Some Characteristics of Universal Cell Nets

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Abstract—In this paper, one-output \( n \)-input combinatorial nondegenerate networks of cells are analyzed. The networks consist of one-output two-input cells that are universal in that they realize all 16 functions of the two-input variables. Such networks have a number of interesting properties, among which is the fact that if network \( N \) realizes \( F(x) \), it also realizes the dual function \( F^D(x) \).

Besides the basic properties shown to be possessed by all networks, the number of functions realized by certain network forms are calculated. The network forms discussed lead naturally to a class of networks called disjunctive networks. These are characterized by cell interconnections where cell outputs and network inputs connect to at most one cell input. It is shown that disjunctive networks possess the interesting property that all networks of \( c \) cells realize the same number of functions (but not necessarily the same functions). Further, it is shown that all disjunctive networks of \( c \) cells realize the same number of functions dependent on all input variables.

Index Terms—Decomposition, disjunctive networks, network forms, polyfunctional nets, universal cells.

I. INTRODUCTION

URING the past decade, a number of studies have been conducted on networks of cells, where each cell realizes a function from some set of functions. One of the first networks to be considered was the cascade of Maitra [1]. This network consisted of a row-like arrangement of two-input one-output universal cells; that is, each cell realized one of the 16 possible functions of two variables. The Maitra cascade received considerable attention. Sklansky [2], Levy, Winder, and Mott [3], Minnick [4], and Stone and Korenjak [5] contributed to the theory of cascades. In addition, Minnick [4] introduced the cutpoint cell and investigated the properties of cutpoint networks. Urbano [6] developed measures of network function complexity that described the relationship between the functions realized by the network and the functions realized by the component cells. The characterizations presented applied to general networks, not just cascades.

Universal two-input cells capable of learning have also been studied. The Artron, a stochastically trainable cell put forth by Lee [7], has been studied by Fuhr [8]. A cell which is trained deterministically, called the Learn-Mod, has been studied by Leakey [9]. This device was used to play simple games, as well as to guide an electronic mouse in a maze.

A survey of the literature indicates that much of the theory of networks with two-input one-output cells has dealt with cascades. The purpose of this paper is to present more general network forms and to establish their properties. In Section II, basic definitions are presented and the functions generated by general networks are characterized. In Section III, a counting procedure is considered that demonstrates the number of functions realized by certain network forms. A special class of network, the disjunctive network, is discussed in Section IV. Disjunctive networks are more general than cascades, but share some of their properties.

II. BASIC PROPERTIES

Fig. 1 shows a two-input universal cell whose input/output relationship is given as

\[
F = k_0 \bar{x}_1 \bar{x}_2 + k_1 \bar{x}_1 x_2 + k_2 x_1 \bar{x}_2 + k_3 x_1 x_2.
\]

(1)

Any of the 16 switching functions on two variables can be realized by assigning the values 0 and 1 to the coefficients \( k_0, k_1, k_2, \) and \( k_3 \). In what follows, the four inputs corresponding to these coefficients will not be shown explicitly in the figures.

The \( n \)-input one-output networks described in this paper are strictly combinatorial; that is, there are no closed loops. Furthermore, the networks are nondegenerate in the sense that all network inputs connect to cell inputs and all cell outputs connect either to the inputs of other cells or to the network output. Only one cell connects to the network output, and this cell is called, appropriately, the output cell.

An examination of (1) shows that if a universal cell realizes some function \( F(x_1,x_2) \), it can be made to realize the complement function \( \bar{F}(x_1,x_2) \) by simply complementing the coefficients \( k_0, k_1, k_2, \) and \( k_3 \). Thus the complement of a function realized by a network of universal cells is obtained by simply complementing the constants \( k_0, k_1, k_2, \) and \( k_3 \) of the output cell. This result is stated formally as follows.

Lemma 1: If a network of universal cells realizes the function \( f(x) = f(x_1, \ldots, x_n) \), then the same network also realizes the function \( \bar{f}(x) \).

