of the test by $u_0$ of $u_{10}$ is an incorrect 0. Thus, there are three faulty processors and one incorrect test result. Since it is impossible to determine which of $u_0$ and $u_{10}$ is faulty, $S_2$ is not 3/1-diagnosable. This proves Observation 3.

Having examined systems which are $t_0/\tau$-diagnosable, consider systems which are $t_0/\tau$-diagnosable. With respect to the question of whether such systems are also $(t' + \tau)$-diagnosable, we have

Observation 4 [2]. There exists a system $(S_3)$ which is $t_0/\tau$-diagnosable but not $(t' + \tau)$-diagnosable, for some $t'$, $\tau > 1$.

Fig. 5 shows $S_3$, a complete digraph on $n = 5$ vertices. $S_3$ is not 3-diagnosable, since $t = 3$ violates the condition $n \geq 2t + 1$, necessary for a system to be $t$-diagnosable. However, $S_3$ is $t_0/2$-diagnosable, as follows.

If all test outcomes by fault-free processors were correct, then a single faulty processor would fail all four tests applied to it, while all other (fault-free) processors would fail at most one test. If there are at most two test outcomes of tests of the faulty processor which are incorrect 0's, then the faulty processor fails at least two tests and can thus be identified. This confirms Observation 4. An important characteristic of $S_3$ is the large number of tests per processor. Thus, even if several are incorrect, sufficiently many valid test results remain to determine which processor is faulty.

With respect to the question of whether there is a $t_0/\tau$-diagnosable system which is not $t_0/\tau$-diagnosable, we have the following
**Theorem 2** [2]. *If $S$ is $t_2/\tau$-diagnosable, then $S$ is $t_2/\tau$-diagnosable.*

Theorem 2 can be proven by contradiction in a manner similar to the proof of Theorem 1. On an intuitive level, in a system which is $t_2/\tau$-diagnosable, up to $t$ faulty processors can be uniquely identified, in spite of $\tau$ incorrect test results. Thus, it is reasonable to expect that up to $t'$ faults can be uniquely determined when there are the same number of missing test results.

### 4. Diagnosis in systems where no two processors test each other

Fig. 6 summarizes the results of this section. That is, for the special case of systems in which no two processors test each other, diagnosability of one type implies diagnosability of the other two types. It is convenient to begin by showing a result that holds in general systems.

![Diagram](image-url)
Lemma 1. In a $t^0_r/t^r$, $(t' + r)$-, and $t^0_r/r$-diagnosable system, each processor is tested by at least $t' + r$ other processors.

Proof. Theorem 2 of [7] shows that in a $(t' + r)$-diagnosable system, each processor is tested by at least $t' + r$ other processors. The proof for the other two types of diagnosability follows in a similar manner. On the contrary, if there exists a processor $u$ tested by $k$ other processors, where $k < t' + r$, each of which is faulty or is fault-free, and either produces $0'$ or $2$ test outcomes then it is impossible to determine if $u$ is faulty or fault-free. That is, there exists two fault patterns containing $t'$ or fewer faulty processors which produce the same set of test outcomes. It must be therefore, that all processors are tested by $t' + r$ or more other processors.

Consider the relationship between $t^0_r/r$- and $(t' + r)$-diagnosability in the class of systems where no two processors test each other. If $S$ is a system in this class, then, from Theorem 1 of [3], $S$ is $t$-diagnosable if every processor is tested by at least $t$ other units. Thus, from Lemma 1 above, if $S$ is $t^0_r/r$-diagnosable, it is also $(t' + r)$-diagnosable. From this and Theorem 1 above, we have

Theorem 3 (due to Hakimi [4]). Let $S$ be a system in which no two processors test each other. $S$ is $t^0_r/r$-diagnosable iff $S$ is $(t' + r)$-diagnosable.

Therefore, the only instances of systems which are $t^0_r/r$-diagnosable but not $(t' + r)$-diagnosable occur in the class of systems where processors test each other.

Consider the relationship between $t^0_r/r$- and $(t' + r)$-diagnosability. From Lemma 1 above, each unit in a $t^0_r/r$-diagnosable system is tested by $t' + r$ other units. From Theorem 1 of [3] it follows that if $S$ is a system in which no two processors test each other, then $S$ is also $(t' + r)$-diagnosable. This proves the only if part of

Theorem 4 [2]. Let $S$ be a system in which no two processors test each other. $S$ is $t'/r$-diagnosable iff $S$ is $(t' + r)$-diagnosable.

Proof. (if) Let $S$ be a $t$-diagnosable system in which no two processors test each other. $S$ is shown to be $t^0_r/r$-diagnosable for $t = t' + r$ by contradiction. That is, assume, on the contrary, $S$ is not $t^0_r/r$-diagnosable. We proceed by showing that not all processors in $S$ are tested by $t' + r$ other processors, and so $S$ is not $(t' + r)$-diagnosable, as assumed.

