# NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS MA 3243 SOLUTIONS OF PROBLEMS IN LECTURE NOTES

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**CHAPTER 1**

# 1 Introduction and Applications

## 1.1 Basic Concepts and Definitions

### Problems

1. Give the order of each of the following PDEs

  a. $u_{xx} + u_{yy} = 0$
  b. $u_{xxx} + u_{xy} + a(x)u_y + \log u = f(x,\, y)$
  c. $u_{xxx} + u_{xyyy} + a(x)u_{xxy} + u^2 = f(x,\, y)$
  d. $u\, u_{xx} + u_{yy}^2 + e^u = 0$
  e. $u_x + cu_y = d$

2. Show that

$$u(x,\, t) = \cos(x - ct)$$

is a solution of

$$u_t + cu_x = 0$$

3. Which of the following PDEs is linear? quasilinear? nonlinear? If it is linear, state whether it is homogeneous or not.

  a. $u_{xx} + u_{yy} - 2u = x^2$
  b. $u_{xy} = u$
  c. $u\, u_x + x\, u_y = 0$
  d. $u_x^2 + \log u = 2xy$
  e. $u_{xx} - 2u_{xy} + u_{yy} = \cos x$
  f. $u_x(1 + u_y) = u_{xx}$
  g. $(\sin u_x)u_x + u_y = e^x$
  h. $2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0$
  i. $u_x + u_x u_y - u_{xy} = 0$

4. Find the general solution of

$$u_{xy} + u_y = 0$$

(Hint: Let $v = u_y$)

5. Show that

$$u = F(xy) + x\, G(\frac{y}{x})$$

is the general solution of

$$x^2 u_{xx} - y^2 u_{yy} = 0$$

1. a. Second order
   b. Third order
   c. Fourth order
   d. Second order
   e. First order

2. $u = \cos(x - ct)$

$$u_t = -c \cdot (-\sin(x - ct)) = c \sin(x - ct)$$
$$u_x = 1 \cdot (-\sin(x - ct)) = -\sin(x - ct)$$
$$\Rightarrow u_t + cu_x = c \sin(x - ct) - c \sin(x - ct) = 0.$$

3. a. Linear, inhomogeneous
   b. Linear, homogeneous
   c. Quasilinear, homogeneous
   d. Nonlinear, inhomogeneous
   e. Linear, inhomogeneous
   f. Quasilinear, homogeneous
   g. Nonlinear, inhomogeneous
   h. Linear, homogeneous
   i. Quasilinear, homogeneous

4.

$$u_{xy} + u_y = 0$$

Let $v = u_y$ then the equation becomes

$$v_x + v = 0$$

For fixed $y$, this is a separable ODE

$$\frac{dv}{v} = -dx$$

$$\ln v = -x + C(y)$$

$$v = K(y)\, e^{-x}$$

In terms of the original variable $u$ we have

$$u_y = K(y)\, e^{-x}$$

$$u = e^{-x}\, q(y) + p(x)$$

You can check your answer by substituting this solution back in the PDE.

5.

$$u = F(xy) + x\,G\left(\frac{y}{x}\right)$$

$$u_x = y\,F'(xy) + G\left(\frac{y}{x}\right) + x\left(-\frac{y}{x^2}\right)G'\left(\frac{y}{x}\right)$$

$$u_{xx} = y^2\,F''(xy) + \left(-\frac{y}{x^2}\right)G'\left(\frac{y}{x}\right) - \frac{y}{x}\left(-\frac{y}{x^2}\right)G''\left(\frac{y}{x}\right) + \left(\frac{y}{x^2}\right)G'\left(\frac{y}{x}\right)$$

$$u_{xx} = y^2\,F''(xy) + \frac{y^2}{x^3}\,G''\left(\frac{y}{x}\right)$$

$$u_y = x\,F'(xy) + x\,\frac{1}{x}G'\left(\frac{y}{x}\right)$$

$$u_{yy} = x^2\,F''(xy) + \frac{1}{x}G''\left(\frac{y}{x}\right)$$

$$x^2u_{xx} - y^2u_{yy} = x^2\left(y^2\,F'' + \frac{y^2}{x^3}\,G''\right) - y^2\left(x^2\,F'' + \frac{1}{x}\,G''\right)$$

Expanding one finds that the first and third terms cancel out and the second and last terms cancel out and thus we get zero.

## 1.2 Applications

## 1.3 Conduction of Heat in a Rod

## 1.4 Boundary Conditions

### Problems

1. Suppose the initial temperature of the rod was

$$u(x,\,0) = \begin{cases} 2x & 0 \le x \le 1/2 \\ 2(1-x) & 1/2 \le x \le 1 \end{cases}$$

and the boundary conditions were

$$u(0,\,t) = u(1,\,t) = 0\ ,$$

what would be the behavior of the rod's temperature for later time?

2. Suppose the rod has a constant internal heat source, so that the equation describing the heat conduction is

$$u_t = k u_{xx} + Q, \qquad 0 < x < 1\ .$$

Suppose we fix the temperature at the boundaries

$$\begin{aligned} u(0,\,t) &= 0 \\ u(1,\,t) &= 1\ . \end{aligned}$$

What is the steady state temperature of the rod? (Hint: set $u_t = 0$ .)

3. Derive the heat equation for a rod with thermal conductivity $K(x)$.

4. Transform the equation

$$u_t = k(u_{xx} + u_{yy})$$

to polar coordinates and specialize the resulting equation to the case where the function $u$ does NOT depend on $\theta$. (Hint: $r = \sqrt{x^2 + y^2}$, $\tan\theta = y/x$)

5. Determine the steady state temperature for a one-dimensional rod with constant thermal properties and

a. $Q = 0,\quad u(0) = 1,\quad u(L) = 0$
b. $Q = 0,\quad u_x(0) = 0,\quad u(L) = 1$
c. $Q = 0,\quad u(0) = 1,\quad u_x(L) = \varphi$

d. $\frac{Q}{k} = x^2,\quad u(0) = 1,\quad u_x(L) = 0$

e. $Q = 0,\quad u(0) = 1,\quad u_x(L) + u(L) = 0$

1. Since the temperature at both ends is zero (boundary conditions), the temperature of the rod will drop until it is zero everywhere.

2.

$$k\,u_{xx} + Q = 0$$
$$u(0.t) = 0$$
$$u(1,t) = 1$$

$\Rightarrow \qquad u_{xx} = -\frac{Q}{k}$

Integrate with respect to $x$

$$u_x = -\frac{Q}{k}\,x + A$$

Integrate again

$$u = -\frac{Q}{k}\frac{x^2}{2} + A\,x + B$$

Using the first boundary condition $u(0) = 0$ we get $B = 0$. The other boundary condition will yield

$$-\frac{Q}{k}\frac{1}{2} + A = 1$$

$$\Rightarrow \qquad A = \frac{Q}{2k} + 1$$

$$\Rightarrow \qquad u(x) = \left(1 + \frac{Q}{2k}\right)x - \frac{Q}{2k}\,x^2$$

3. Follow class notes.

4.