Another useful fact derivable from (1) deals with the complementation of one or more inputs to the network. Let \( f(x| x_i \rightarrow y) \) denote the function derived from \( f(x) \) by replacing the variable \( x_i \) everywhere by the new variable \( y \). From (1), it is clear that to form \( f(x_1, x_2 | x_1 \rightarrow \bar{x}_1) \) it is necessary to interchange the value of \( k_0 \) with that of \( k_2 \), and the value of \( k_1 \) with that of \( k_3 \). Similarly, to complement \( x_2 \), interchange...
the value of \( k_0 \) with that of \( k_1 \) and the value of \( k_2 \) with that of \( k_3 \). Thus, in order to form \( f(x_1 | x_2 \rightarrow \bar{x}_3) \), one need only make the appropriate interchange of constants at those universal cells having \( x_i \) as an input. Stated formally, this result becomes the following.

**Lemma 2:** If a network of universal cells realizes the function \( f(x) \), then the same network also realizes the function \( f(x_1 | x_2 \rightarrow \bar{x}_3) \).

From Lemmas 1 and 2 and from the fact that \( f^d(x) = \bar{f}(\bar{x}) \), where \( f^d(x) \) is the dual of \( f(x) \) and \( f(\bar{x}) = f(x_1 | x_2 \rightarrow \bar{x}_3) \), \( i = 1, 2, \cdots, n \), the following corollary results.

**Corollary 1:** If a network of universal cells realizes the function \( f(x) \) then the same network also realizes \( f^d(x) \). Another useful result is the following.

**Lemma 3:** If a network of universal cells realizes the function \( f(x) \), then the network also realizes \( f(x_1 | x_2 \rightarrow 1) \) and \( f(x | x_1 \rightarrow 0) \).

**Proof:** First consider a single cell having inputs \( x \) and \( y \). Such a cell satisfies the equation

\[
 f(x, y) = x(k_0 \bar{y} + k_2 y) + x(k_1 \bar{y} + k_3 y). 
\]

The function \( f(x, y | y \rightarrow 1) \) may be obtained from (i) as follows. If \( k_2 = 1 \), then set \( k_0 = 1 \); otherwise set \( k_0 = 0 \), and if \( k_3 = 1 \), set \( k_1 = 1 \); otherwise set \( k_1 = 0 \). By applying this procedure to all cells in the network having \( x_1 \) as an input, the result follows immediately.

The case of \( f(x | x_1 \rightarrow 0) \) now follows directly from Lemma 2. Q.E.D.

In general, not all functions on the \( n \)-input variables to a given universal cell network are realized. For example, the two-cell network on three variables shown in Fig. 2 will not realize the function

\[
 f(A, B, C) = ABC + \bar{A}\bar{B}C. 
\]

This is easily seen in the following way. Due to the topology of the network, all realized functions must be decomposable as \( f(A, B, C) = h(f(A, B), C) \), where \( h \) is the function realized by the output cell, \( g \) is the function realized by the leftmost cell, and \( f \) is the function realized by the network. However, for the particular function of (2), there is no such decomposition and so (2) is not realizable. On the other hand, the network shown in Fig. 2 will realize the trivial functions \( f(A, B, C) = 0 \) and \( f(A, B, C) = 1 \) by setting the constant inputs at the output cell to all 0’s and all 1’s, respectively. By the same reasoning, it is easily observed that any network of universal cells can realize the trivial functions. There are, of course, other functions that any network of universal cells can realize. Any product or any sum of subsets of the input variables is also realizable by an arbitrary network of universal cells. This is proved in Lemma 4.