If $S$ is not $t^0_r/r$-diagnosable, then there exists two fault patterns, $FP_1$ and $FP_2$, each with $t'$ or fewer faulty processors, which produce the same set of test outcomes containing no more than $r$ incorrect test results. Fig. 7 shows the situation.

Here, $FP_1 = V_2 \cup V_4$ and $FP_2 = V_3 \cup V_4$. Let $\Gamma_u(V)$ be the number of tests applied to processors in $V$. An upper bound on $\Gamma_u(V_2 \cup V_3)$ can be expressed as

$$\Gamma_u(V_2 \cup V_3) \leq \tau_1 + \tau_2 + |V_2 \cup V_3||V_4| + |V_2 \cup V_3||V_2 \cup V_3| - 1)/2$$ (1)
where $\tau_1$ (≤τ) and $\tau_2$ (≤τ) are the number of incorrect test outcomes of tests applied to $V_2$ and $V_3$, respectively. $\tau_1 + \tau_2$ represents the maximum number of tests which can be applied by processors in $V_1$. $|V_2 \cup V_3||V_4|$ represents the maximum number of tests which can be applied by processors in $V_4$. $|V_2 \cup V_3||V_2 \cup V_3| - 1)/2$ represents the maximum number of tests applied by processors in $V_2 \cup V_3$, there being no two processors which test each other. Consider two cases.

Case 1. $|V_2 \cup V_3| \geq 2$. Dividing both sides of (1) by $|V_2 \cup V_3|$ yields

$$\frac{\Gamma_n(V_2 \cup V_3)}{|V_2 \cup V_3|} \leq \frac{\tau_1 + \tau_2}{|V_2 \cup V_3|} + \frac{2|V_4| + |V_2 \cup V_3| - 1}{2}.$$ 

Since $|V_2 \cup V_3| \geq 2$ and $|V_2 \cup V_3| \leq |V_2| + |V_3|$, we have

$$\frac{\Gamma_n(V_2 \cup V_3)}{|V_2 \cup V_3|} \leq \left(\tau_1 + |V_2| + |V_4|\right) + \frac{|V_2| + |V_3| + |V_4| - 1}{2} \leq (\tau' + \tau) - \frac{1}{2}. \quad (2)$$

The only tests of processors in $V_2 \cup V_1$ by processors in $V_1$ are those which produce a 0 result that is incorrect in one of two fault patterns. A 1 test outcome never occurs, because in one of the two fault patterns both of the processors involved are fault-free.
Because the average number of tests per processor in $V_2 \cup V_4$ as expressed on the left side in (2) is less than $t' + \tau$, at least one is tested by fewer than $t' + \tau$ processors. Thus, $S$ is not $(t' + \tau)$-diagnosable.

Case 2: $|V_2 \cup V_4| = 1$. Assume that $|V_2| = 1$ and $|V_4| = 0$. We have

$$\Gamma(V_2) < r_1 + |V_4|$$

and there are two fault patterns, one with $|V_4|$ faulty processors and one with $|V_4| + 1$ faulty processors which produce the same set of test outcomes. Thus, $S$ is not $(t' + \tau)$-diagnosable, where $t' + \tau = |V_4| + 1 + r_1$.

The relation between $t_2/\tau$- and $t_0/\tau$-diagnosability for systems in which no two processors test each other can be seen immediately from Theorems 3 and 4.

**Theorem 5** [2]. Let $S$ be a system in which no two processors test each other. $S$ is $t_2/\tau$-diagnosable if $S$ is $t_0/\tau$-diagnosable.

This justifies the third implication pair in Fig. 6.

5. Concluding remarks

Three types of diagnosability have been considered.

1. $(t' + \tau)$-diagnosability corresponding to the case of permanently faulty processors and two-valued test outcomes – 0 (pass) and 1 (fail).

2. $t_2/\tau$-diagnosability corresponding to the case of permanently faulty processors and three-valued test outcomes – (pass) and 1 (fail), and 2 (missing).

3. $t_0/\tau$-diagnosability corresponding to the case of intermittently faulty processors and three-valued test outcomes – 0 (pass), 1 (fail), and 0’ (incorrect pass).

It is shown that a general system which is either $t_0/\tau$- or $(t' + \tau)$-diagnosable is also $t_2/\tau$-diagnosable.

For the special case where no two processors test each other, the existence of any one type of diagnosability implies the existence of the other two types. It follows that for such systems, a test for $(t' + \tau)$-diagnosability is also a test for $t_2/\tau$- and $t_0/\tau$-diagnosability. In systems where two processors test each other, a test for $(t' + \tau)$-diagnosability is not quite as useful. If a test determines a system $S$ to be $(t' + \tau)$-diagnosable, it can be concluded that $S$ is also $t_2/\tau$-diagnosable. However, if the test fails, $S$ may still be $t_2/\tau$-diagnosable. Complete tests for both $t_2/\tau$- and $t_0/\tau$-diagnosability have yet to be developed.

**References**


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