$$r = \left(x^2 + y^2\right)^{\frac{1}{2}}$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

$$r_x = \frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}} \cdot 2x \;=\; \frac{x}{\sqrt{x^2 + y^2}}$$

$$\theta_x = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) \;=\; -\frac{y}{x^2 + y^2}$$

$$r_y = \frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}} \cdot 2y \;=\; \frac{y}{\sqrt{x^2 + y^2}}$$

$$\theta_y = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) \;=\; \frac{x}{x^2 + y^2}$$

$$u_x = u_r r_x \,+\, u_\theta \theta_x \;=\; \frac{x}{\sqrt{x^2 + y^2}}\, u_r \,-\, \frac{y}{x^2 + y^2}\, u_\theta$$

$$u_y = u_r r_y \,+\, u_\theta \theta_y \;=\; \frac{y}{\sqrt{x^2 + y^2}}\, u_r \,+\, \frac{x}{x^2 + y^2}\, u_\theta$$

$$u_{xx} \;=\; \left(\frac{x}{\sqrt{x^2 + y^2}}\right)_x u_r \,+\, \frac{x}{\sqrt{x^2 + y^2}}\,(u_r)_x \,-\, \left(\frac{y}{x^2 + y^2}\right)_x u_\theta \,-\, \frac{y}{x^2 + y^2}\,(u_\theta)_x$$

$$u_{xx} \;=\; \frac{\sqrt{x^2 + y^2} \,-\, x\frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}} \cdot 2x}{x^2 + y^2}\, u_r \,+\, \frac{x}{\sqrt{x^2 + y^2}}\left[\frac{x}{\sqrt{x^2 + y^2}}\, u_{rr} \,-\, \frac{y}{x^2 + y^2}\, u_{r\theta}\right]$$

$$-\,\frac{-2xy}{\left(x^2 + y^2\right)^2}\, u_\theta \,-\, \frac{y}{x^2 + y^2}\left[\frac{x}{\sqrt{x^2 + y^2}}\, u_{r\theta} \,-\, \frac{y}{x^2 + y^2}\, u_{\theta\theta}\right]$$

$$u_{xx} \;=\; \frac{x^2}{x^2 + y^2}\, u_{rr} \,-\, \frac{2xy}{\left(x^2 + y^2\right)^{\frac{3}{2}}}\, u_{r\theta} \,+\, \frac{y^2}{\left(x^2 + y^2\right)^2}\, u_{\theta\theta} \,+\, \frac{y^2}{\left(x^2 + y^2\right)^{\frac{3}{2}}}\, u_r \,+\, \frac{2xy}{\left(x^2 + y^2\right)^2}\, u_\theta$$

$$u_{yy} \;=\; \left(\frac{y}{\sqrt{x^2 + y^2}}\right)_y u_r \,+\, \frac{y}{\sqrt{x^2 + y^2}}\,(u_r)_y \,+\, \left(\frac{x}{x^2 + y^2}\right)_y u_\theta \,+\, \frac{x}{x^2 + y^2}\,(u_\theta)_y$$

$$u_{yy} \;=\; \frac{\sqrt{x^2 + y^2} \,-\, y\frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}} \cdot 2y}{x^2 + y^2}\, u_r \,+\, \frac{y}{\sqrt{x^2 + y^2}}\left[\frac{y}{\sqrt{x^2 + y^2}}\, u_{rr} \,+\, \frac{x}{x^2 + y^2}\, u_{r\theta}\right]$$

$$+\,\frac{-2xy}{\left(x^2 + y^2\right)^2}\, u_\theta \,+\, \frac{x}{x^2 + y^2}\left[\frac{y}{\sqrt{x^2 + y^2}}\, u_{r\theta} \,+\, \frac{x}{x^2 + y^2}\, u_{\theta\theta}\right]$$

$$u_{yy} \;=\; \frac{y^2}{x^2 + y^2}\, u_{rr} \,+\, \frac{2xy}{\left(x^2 + y^2\right)^{\frac{3}{2}}}\, u_{r\theta} \,+\, \frac{x^2}{\left(x^2 + y^2\right)^2}\, u_{\theta\theta} \,+\, \frac{x^2}{\left(x^2 + y^2\right)^{\frac{3}{2}}}\, u_r \,-\, \frac{2xy}{\left(x^2 + y^2\right)^2}\, u_\theta$$

$$\Rightarrow u_{xx} + u_{yy} = u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r$$

$$\boxed{u_t = k\left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}\right)}$$

In the case $u$ is <u>independent</u> of $\theta$:

$$\boxed{u_t = k\left(u_{rr} + \frac{1}{r} u_r\right)}$$

5. $k\,u_{xx} + Q = 0$

a. $k\,u_{xx} = 0$

Integrate twice with respect to $x$

$$u(x) = Ax + B$$

Use the boundary conditions

$$u(0) = 1 \qquad \text{implies } B = 1$$

$$u(L) = 0 \qquad \text{implies } AL + B = 0 \qquad \text{that is } A = -\frac{1}{L}$$

Therefore

$$\boxed{u(x) = -\frac{x}{L} + 1}$$

b. $k\,u_{xx} = 0$

Integrate twice with respect to $x$ as in the previous case

$$u(x) = Ax + B$$

Use the boundary conditions

$$u_x(0) = 0 \qquad \text{implies } A = 0$$

$$u(L) = 1 \qquad \text{implies } AL + B = 1 \qquad \text{that is } B = 1$$

Therefore

$$\boxed{u(x) = 1}$$

c. $k\,u_{xx} = 0$

Integrate twice with respect to $x$ as in the previous case

$$u(x) = Ax + B$$

Use the boundary conditions

$$u(0) = 1 \qquad \text{implies } B = 1$$

$$u_x(L) = \varphi \qquad \text{implies } A = \varphi$$

Therefore

$$\boxed{u(x) = \varphi\, x + 1}$$

d. $\quad k\, u_{xx} \,+\, Q \,=\, 0$

$u_{xx} \,=\, -\,\frac{Q}{k} \,=\, -\,x^2$

Integrate with respect to $x$ we get

$$u_x(x) \,=\, -\,\frac{1}{3}\,x^3 \,+\, A$$

Use the boundary condition

$$u_x(L) \,=\, 0 \qquad \text{implies} \;\; -\,\frac{1}{3}\,L^3 \,+\, A \,=\, 0 \qquad \text{that is } A \,=\, \frac{1}{3}\,L^3$$

Integrating again with respect to $x$

$$u \,=\, -\,\frac{x^4}{12} \,+\, \frac{1}{3}\,L^3\,x \,+\, B$$

Use the second boundary condition

$$u(0) \,=\, 1 \qquad \text{implies } B \,=\, 1$$

Therefore

$$\boxed{u(x) \,=\, -\frac{x^4}{12} \,+\, \frac{L^3}{3}\,x \,+\, 1}$$

e. $\quad k\, u_{xx} \,=\, 0$

Integrate twice with respect to $x$ as in the previous case

$$u(x) \,=\, Ax \,+\, B$$

Use the boundary conditions

$$u(0) \,=\, 1 \qquad \text{implies } B \,=\, 1$$

$$u_x(L) \,+\, u(L) = 0 \qquad \text{implies } A \,+\, (AL \,+\, 1) \,=\, 0 \qquad \text{that is } A \,=\, -\,\frac{1}{L+1}$$

Therefore

$$\boxed{u(x) \,=\, -\,\frac{1}{L+1}\,x \,+\, 1}$$

## 1.5 A Vibrating String

### Problems

1. Derive the telegraph equation

$$u_{tt} + au_t + bu = c^2 u_{xx}$$

by considering the vibration of a string under a damping force proportional to the velocity and a restoring force proportional to the displacement.

2. Use Kirchoff's law to show that the current and potential in a wire satisfy

$$\begin{aligned} i_x + C\,v_t + Gv &= 0 \\ v_x + L\,i_t + Ri &= 0 \end{aligned}$$

where $i$ = current, $v = L$ = inductance potential, $C$ = capacitance, $G$ = leakage conductance, $R$ = resistance,

b. Show how to get the one dimensional wave equations for $i$ and $v$ from the above.