**Lemma 4:** Let \( N \) be an arbitrary combinatorial network of universal cells having a single output and \( n \)-input variables \( x_1, x_2, \cdots, x_n \). Then \( N \) can realize the functions 0, 1, \( x_1^* x_2^* \cdots x_n^* \), \( x_1^* + x_2^* \cdots x_n^* \), \( x_1^* + x_2^* \cdots x_n^* \), \( x_1^* \circ x_2^* \circ \cdots \circ x_n^* \), \( x_1^* \circ x_2^* \circ \cdots \circ x_n^* \), and \( x_1^* \circ x_2^* \circ \cdots \circ x_n^* \) for \( n = 1, 2, \cdots, n, n, \cdots, n, n, n, n, n, n, n, n \) and \( x_i^* \in \{ x, \bar{x} \} \) and where \( \circ \) and \( \circ \) are the EXCLUSIVE OR and the coincident product, respectively.

**Proof:** The proof of trivial cases \( f(x) = 0 \) and \( f(x) = 1 \), has already been discussed. Therefore, without loss of generality, assume the function \( x_1 x_2 \cdots x_n \) is to be realized. Since the network is made up of universal cells, each cell may be made to realize the AND function and thus the output of the network having the \( n \)-input variables is the product of these variables.

Lemma 3 allows the product of any subset of these variables to be obtained and Lemma 2 allows the complementation of any variable. Thus all products of literals may be obtained.

By application of Lemma 1 to the product terms, the sum term is produced.

By identical arguments, the functions \( x_1^* \circ \cdots \circ x_n^* \) and \( x_1^* \circ \cdots \circ x_n^* \) are also realizable. Q.E.D.

There are, of course, a number of ways to generate a network of universal cells to realize an arbitrary function or \( n \) variables. One, in particular, is to form the desired product terms as in Lemma 4 and then form the logical sum of these terms again as in Lemma 4. Another approach is to attempt repeated functional decomposition [10]. A function on \( n \) variables is said to have a simple disjunctive decomposition if it can be written as

\[
 f(x_1, x_2, \cdots, x_n) = \phi(x_{i_1}, \cdots, x_{i_p}, x_{i_{p+1}}, \cdots, x_{i_n}) 
\]

where \( x_{i_j} \neq x_{i_k} \) for \( j \neq k \). This function may be rewritten by the Shannon [11] expansion theorem as

\[
 f(x) = \phi(x_{i_1}, \cdots, x_{i_p}) g_1(x_{i_{p+1}}, \cdots, x_{i_n}) 
\]

\[
 + \bar{\phi}(x_{i_1}, \cdots, x_{i_p}) g_2(x_{i_{p+1}}, \cdots, x_{i_n}). 
\]

It should be obvious from this equation that all functions on \( n \) variables are trivially disjunctively decomposable upon letting \( \phi \) be a function of a single variable.

A procedure, then, for realizing in a universal cell network an arbitrary function on \( n \) variables is as follows.

**Step 1:** Find a disjunctive decomposition 2 of the function having \( r \) in (3) and (4) maximum.

**Step 2:** Find networks of universal cells that realize the functions \( \phi, g_1, \) and \( g_2 \) of (4) by repeating Steps 1 and 2, if necessary.

**Step 3:** Connect the realizing networks as shown in Fig. 3 to form the desired network of universal cells.

1It has been shown by Curtis [16] that if (3) holds then \( f(x) \) may be written as \( f(x_1, \cdots, x_n) = \phi(x_{i_1}, \cdots, x_{i_p}) + \bar{\phi}(x_{i_1}, \cdots, x_{i_p}) \) where \( x_{i_1}, x_{i_2} \in \{ -1, 0, 1 \} \). Using this expression instead of (4), along with the specified realization procedure, a network generally having fewer elements may be realized.

2For a discussion of methods of finding disjunctive decomposition refer to Kohavi [10, pp. 103-114].
III. NUMBER OF FUNCTIONS REALIZED

As pointed out earlier, an arbitrary network of \( c \) universal cells having \( n \)-input variables does not, in general, realize all of the functions on these \( n \) variables. It is of interest to inquire as to the number of functions such a network does realize. In general, a function that has a simple disjunctive decomposition may be thought to have a network realization as shown in Fig. 4. By knowing the number of functions realized by each of the subnetworks \( N_1 \) and \( N_2 \), it is possible to compute the number of functions realized by the whole network of Fig. 4.