1. Follow class notes.

$a$, $b$ are the proportionality constants for the forces mentioned in the problem.

2. a. Check any physics book on Kirchoff's law.

b. Differentiate the first equation with respect to $t$ and the second with respect to $x$

$$i_{xt} + C\,v_{tt} + G\,v_t = 0$$
$$v_{xx} + L\,i_{tx} + R\,i_x = 0$$

Solve the first for $i_{xt}$ and substitute in the second

$$i_{xt} = -C\,v_{tt} - G\,v_t$$

$$\Rightarrow v_{xx} - CL\,v_{tt} - GL\,v_t + R\,i_x = 0$$

$i_x$ can be solved for from the original first equation

$$i_x = -C\,v_t - G\,v$$

$$\Rightarrow v_{xx} - CL\,v_{tt} - GL\,v_t - RC\,v_t - RG\,v = 0$$

Or

$$v_{tt} + \left(\frac{G}{C} + \frac{R}{L}\right) v_t + \frac{RG}{CL} v = \frac{1}{CL} v_{xx}$$

which is the telegraph equation.

In a similar fashion, one can get the equation for $i$.

# 2 Separation of Variables-Homogeneous Equations

## 2.1 Parabolic equation in one dimension

## 2.2 Other Homogeneous Boundary Conditions

### Problems

1. Consider the differential equation

$$X''(x) + \lambda X(x) = 0$$

Determine the eigenvalues $\lambda$ (assumed real) subject to

a. $X(0) = X(\pi) = 0$

b. $X'(0) = X'(L) = 0$

c. $X(0) = X'(L) = 0$

d. $X'(0) = X(L) = 0$

e. $X(0) = 0$ and $X'(L) + X(L) = 0$

Analyze the cases $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$.

1. a.

$$X'' + \lambda X = 0$$
$$X(0) = 0$$
$$X(\pi) = 0$$

Try $e^{rx}$. As we know from ODEs, this leads to the characteristic equation for $r$

$$r^2 + \lambda = 0$$

Or

$$r = \pm\sqrt{-\lambda}$$

We now consider three cases depending on the sign of $\lambda$

Case 1: $\lambda < 0$

In this case $r$ is the square root of a positive number and thus we have two real roots. In this case the solution is a linear combination of two real exponentials

$$X = A_1 e^{\sqrt{-\lambda}x} + B_1 e^{-\sqrt{-\lambda}x}$$

It is well known that the solution can also be written as a combination of hyperbolic sine and cosine, i.e.

$$X = A_2 \cosh\sqrt{-\lambda}x + B_2 \sinh\sqrt{-\lambda}x$$

The other two forms are may be less known, but easily proven. The solution can be written as a shifted hyperbolic cosine (sine). The proof is straight forward by using the formula for $\cosh(a+b)$ $(\sinh(a+b))$

$$X = A_3 \cosh\left(\sqrt{-\lambda}x + B_3\right)$$

Or

$$X = A_4 \sinh\left(\sqrt{-\lambda}x + B_4\right)$$

Which form to use, depends on the boundary conditions. Recall that the hypebolic sine vanishes ONLY at $x = 0$ and the hyperbolic cosine is always positive. If we use the last form of the general solution then we immediately find that $B_4 = 0$ is a result of the first boundary condition and clearly to satisfy the second boundary condition we must have $A_4 = 0$ (recall $\sinh x = 0$ only for $x = 0$ and the second boundary condition reads $A_4 \sinh\sqrt{-\lambda}\pi = 0$, thus the coefficient $A_4$ must vanish).

Any other form will yields the same trivial solution, may be with more work!!!

Case 2: $\lambda = 0$

In this case we have a double root $r = 0$ and as we know from ODEs the solution is

$$X = Ax + B$$

The boundary condition at zero yields

$$X(0) = B = 0$$

and the second condition

$$X(\pi) = A\pi = 0$$

This implies that $A = 0$ and therefore we again have a trivial solution.

Case 3: $\lambda > 0$
In this case the two roots are imaginary

$$r = \pm i\sqrt{\lambda}$$

Thus the solution is a combination of sine and cosine

$$X = A_1 \cos\sqrt{\lambda}x + B_1 \sin\sqrt{\lambda}x$$

Substitute the boundary condition at zero

$$X(0) = A_1$$

Thus $A_1 = 0$ and the solution is

$$X = B_1 \sin\sqrt{\lambda}x$$

Now use the condition at $\pi$

$$X(\pi) = B_1 \sin\sqrt{\lambda}\pi = 0$$

If we take $B_1 = 0$, we get a trivial solution, but we have another choice, namely

$$\sin\sqrt{\lambda}\pi = 0$$

This implies that the argument of the sine function is a multiple of $\pi$

$$\sqrt{\lambda_n}\pi = n\pi \qquad n = 1, 2, \ldots$$

Notice that since $\lambda > 0$ we must have $n > 0$. Thus

$$\sqrt{\lambda_n} = n \qquad n = 1, 2, \ldots$$

Or

$$\lambda_n = n^2 \qquad n = 1, 2, \ldots$$

The solution is then depending on $n$, and obtained by substituting for $\lambda_n$

$$X_n(x) = \sin nx$$

Note that we ignored the constant $B_1$ since the eigenfunctions are determined up to a multiplicative constant. (We will see later that the constant will be incorporated with that of the linear combination used to get the solution for the PDE)

1. b.

$$X'' + \lambda X = 0$$
$$X'(0) = 0$$
$$X'(L) = 0$$

Try $e^{rx}$. As we know from ODEs, this leads to the characteristic equation for $r$

$$r^2 + \lambda = 0$$

Or

$$r = \pm\sqrt{-\lambda}$$

We now consider three cases depending on the sign of $\lambda$

Case 1: $\lambda < 0$

In this case $r$ is the square root of a positive number and thus we have two real roots. In this case the solution is a linear combination of two real exponentials

$$X = A_1 e^{\sqrt{-\lambda}x} + B_1 e^{-\sqrt{-\lambda}x}$$

It is well known that the solution can also be written as a combination of hyperbolic sine and cosine, i.e.

$$X = A_2 \cosh\sqrt{-\lambda}x + B_2 \sinh\sqrt{-\lambda}x$$

The other two forms are may be less known, but easily proven. The solution can be written as a shifted hyperbolic cosine (sine). The proof is straight forward by using the formula for $\cosh(a+b)$ $(\sinh(a+b))$

$$X = A_3 \cosh\left(\sqrt{-\lambda}x + B_3\right)$$

Or

$$X = A_4 \sinh\left(\sqrt{-\lambda}x + B_4\right)$$

Which form to use, depends on the boundary conditions. Recall that the hypebolic sine vanishes ONLY at $x = 0$ and the hyperbolic cosine is always positive. If we use the following form of the general solution

$$X = A_3 \cosh\left(\sqrt{-\lambda}x + B_3\right)$$

then the derivative $X'$ wil be

$$X' = \sqrt{-\lambda}A_3 \sinh\left(\sqrt{-\lambda}x + B_3\right)$$

The first boundary condition $X'(0) =$ yields $B_3 = 0$ and clearly to satisfy the second boundary condition we must have $A_3 = 0$ (recall $\sinh x = 0$ only for $x = 0$ and the second boundary condition reads $\sqrt{-\lambda}A_3 \sinh\sqrt{-\lambda}L = 0$, thus the coefficient $A_3$ must vanish).

Any other form will yields the same trivial solution, may be with more work!!!

Case 2: $\lambda = 0$

In this case we have a double root $r = 0$ and as we know from ODEs the solution is

$$X = Ax + B$$

The boundary condition at zero yields

$$X'(0) = A = 0$$

and the second condition

$$X'(L) = A = 0$$

This implies that $A = 0$ and therefore we have no restriction on $B$. Thus in this case the solution is a constant and we take

$$X(x) = 1$$

Case 3: $\lambda > 0$

In this case the two roots are imaginary

$$r = \pm i\sqrt{\lambda}$$

Thus the solution is a combination of sine and cosine

$$X = A_1 \cos\sqrt{\lambda}x + B_1 \sin\sqrt{\lambda}x$$

Differentiate

$$X' = -\sqrt{\lambda}A_1 \sin\sqrt{\lambda}x + \sqrt{\lambda}B_1 \cos\sqrt{\lambda}x$$

Substitute the boundary condition at zero

$$X'(0) = \sqrt{\lambda}B_1$$

Thus $B_1 = 0$ and the solution is

$$X = A_1 \cos\sqrt{\lambda}x$$

Now use the condition at $L$

$$X'(L) = -\sqrt{\lambda}A_1 \sin\sqrt{\lambda}L = 0$$

If we take $A_1 = 0$, we get a trivial solution, but we have another choice, namely

$$\sin\sqrt{\lambda}L = 0$$

This implies that the argument of the sine function is a multiple of $\pi$

$$\sqrt{\lambda_n}L = n\pi \qquad n = 1, 2, \ldots$$

Notice that since $\lambda > 0$ we must have $n > 0$. Thus

$$\sqrt{\lambda_n} = \frac{n\pi}{L} \qquad n = 1, 2, \ldots$$

Or

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, \ldots$$

The solution is then depending on $n$, and obtained by substituting $\lambda_n$

$$X_n(x) = \cos\frac{n\pi}{L}x$$

1. c.

$$X'' + \lambda X = 0$$

$$X(0) = 0$$

$$X'(L) = 0$$

Try $e^{rx}$. As we know from ODEs, this leads to the characteristic equation for $r$

$$r^2 + \lambda = 0$$

Or

$$r = \pm\sqrt{-\lambda}$$

We now consider three cases depending on the sign of $\lambda$

Case 1: $\lambda < 0$

In this case $r$ is the square root of a positive number and thus we have two real roots. In this case the solution is a linear combination of two real exponentials

$$X = A_1 e^{\sqrt{-\lambda}x} + B_1 e^{-\sqrt{-\lambda}x}$$

It is well known that the solution can also be written as a combination of hyperbolic sine and cosine, i.e.

$$X = A_2 \cosh\sqrt{-\lambda}x + B_2 \sinh\sqrt{-\lambda}x$$

The other two forms are may be less known, but easily proven. The solution can be written as a shifted hyperbolic cosine (sine). The proof is straight forward by using the formula for $\cosh(a+b)$ $(\sinh(a+b))$

$$X = A_3 \cosh\left(\sqrt{-\lambda}x + B_3\right)$$

Or

$$X = A_4 \sinh\left(\sqrt{-\lambda}x + B_4\right)$$

Which form to use, depends on the boundary conditions. Recall that the hypebolic sine vanishes ONLY at $x = 0$ and the hyperbolic cosine is always positive. If we use the following form of the general solution

$$X = A_4 \sinh\left(\sqrt{-\lambda}x + B_4\right)$$

then the derivative $X'$ wil be

$$X' = \sqrt{-\lambda}A_4 \cosh\left(\sqrt{-\lambda}x + B_4\right)$$

The first boundary condition $X(0) =$ yields $B_4 = 0$ and clearly to satisfy the second boundary condition we must have $A_4 = 0$ (recall $\cosh x$ is never zero thus the coefficient $A_4$ must vanish).

Any other form will yields the same trivial solution, may be with more work!!!

Case 2: $\lambda = 0$

In this case we have a double root $r = 0$ and as we know from ODEs the solution is

$$X = Ax + B$$

The boundary condition at zero yields

$$X(0) = B = 0$$

and the second condition

$$X'(L) = A = 0$$

This implies that $B = A = 0$ and therefore we have again a trivial solution.

Case 3: $\lambda > 0$

In this case the two roots are imaginary

$$r = \pm i\sqrt{\lambda}$$

Thus the solution is a combination of sine and cosine

$$X = A_1 \cos\sqrt{\lambda}x + B_1 \sin\sqrt{\lambda}x$$

Differentiate

$$X' = -\sqrt{\lambda}A_1 \sin\sqrt{\lambda}x + \sqrt{\lambda}B_1 \cos\sqrt{\lambda}x$$

Substitute the boundary condition at zero

$$X(0) = A_1$$

Thus $A_1 = 0$ and the solution is

$$X = B_1 \sin\sqrt{\lambda}x$$

Now use the condition at $L$

$$X'(L) = \sqrt{\lambda}B_1 \cos\sqrt{\lambda}L = 0$$

If we take $B_1 = 0$, we get a trivial solution, but we have another choice, namely

$$\cos\sqrt{\lambda}L = 0$$

This implies that the argument of the cosine function is a multiple of $\pi$ plus $\pi/2$

$$\sqrt{\lambda_n}L = \left(n + \frac{1}{2}\right)\pi \qquad n = 0, 1, 2, \ldots$$

Notice that since $\lambda > 0$ we must have $n \geq 0$. Thus

$$\sqrt{\lambda_n} = \frac{\left(n + \frac{1}{2}\right)\pi}{L} \qquad n = 0, 1, 2, \ldots$$

Or

$$\lambda_n = \left(\frac{\left(n + \frac{1}{2}\right)\pi}{L}\right)^2 \qquad n = 0, 1, 2, \ldots$$

The solution is then depending on $n$, and obtained by substituting $\lambda_n$

$$X_n(x) = \sin\frac{\left(n + \frac{1}{2}\right)\pi}{L}x$$

1. d.

$$X'' + \lambda X = 0$$
$$X'(0) = 0$$
$$X(L) = 0$$

Try $e^{rx}$. As we know from ODEs, this leads to the characteristic equation for $r$

$$r^2 + \lambda = 0$$

Or

$$r = \pm\sqrt{-\lambda}$$

We now consider three cases depending on the sign of $\lambda$

Case 1: $\lambda < 0$

In this case $r$ is the square root of a positive number and thus we have two real roots. In this case the solution is a linear combination of two real exponentials

$$X = A_1 e^{\sqrt{-\lambda}x} + B_1 e^{-\sqrt{-\lambda}x}$$

It is well known that the solution can also be written as a combination of hyperbolic sine and cosine, i.e.

$$X = A_2 \cosh\sqrt{-\lambda}x + B_2 \sinh\sqrt{-\lambda}x$$

The other two forms are may be less known, but easily proven. The solution can be written as a shifted hyperbolic cosine (sine). The proof is straight forward by using the formula for $\cosh(a+b)$ $(\sinh(a+b))$

$$X = A_3 \cosh\left(\sqrt{-\lambda}x + B_3\right)$$

Or

$$X = A_4 \sinh\left(\sqrt{-\lambda}x + B_4\right)$$

Which form to use, depends on the boundary conditions. Recall that the hypebolic sine vanishes ONLY at $x = 0$ and the hyperbolic cosine is always positive. If we use the following form of the general solution

$$X = A_3 \cosh\left(\sqrt{-\lambda}x + B_3\right)$$

then the derivative $X'$ wil be

$$X' = \sqrt{-\lambda}A_3 \sinh\left(\sqrt{-\lambda}x + B_3\right)$$

The first boundary condition $X'(0) =$ yields $B_3 = 0$ and clearly to satisfy the second boundary condition we must have $A_3 = 0$ (recall $\cosh x$ is never zero thus the coefficient $A_3$ must vanish).