Lemma 5: The number of functions \( n \) realized by the network of Fig. 4, not counting functions obtained from other functions of variable permutations\(^3\) is

\[
n = \left( \frac{n_1}{2} - 1 \right) (n_2 - m_\phi) + m_\phi
\]

where \( n_1 \) and \( n_2 \) are the number of functions realized by networks \( N_1 \) and \( N_2 \), respectively, and \( m_\phi \) is the number of functions realized by \( N_2 \) which are independent of the input variable \( \phi \).

Proof: Let \( A_1 \) be the set of functions realized by network \( N_1 \) and let \( B_1 = \{ g \in A_1 | g \notin B_1 \} \). By Lemma 2, network \( N_2 \) realizes \( f(\phi, x_{i_{r+1}}, \ldots, x_{i_n}) \) whenever it realizes \( f(\phi, x_{i_{r+1}}, \ldots, x_{i_n} | \phi \to 0) \) and \( f(\phi, x_{i_{r+1}}, \ldots, x_{i_n} | \phi \to 1) \) and thus only the set \( B_1 \) need be considered as input to \( N_2 \). Further, by Lemma 3, \( N_2 \) realizes \( f(\phi, x_{i_{r+1}}, \ldots, x_{i_n} | \phi \to 1) \) and \( f(\phi, x_{i_{r+1}}, \ldots, x_{i_n}) \) and thus only the nontrivial functions of \( B_1 \) need to be considered. The number of nontrivial functions in \( B_1 \) is \( (n_1/2) - 1 \). Network \( N_2 \) realizes \( n_2 - m_\phi \) functions which are dependent on the output of network \( N_1 \). Thus there are \( ((n_1/2) - 1)(n_2 - m_\phi) \) functions realized that are dependent on \( \phi \). Add to this the number of functions independent of \( \phi \) and the desired result is obtained.

Q.E.D.

A special case of the network shown in Fig. 4 is the one shown in Fig. 5. This realization is appropriate to functions that have the compound disjunctive decomposition shown in (6).

\[
f(x_1, \ldots, x_n) = F(g_1(x_1, \ldots, x_r), g_2(x_{r+1}, \ldots, x_n)).
\]

\(^3\)Actually as the lemmas are presented, the restriction on variable permutation only applies to the interchange of an input variable of network \( N_1 \) with an input variable of \( N_2 \). A similar statement can be made concerning Lemma 6.

The network enclosed by the broken line shows the network equivalent to \( N_2 \) of Fig. 4, called \( N_2^* \). The number of functions realized by \( N_2^* \) (not counting variable permutations) is found as follows. There are two trivial functions, 0 and 1. There are also two functions \( g_1 \) and \( g_1 \) independent of \( g_2 \). Of the 16 functions on the two variables \( g_1 \) and \( g_2 \), 12 remain that are dependent on \( g_2 \). Because of Lemmas 1 and 2 it is necessary to consider only half of the set of functions generated by \( N_2 \) and only the nontrivial ones at that. This number is \( (n_1/2) - 1 \). Thus the number of functions generated by \( N_2^* \) is

\[
n^*_2 = 12 \left( \frac{n_2}{2} - 1 \right) + 2 + 2 = 6n_2 - 8.
\]

From this Lemma 6 results.

Lemma 6: The number of functions \( n \) realized by the network of Fig. 5 not counting functions obtained by permuting the variables is

\[
n = \frac{5}{2} n_1 n_2 - 4(n_1 + n_2) + 8
\]

where \( n_1 \) and \( n_2 \) are the number of functions realized by networks \( N_1 \) and \( N_2 \), respectively.

Proof: From Lemma 5 and Fig. 5 it is clear that the number of functions realized by \( N_1 \) and \( N_2^* \) is

\[
n = \left( \frac{n_1}{2} - 1 \right) (n_2^* - m_\phi) + m_\phi.
\]

From (7), \( n^*_2 = 6n_2 - 8 \), and since there are \( n_2 \) functions that are independent of \( g_1 \) in Fig. 5, \( m_\phi = n_2 \). Putting these numbers into (9) gives the required result.