Any other form will yields the same trivial solution, may be with more work!!!

Case 2: $\lambda = 0$

In this case we have a double root $r = 0$ and as we know from ODEs the solution is

$$X = Ax + B$$

The boundary condition at zero yields

$$X'(0) = A = 0$$

and the second condition

$$X(L) = B = 0$$

This implies that $B = A = 0$ and therefore we have again a trivial solution.

Case 3: $\lambda > 0$

In this case the two roots are imaginary

$$r = \pm i\sqrt{\lambda}$$

Thus the solution is a combination of sine and cosine

$$X = A_1 \cos\sqrt{\lambda}x + B_1 \sin\sqrt{\lambda}x$$

Differentiate

$$X' = -\sqrt{\lambda}A_1 \sin\sqrt{\lambda}x + \sqrt{\lambda}B_1 \cos\sqrt{\lambda}x$$

Substitute the boundary condition at zero

$$X'(0) = \sqrt{\lambda}B_1$$

Thus $B_1 = 0$ and the solution is

$$X = A_1 \cos\sqrt{\lambda}x$$

Now use the condition at $L$

$$X(L) = A_1 \cos\sqrt{\lambda}L = 0$$

If we take $A_1 = 0$, we get a trivial solution, but we have another choice, namely

$$\cos\sqrt{\lambda}L = 0$$

This implies that the argument of the cosine function is a multiple of $\pi$ plus $\pi/2$

$$\sqrt{\lambda_n}L = \left(n + \frac{1}{2}\right)\pi \qquad n = 0, 1, 2, \ldots$$

Notice that since $\lambda > 0$ we must have $n \geq 0$. Thus

$$\sqrt{\lambda_n} = \frac{\left(n + \frac{1}{2}\right)\pi}{L} \qquad n = 0, 1, 2, \ldots$$

Or

$$\lambda_n = \left(\frac{\left(n + \frac{1}{2}\right)\pi}{L}\right)^2 \qquad n = 0, 1, 2, \ldots$$

The solution is then depending on $n$, and obtained by substituting $\lambda_n$

$$X_n(x) = \cos\frac{\left(n + \frac{1}{2}\right)\pi}{L}x$$

1. e.

$$X'' + \lambda X = 0$$

$$X(0) = 0$$

$$X'(L) + X(L) = 0$$

Try $e^{rx}$. As we know from ODEs, this leads to the characteristic equation for $r$

$$r^2 + \lambda = 0$$

Or

$$r = \pm\sqrt{-\lambda}$$

We now consider three cases depending on the sign of $\lambda$

Case 1: $\lambda < 0$

In this case $r$ is the square root of a positive number and thus we have two real roots. In this case the solution is a linear combination of two real exponentials

$$X = A_1 e^{\sqrt{-\lambda}x} + B_1 e^{-\sqrt{-\lambda}x}$$

It is well known that the solution can also be written as a combination of hyperbolic sine and cosine, i.e.

$$X = A_2 \cosh\sqrt{-\lambda}x + B_2 \sinh\sqrt{-\lambda}x$$

The other two forms are may be less known, but easily proven. The solution can be written as a shifted hyperbolic cosine (sine). The proof is straight forward by using the formula for $\cosh(a + b)$ $(\sinh(a + b))$

$$X = A_3 \cosh\left(\sqrt{-\lambda}x + B_3\right)$$

Or

$$X = A_4 \sinh\left(\sqrt{-\lambda}x + B_4\right)$$

Which form to use, depends on the boundary conditions. Recall that the hypebolic sine vanishes ONLY at $x = 0$ and the hyperbolic cosine is always positive. If we use the following form of the general solution

$$X = A_4 \sinh\left(\sqrt{-\lambda}x + B_4\right)$$

then the derivative $X'$ wil be

$$X' = \sqrt{-\lambda}A_4 \cosh\left(\sqrt{-\lambda}x + B_4\right)$$

The first boundary condition $X(0) = 0$ yields $B_4 = 0$ and clearly to satisfy the second boundary condition

$$\sqrt{-\lambda}A_4 \cosh\sqrt{-\lambda}L = 0$$

we must have $A_4 = 0$ (recall $\cosh x$ is never zero thus the coefficient $A_4$ must vanish).

Any other form will yields the same trivial solution, may be with more work!!!

Case 2: $\lambda = 0$
In this case we have a double root $r = 0$ and as we know from ODEs the solution is

$$X = Ax + B$$

The boundary condition at zero yields

$$X(0) = B = 0$$

and the second condition

$$X'(L) + X(L) = A + AL = 0$$

Or

$$A(1 + L) = 0$$

This implies that $B = A = 0$ and therefore we have again a trivial solution.

Case 3: $\lambda > 0$
In this case the two roots are imaginary

$$r = \pm i\sqrt{\lambda}$$

Thus the solution is a combination of sine and cosine

$$X = A_1 \cos\sqrt{\lambda}x + B_1 \sin\sqrt{\lambda}x$$

Differentiate

$$X' = -\sqrt{\lambda}A_1 \sin\sqrt{\lambda}x + \sqrt{\lambda}B_1 \cos\sqrt{\lambda}x$$

Substitute the boundary condition at zero

$$X(0) = A_1$$

Thus $A_1 = 0$ and the solution is

$$X = B_1 \sin\sqrt{\lambda}x$$

Now use the condition at $L$

$$X'(L) + X(L) = \sqrt{\lambda}B_1 \cos\sqrt{\lambda}L + B_1 \sin\sqrt{\lambda}L = 0$$

If we take $B_1 = 0$, we get a trivial solution, but we have another choice, namely

$$\sqrt{\lambda}\cos\sqrt{\lambda}L + \sin\sqrt{\lambda}L = 0$$

If $\cos\sqrt{\lambda}L = 0$ then we are left with $\sin\sqrt{\lambda}L = 0$ which is not possible (the cosine and sine functions do not vanish at the same points).

Thus $\cos\sqrt{\lambda}L \neq 0$ and upon dividing by it we get

$$-\sqrt{\lambda} = \tan\sqrt{\lambda}L$$

This can be solved graphically or numerically (see figure). The points of intersection are values of $\sqrt{\lambda_n}$. The solution is then depending on $n$, and obtained by substituting $\lambda_n$

$$X_n(x) = \sin\sqrt{\lambda_n}x$$

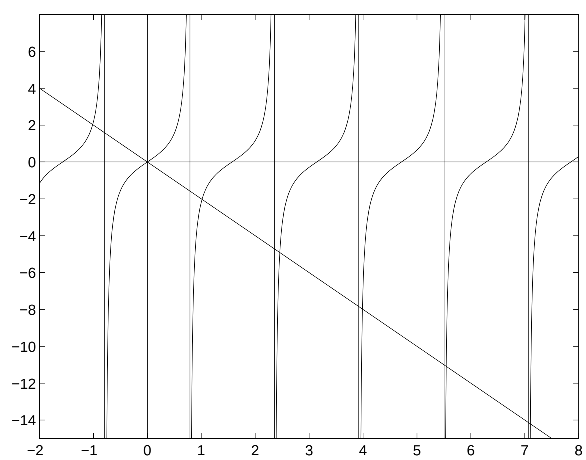


Figure 1: Graphical solution of the eigenvalue problem

# 3 Fourier Series

## 3.1 Introduction

## 3.2 Orthogonality

## 3.3 Computation of Coefficients

### Problems

1. For the following functions, sketch the Fourier series of $f(x)$ on the interval $[-L, L]$. Compare $f(x)$ to its Fourier series

a. $f(x) = 1$

b. $f(x) = x^2$

c. $f(x) = e^x$

d.