Q.E.D.

As an example of the application of Lemma 6, consider the four-input three-cell network shown in Fig. 6(a). This network...
has the form of Fig. 5 and thus Lemma 6 holds. In particular, since the two-input cells, labeled \( N_1 \) and \( N_2 \), each realize 16 functions, this network realizes \( \frac{5}{2} \cdot 16 \cdot \frac{16}{8} = 520 \) functions. The network shown in Fig. 2 is also of this type. Here, however, one of the input networks is simply a single wire. One way of handling this situation is to replace the wire by a universal cell having both inputs tied together. This is shown in Fig. 7. The justification for this comes from the fact that cell \( C \) in Fig. 7(b) realizes the functions \( f(A, 1) \), \( f(A, 0) \), \( f(A, g) \), and \( f(A, \bar{g}) \) by Lemmas 2 and 3. Thus only the functions \( g = B \) need be considered as an input function to cell \( C \). Cell \( C' \), however, realizes four functions and therefore the total number of functions realized by the network of Fig. 2 is \( (5/2)(4)(16) - 4(16 + 4) + 8 = 88 \).

IV. DISJUNCTIVE NETWORKS

It may be easily verified by applying Lemma 5 twice that the number of functions realized by the network of Fig. 6(b) is the same as the number realized by Fig. 6(a), viz. 520, although the networks are topologically different. One naturally wonders whether this is a property of networks of universal cells of this type or whether it is simply a coincidence. Clearly both of the networks are topologically similar to that of Fig. 5. Furthermore, it may be observed that the input variables are connected to at most one cell and that the cell outputs serve as cell inputs to at most one cell. Combinatorial networks of this type will be called disjunctive networks. (Such networks have been studied by Levy, Winder, and Mott [3]). Another way of defining a disjunctive network is to say that it is disjunctive if it has a form as in Fig. 5 and each of the subnetworks \( N_1 \) and \( N_2 \) are disjunctive. This leads naturally to a type of functional decomposition called a basic disjunctive decomposition. A function has a basic disjunctive decomposition if it has a compound disjunctive decomposition [as in (6)] and if each of the two subfunctions have basic disjunctive decompositions. From these definitions, the following theorem results directly.

**Theorem 1.** A necessary and sufficient condition that a function be realizable by a disjunctive network is that the function have a basic disjunctive decomposition.

Theorem 1 is important in that it not only gives a test for realizability, but also produces an appropriate functional factorization which yields, directly, a network realization.

Disjunctive networks have several interesting properties in terms of the number of functions realized. Consider, for example, the disjunctive cascade of universal cells shown in Fig. 8. This cascade is the disjunctive network of Fig. 5 with \( N_2 \) replaced by a wire. If there are \( C \) cells realizing \( n_C \) functions then the addition of one more cell will produce \( n_{C+1} \) functions, where

\[
n_{C+1} = \frac{5}{2} \cdot 4 \cdot n_C - 4(n_C + 4) + 8 = 6n_C - 8 \quad (10)
\]

The solution to this difference equation yields a result originally due to Maitra.

**Theorem 2 [1]:** The number of functions \( n_r \) realized by a disjunctive cascade of \( r \) universal cells, not counting variable permutations, is

\[
n_r = \frac{12 \cdot 6^r + 8}{5} \quad (11)
\]

In the example cited at the beginning of this section, it was pointed out that for two topologically dissimilar disjunctive networks on the same number of variables, the same number of functions was realized in each case. Theorem 3 shows that, in fact, this is a general property of disjunctive networks.