$$f(x) = \begin{cases} \frac{1}{2}x & x < 0 \\ 3x & x > 0 \end{cases}$$

e.

$$f(x) = \begin{cases} 0 & x < \frac{L}{2} \\ & \\ x^2 & x > \frac{L}{2} \end{cases}$$

2. Sketch the Fourier series of $f(x)$ on the interval $[-L, L]$ and evaluate the Fourier coefficients for each

a. $f(x) = x$

b. $f(x) = \sin \frac{\pi}{L} x$

c.

$$f(x) = \begin{cases} 1 & |x| < \frac{L}{2} \\ & \\ 0 & |x| > \frac{L}{2} \end{cases}$$

3. Show that the Fourier series operation is linear, i.e. the Fourier series of $\alpha f(x) + \beta g(x)$ is the sum of the Fourier series of $f(x)$ and $g(x)$ multiplied by the corresponding constant.

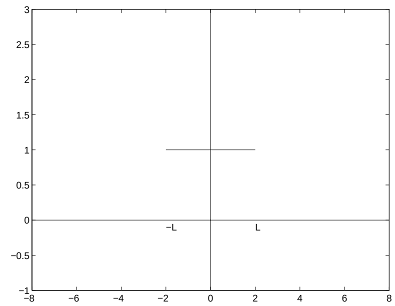


Figure 2: Graph of $f(x) = 1$

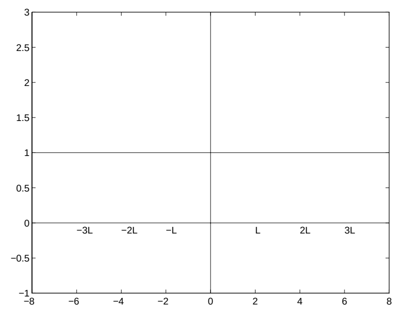


Figure 3: Graph of its periodic extension

1. a. $f(x) = 1$

Since the periodic extension of $f(x)$ is continuous, the Fourier series is identical to (the periodic extension of) $f(x)$ everywhere.

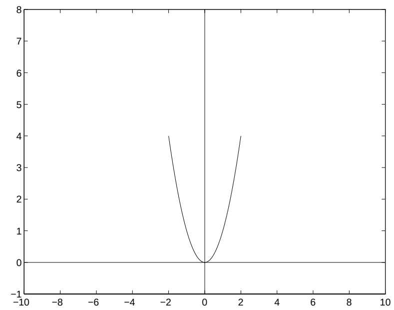


Figure 4: Graph of $f(x) = x^2$

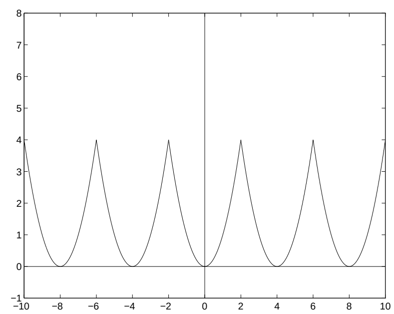


Figure 5: Graph of its periodic extension

1. b. $f(x) = x^2$

Since the periodic extension of $f(x)$ is continuous, the Fourier series is identical to (the periodic extension of) $f(x)$ everywhere.

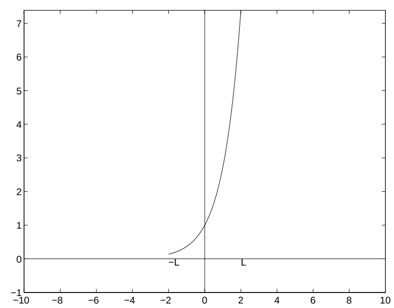


Figure 6: Graph of $f(x) = e^x$

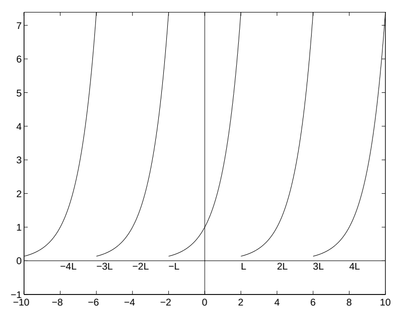


Figure 7: Graph of its periodic extension

1. c. $f(x) = e^x$

Since the periodic extension of $f(x)$ is discontinuous, the Fourier series is identical to (the periodic extension of) $f(x)$ everywhere except for the points of discontinuities. At $x = \pm L$ (and similar points in each period), we have the average value, i.e.

$$\frac{e^L + e^{-L}}{2} = \cosh L$$

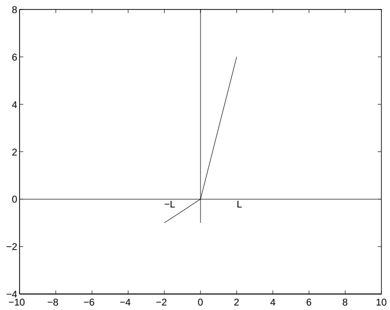


Figure 8: Graph of $f(x)$

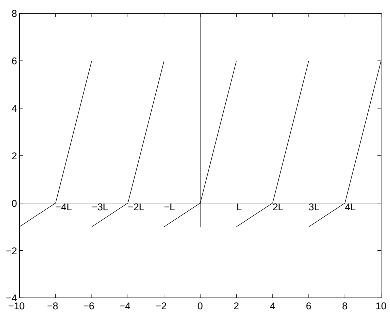


Figure 9: Graph of its periodic extension

1. d.

Since the periodic extension of $f(x)$ is discontinuous, the Fourier series is identical to (the periodic extension of) $f(x)$ everywhere except at the points of discontinuities. At those points $x = \pm L$ (and similar points in each period), we have

$$\frac{3L + (-\frac{1}{2}L)}{2} = \frac{5}{4}L$$

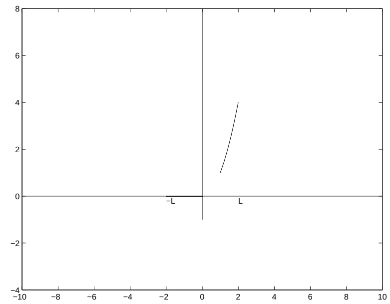


Figure 10: Graph of $f(x)$

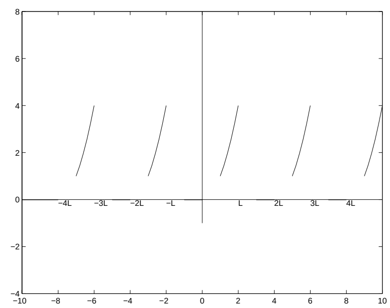


Figure 11: Graph of its periodic extension

1. e.