It has been shown by a number of authors [12], [14], and [15] that a function has a compound disjunctive decomposition if there exists a partition matrix having row and column multiplicity of at most 2. This test may then be used repeatedly to determine basic decomposability. Again, refer to Kohavi [10] for details of disjunctive decomposition.
Theorem 3: The number \( n_c \) of functions realized, not counting variable permutations, by a disjunctive network of \( c \) cells is

\[
n_c = \frac{12 \cdot 6^c + 8}{5}.
\]

Proof: The proof is by induction on \( c \). The case of \( c = 0 \) corresponds to a single wire and this is a rather special case, since degenerate networks of this type have been specifically excluded from consideration. However, for purposes of this theorem, it is appropriate in what follows to treat the wire as was done in Fig. 7. In such a case, four functions are realized which is

\[
n_0 = \frac{12 \cdot 6^0 + 8}{5} = \frac{20}{5} = 4.
\]

For \( c = 1 \)

\[
n_1 = \frac{12 \cdot 6^1 + 8}{5} = \frac{72 + 8}{5} = 16
\]

and thus the theorem is true for \( c = 1 \). Assume the theorem to be true for all \( c < p \). Consider a disjunctive network of \( p + 1 \) cells. Since the network is disjunctive it may be realized as in Fig. 5 where networks \( N_1 \) and \( N_2 \) have, respectively, \( p_1 > 0 \) and \( p_2 > 0 \) cells. (Note that the comments made previously concerning \( n_0 \) are now directly applicable for the case where \( p_1 \) or \( p_2 \) equals \( 0 \).) Thus the total number of cells is \( p_1 + p_2 + 1 = p + 1 \). By assumption, the two networks realize

\[
n_{p_1} = \frac{12 \cdot 6^{p_1} + 8}{5}
\]

functions. From Lemma 6 the total number of functions realized is

\[
n_{p+1} = \frac{5}{2} n_{p_1} n_{p_2} - 4 (n_{p_1} + n_{p_2}) + 8 = \frac{5}{2} \left( \frac{12 \cdot 6^{p_1} + 8}{5} \right) \left( \frac{12 \cdot 6^{p_2} + 8}{5} \right) - 4 \left( \frac{12 \cdot 6^{p_1} + 6^{p_2}}{5} \right) + 8
\]

\[
= \frac{12 \cdot 6^{p+1} + 8}{5}
\]

Q.E.D.

This result is quite interesting in that it states that the number of functions realized by a disjunctive network of universal cells is independent of its topology. Note, however, that this number does not include the functions obtained by variable permutations.

It is clear that independence of the output on some variable \( x_i \) is equivalent to removing that variable from the input set. When this is done the cell to which \( x_i \) originally connected may be removed as mentioned earlier. Thus a network of \( r \) cells realizes \( n_r \) functions which are dependent on all of the input variables. The removal of a cell and thus the removal of dependency on some variable produces, by Theorem 3, \( n_{r-1} \) functions. Thus the following lemma results.

Lemma 7: The number of functions realized by a disjunctive network of \( c \) cells which are independent of one or more variables is

\[
m_1 = \frac{12 \cdot 6^{c-1} + 8}{5}.
\]

An immediate corollary is the following.

Corollary 2: The number of functions realized by a disjunctive network of \( c \) cells which are independent of \( r \) or more variables is

\[
m_r = \frac{12 \cdot 6^{c-r} + 8}{5}.
\]

Lemma 7 also shows that since omission of a variable is equivalent to omitting a cell and since omission of all the cells is equivalent to a function of a single variable, a disjunctive network of \( c \) cells has at most \( c + 1 \) input variables. Thus (11) may be written as

\[
n_n = \frac{2 \cdot 6^n + 8}{5}
\]

where \( n \) is the number of variables.