Since the periodic extension of $f(x)$ is discontinuous, the Fourier series is identical to (the periodic extension of) $f(x)$ everywhere except at the points of discontinuities. At those points $x = \pm L$ (and similar points in each period), we have

$$\frac{L^2 + 0}{2} = \frac{1}{2}L^2$$

At the point $L/2$ and similar ones in each period we have

$$\frac{0 + \frac{1}{4}L^2}{2} = \frac{1}{8}L^2$$

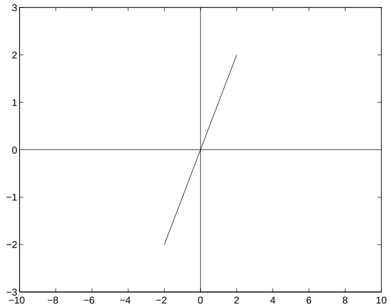

Figure 12: Graph of $f(x) = x$

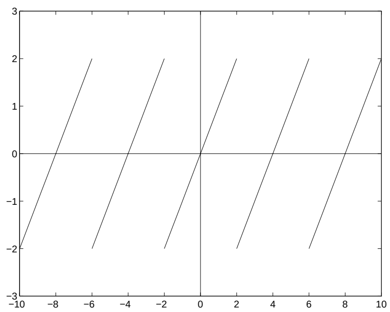

Figure 13: Graph of its periodic extension

2. a. $f(x) = x$

Since the periodic extension of $f(x)$ is discontinuous, the Fourier series is identical to (the periodic extension of) $f(x)$ everywhere except at the points of discontinuities. At those point $x = \pm L$ (and similar points in each period), we have

$$\frac{L + (-L)}{2} = 0$$

Now we evaluate the coefficients.

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_{-L}^{L} x\,dx = 0$$

Since we have integrated an odd function on a symmetric interval. Similarly for all $a_n$.

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\,\sin\frac{n\pi}{L}x\,dx = \frac{1}{L}\left\{\frac{-x\cos\frac{n\pi}{L}x}{\frac{n\pi}{L}}\Big|_{-L}^{L} + \int_{-L}^{L}\frac{\cos\frac{n\pi}{L}x}{\frac{n\pi}{L}}\,dx\right\}$$

This was a result of integration by parts.

$$= \frac{1}{L}\left\{\frac{-L\cos n\pi}{\frac{n\pi}{L}} + \frac{-L\cos(-n\pi)}{\frac{n\pi}{L}} + \frac{\sin\frac{n\pi}{L}x}{\left(\frac{n\pi}{L}\right)^2}\Big|_{-L}^{L}\right\}$$

The last term vanishes at both end points $\pm L$

$$= \frac{1}{L}\frac{-2L\cos n\pi}{\frac{n\pi}{L}} = -\frac{2L}{n\pi}(-1)^n$$

Thus

$$b_n = \frac{2L}{n\pi}(-1)^{n+1}$$

and the Fourier series is

$$x \sim \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{L}x$$

2. b. This function is already in a Fourier sine series form and thus we can read the coefficients

$$\begin{aligned} a_n &= 0 \qquad n = 0, 1, 2, \ldots \\ b_n &= 0 \qquad n \neq 1 \\ b_1 &= 1 \end{aligned}$$

2. c.

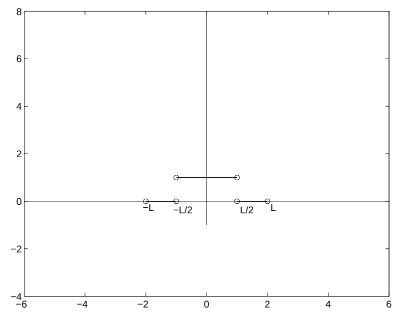


Figure 14: graph of $f(x)$ for problem 2c

Since the function is even, all the coefficients $b_n$ will vanish.

$$a_0 = \frac{1}{L}\int_{-L/2}^{L/2} dx = \frac{1}{L}x|_{-L/2}^{L/2} = \frac{1}{L}\left(\frac{L}{2} - (-\frac{L}{2})\right) = 1$$

$$a_n = \frac{1}{L}\int_{-L/2}^{L/2} \cos\frac{n\pi}{L}x\,dx = \frac{1}{L}\frac{L}{n\pi}\sin\frac{n\pi}{L}x|_{-L/2}^{L/2} = \frac{1}{n\pi}\left(\sin\frac{n\pi}{2} - \sin\frac{-n\pi}{2}\right)$$

Since the sine function is odd the last two terms add up and we have

$$a_n = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

The Fourier series is

$$f(x) \sim \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi}{L}x$$

3.

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x \right)$$

$$g(x) \sim \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi}{L}x + B_n \sin \frac{n\pi}{L}x \right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos \frac{n\pi}{L}x\, dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi}{L}x\, dx$$

$$A_n = \frac{1}{L}\int_{-L}^{L} g(x) \cos \frac{n\pi}{L}x\, dx$$

$$B_n = \frac{1}{L}\int_{-L}^{L} g(x) \sin \frac{n\pi}{L}x\, dx$$

For $\alpha f(x) + \beta g(x)$ we have

$$\frac{1}{2}\gamma_0 + \sum_{n=1}^{\infty} \left( \gamma_n \cos \frac{n\pi}{L}x + \delta_n \sin \frac{n\pi}{L}x \right)$$

and the coefficients are

$$\gamma_0 = \frac{1}{L}\int_{-L}^{L} (\alpha f(x) + \beta g(x))dx$$

which by linearity of the integral is

$$\gamma_0 = \alpha\frac{1}{L}\int_{-L}^{L} f(x)dx + \beta\frac{1}{L}\int_{-L}^{L} g(x)dx = \alpha a_0 + \beta A_0$$

Similarly for $\gamma_n$ and $\delta_n$.

$$\gamma_n = \frac{1}{L}\int_{-L}^{L} (\alpha f(x) + \beta g(x)) \cos \frac{n\pi}{L}x\, dx$$

which by linearity of the integral is

$$\gamma_n = \alpha\frac{1}{L}\int_{-L}^{L} f(x) \cos \frac{n\pi}{L}x\, dx + \beta\frac{1}{L}\int_{-L}^{L} g(x) \cos \frac{n\pi}{L}x\, dx = \alpha a_n + \beta A_n$$

$$\delta_n = \frac{1}{L}\int_{-L}^{L} (\alpha f(x) + \beta g(x)) \sin \frac{n\pi}{L}x\, dx$$

which by linearity of the integral is

$$\delta_n = \alpha\frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi}{L}x\, dx + \beta\frac{1}{L}\int_{-L}^{L} g(x) \sin \frac{n\pi}{L}x\, dx = \alpha b_n + \beta B_n$$

## 3.4 Relationship to Least Squares

## 3.5 Convergence

## 3.6 Fourier Cosine and Sine Series

### Problems

1. For each of the following functions
   i. Sketch $f(x)$
   ii. Sketch the Fourier series of $f(x)$
   iii. Sketch the Fourier sine series of $f(x)$
   iv. Sketch the Fourier cosine series of $f(x)$
   a. $f(x) = \begin{cases} x & x < 0 \\ 1+x & x > 0 \end{cases}$
   b. $f(x) = e^x$
   c. $f(x) = 1 + x^2$
   d. $f(x) = \begin{cases} \frac{1}{2}x + 1 & -2 < x < 0 \\ x & 0 < x < 2 \end{cases}$

2. Sketch the Fourier sine series of

$$f(x) = \cos \frac{\pi}{L} x.$$

Roughly sketch the sum of the first three terms of the Fourier sine series.

3. Sketch the Fourier cosine series and evaluate its coefficients for

$$f(x) = \begin{cases} 1 & x < \frac{L}{6} \\ 3 & \frac{L}{6} < x < \frac{L}{2} \\ 0 & \frac{L}{2} < x \end{cases}$$

4. Fourier series can be defined on other intervals besides $[-L, L]$. Suppose $g(y)$ is defined on $[a, b]$ and periodic with period $b - a$. Evaluate the coefficients of the Fourier series.