Lemma 7 also makes it possible to compute the number of functions realized which are dependent on exactly \( n \) variables as follows. Let \( N(x_i) \) be the number of functions on \( n \) variables independent of \( x_i \). Let \( N_n(x_i, x_j) \) be the number of functions on \( n \) variables independent of \( x_i \) and \( x_j \), etc. Further, let \( N_n(x_i') \) be the number of functions on \( n \) variables dependent on \( x_i \). Then by the principle of inclusion/exclusion [13]:

\[
N_n(x_1, x_2, \ldots, x_n) = Q - \sum_{i=1}^{n} N_n(x_i) + \sum_{i<j} N_n(x_i x_j)
\]

\[
= \frac{n}{5} - \frac{2 \cdot 6^n + 8}{5} + \frac{n}{1} - \frac{2 \cdot 6^{n-1} + 8}{5} + \ldots + \left( \frac{n}{1} \cdot \frac{2 \cdot 6 + 8}{5} \right)
\]

\[
= \frac{n}{5} \left( \sum_{i=0}^{n} \left( \frac{n}{i} \cdot \right) \right) \left( \frac{2 \cdot 6^i + 8}{5} \right)
\]

\[
= \frac{2}{5} \sum_{i=0}^{n} \left( \frac{n}{i} \cdot \right) \left( \frac{2 \cdot 6^i + 8}{5} \right)
\]

\[
= \frac{2}{5} \left( 6 - 1 \right)^n + \frac{8}{5} \left( 1 - 1 \right)^n
\]

Thus Lemma 8.

Lemma 8: The number of functions \( N_n(x_1' \ldots x_n') \) realized by a disjunctive network of universal cells which are dependent on all \( n \)-input variables not counting variable permutations is

\[
N_n(x_1' \ldots x_n') = 2 \cdot 5^{n-1}.
\]

Equation (15) may be rewritten in terms of the number of cells \( c \) in the network as

\[
N_n(x_1' \ldots x_n') = 2 \cdot 5^c.
\]
Further, Lemma 6 indicates that the dependence of a function on a particular variable is the same regardless of which variable is chosen. Thus if there are $n$-input variables to a given disjunctive network then there are

$$N_n(x'_i \cdot \cdots x'_s) = \binom{n}{s} 2^s 5^{s-1} \quad (17)$$

functions on exactly $s$ of the $n$ variables.

V. CONCLUSIONS AND COMMENTS

In this paper some of the properties of networks of two-input one-output universal cells have been presented. With regard to general networks, the functions realized have been characterized. For example, if an arbitrary network realizes $F(x)$, it also realizes the dual function $F^d(x)$.

Further, it has been shown that if a network can be expressed as two subnetworks with distinct inputs, where one network connects to one input of the other, then the number of functions realized by the entire network is expressible in terms of the number of functions realized by the two subnetworks. This result was used in the analysis of networks consisting of two subnetworks with distinct inputs, each connected to the output cell of the network. It was shown that the number of functions depends only on the number of functions realized by the two subnetworks.

In addition to general network forms, networks with specific interconnections were analyzed. These networks, called disjunctive networks, are characterized by their interconnection, as well as by the fact that all functions realized by such networks have a basic disjunctive decomposition. Further, it was found that a disjunctive network of $c$ cells realizes $2 \cdot 5^c$ functions dependent on all inputs and a total of $12 \cdot 6^2 + 8/5$ functions, regardless of its topology.

One aspect of disjunctive networks not considered in this paper is the number of functions $N$ on $n$ variables realized by any disjunctive network. The values of $N_n$ for $1 \leq n \leq 4$ are shown in Table I. In this case, $\tau_n$ is the total number of functions on $n$ variables, i.e., $\tau_n = 2^2n$.

The value of $N_n$ is greater than $n_n$ for $n \geq 4$ because: 1) there exist permutations of input variables which result in a new network (but one which is topologically identical) that realizes a different set of functions than did the original network; and 2) there exist more than one disjunctive network which is topologically different. For $n = 3$ only one topology exists, the two-cell cascade. However, condition 1) holds and so $N_n$ is greater than $n_n$ for $n = 3$ also. In order to derive an expression for $N_n$ it appears that the counting procedures of Section IV must be extended.

Another area not considered in this paper is the class of nondisjunctive networks. This subject is important, in view of the fact that, for large $n$ the number of functions realizable by $n$-input disjunctive networks is small compared to the total number of $n$-variable functions. Certain nondisjunctive networks, on the other hand, are capable of realizing all functions on $n$ variables. Although the analysis of the general nondisjunctive network is complex, a special class of such networks can be analyzed in a particularly tractable manner. The interconnections are restricted in the same way as disjunctive networks, except that network inputs can connect to more than one cell. For a discussion of this, the reader is referred to Butler [12].

### Table I

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<th>$n$</th>
<th>$N_n$</th>
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REFERENCES


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Abstract—A universal logic module (ULM) with interconnected external terminals contains m input terminals and s auxiliary function terminals. The module implements the function \( U(z_1, z_2, \ldots, z_m) \) such that every Boolean function of n variables \( f(x_1, x_2, \ldots, x_n) \) can be realized by an appropriate substitution of an element of \( \delta = \{x_1, x_1, x_2, x_2, \ldots, x_n, \overline{x_n}, s_1, s_2, \ldots, s_s\} \) for each \( z_j \). An improved lower bound on the minimum number of terminals of a ULM of this type is derived. It is shown that certain of the “best-known” designs are in fact optimal. Improved designs are presented for ULM’s of nine and ten arguments.

Index Terms—Combinational networks, functional standardization, logical design, number of terminals, universal logic modules.

I. INTRODUCTION

UNIVERSAL logic modules (ULM’s) have received considerable attention in recent years [1]-[7], due in large part to their potential applicability to logic design using integrated circuit and large-scale integrated circuit packages. Much of this effort has gone into decreasing the number of terminal pins on the ULM since high gate-pin ratios and fewer external interconnections seem to result in circuits that have fewer modules, simpler interconnection patterns, and are more reliable.

The ULM has undergone several refinements since it first appeared in the literature [1]. Yau and Tang [2] provided the first systematic approach to the design of ULM’s and Preparata and Muller [3] developed a very ingenious minterm partitioning scheme that significantly reduced the number of terminals required to realize a ULM of any number of arguments. This paper considers the generalization defined by Patt [4] and studied by Preparata [5].

A ULM of this type consists of a universal Boolean function \( U(z_1, z_2, \ldots, z_m) \) and \( s \) auxiliary functions \( g_1(z_1, z_2, \ldots, z_n) \)

\( g_2(z_1, \ldots, z_n), \ldots, g_s(z_1, \ldots, z_n) \); see Fig. 1. A Boolean function \( U(z_1, z_2, \ldots, z_m) \) is universal in \( n \) variables \( x_1, x_2, \ldots, x_n \) if every Boolean function of \( n \) variables \( f(x_1, x_2, \ldots, x_n) \) can be realized by an appropriate substitution of a member of some set \( \delta \) for each \( z_j \). In this model, the set \( \delta = \{x_1, x_1, x_2, x_2, \ldots, x_n, \overline{x_n}, 0, 1, g_1, g_2, \ldots, g_s\} \). The arguments of \( g_1, g_2, \ldots, g_s \) and the first \( n \) arguments of \( U \) are restricted to the corresponding uncomplemented \( x_i \); i.e., \( z_i = x_i, 1 \leq i \leq n \). Note that if \( s = 0 \), the model reduces to that of [3].

The structure of \( U \) is that introduced in [3] and used in [4]-[6]. The \( 2^n \) minterms of \( n \) variables \( z_1, z_2, \ldots, z_n \) are partitioned into unions \( R_1, R_2, \ldots, R_t \), called blocks. Each block \( R_t \) has the property that the union of any arbitrary subset of the set of minterms of \( R_t \) can be obtained by forming the intersection of \( R_t \) with the appropriate member of \( \delta \). \( U \) has the form:

\[ z_{n+1} R_1 + z_{n+2} R_2 + \cdots + z_{n+t} R_t. \]

The ULM, therefore, has \( n \) (independent variables) + \( t \) (controls) + \( s \) (auxiliary functions) = \( p \) terminals. In Section II, a lower bound for \( p \) is derived which is a slight improvement on the previous best lower bound, which is found in [5]. In Section III, an improved design for the ULM of nine argu-