5. Expand

$$f(x) = \begin{cases} 1 & 0 < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

in a series of $\sin nx$.

   a. Evaluate the coefficients explicitly.

   b. Graph the function to which the series converges to over $-2\pi < x < 2\pi$.

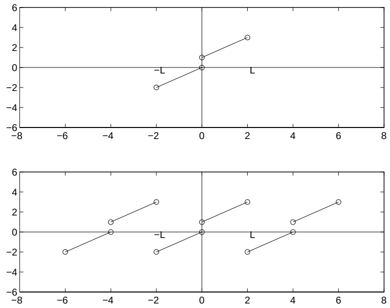


Figure 15: Sketch of $f(x)$ and its periodic extension for 1a

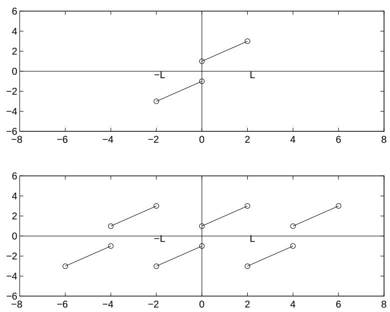


Figure 16: Sketch of the odd extension and its periodic extension for 1a

1. a.

The Fourier series is the same as the periodic extension except for the points of discontinuities where the Fourier series yields

$$\frac{1\,+\,0}{2}\,=\,\frac{1}{2}$$

For the Fourier sine series we take ONLY the right branch on the interval $[0, L]$ and extend it as an odd function.

Now the same discontinuities are there but the value of the Fourier series at those points is

$$\frac{1\,+\,(-1)}{2}\,=\,0$$

For the Fourier cosine series we need an even extension

Note that the periodic extension IS continuous and the Fourier series gives the exact same sketch.

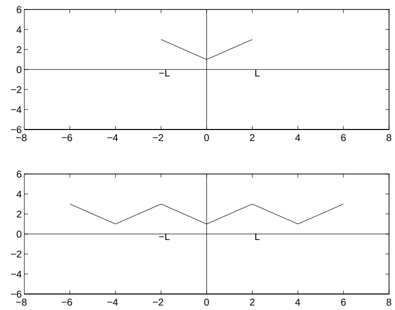


Figure 17: Sketch of the even extension and its periodic extension for 1a

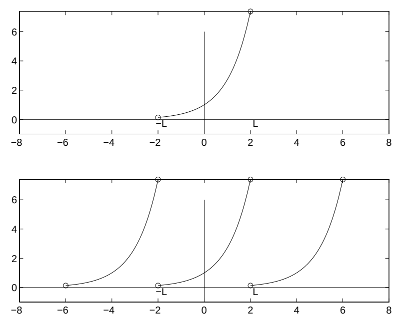


Figure 18: Sketch of $f(x)$ and its periodic extension for 1b

1. b.

The Fourier series is the same as the periodic extension except for the points of discontinuities where the Fourier series yields

$$\frac{e^L + e^{-L}}{2} = \cosh L$$

For the Fourier sine series we take ONLY the right branch on the interval $[0, L]$ and extend it as an odd function.

Now the same discontinuities are there but the value of the Fourier series at those points

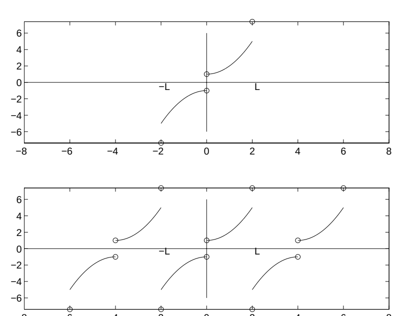


Figure 19: Sketch of the odd extension and its periodic extension for 1b

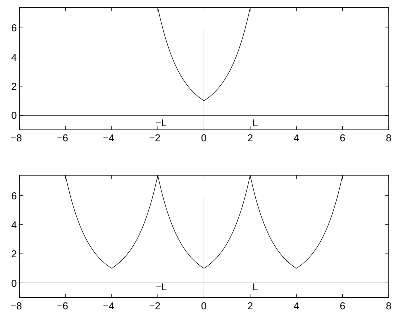


Figure 20: Sketch of the even extension and its periodic extension for 1b

is

$$\frac{1 + (-1)}{2} = 0$$

For the Fourier cosine series we need an even extension

Note that the periodic extension IS continuous and the Fourier series gives the exact same sketch.

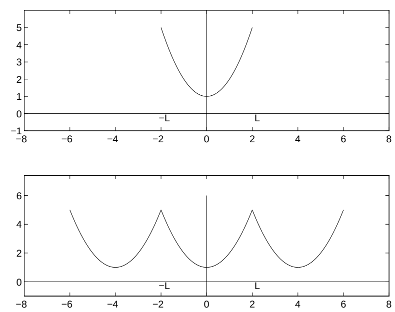


Figure 21: Sketch of $f(x)$ and its periodic extension for 1c

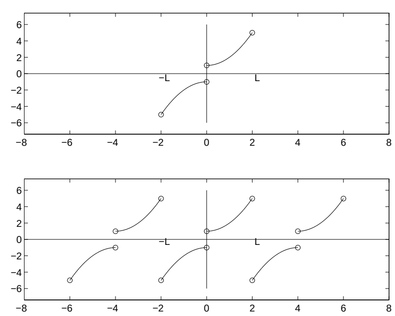


Figure 22: Sketch of the odd extension and its periodic extension for 1c

1. c.

The Fourier series is the same as the periodic extension. In fact the Fourier cosine series is the same!!!

For the Fourier sine series we take ONLY the right branch on the interval $[0, L]$ and extend it as an odd function.

Now the same discontinuities are there but the value of the Fourier series at those points is

$$\frac{1 + (-1)}{2} = 0$$

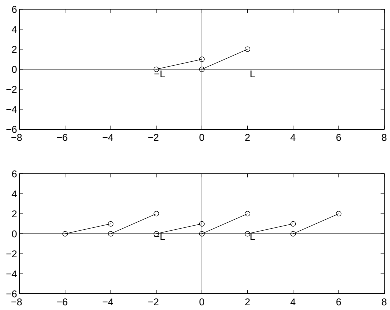


Figure 23: Sketch of $f(x)$ and its periodic extension for 1d

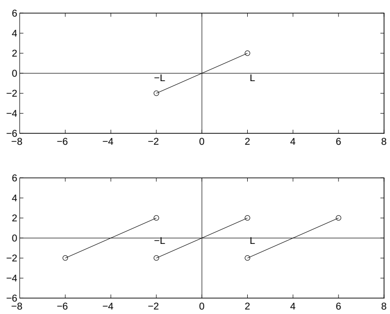


Figure 24: Sketch of the odd extension and its periodic extension for 1d

1. d.

The Fourier series is the same as the periodic extension except for the points of discontinuities where the Fourier series yields

$$\frac{1\,+\,0}{2} = \frac{1}{2} \qquad \text{for } x = 0 + \text{multiples of } 4$$

$$\frac{1\,+\,2}{2} = \frac{3}{2} \qquad \text{for } x = 2 + \text{multiples of } 4$$

For the Fourier sine series we take ONLY the right branch on the interval $[0, L]$ and extend it as an odd function.

Now some of the same discontinuities are there but the value of the Fourier series at those points is

$$\frac{2\,+\,(-2)}{2} = 0 \qquad \text{for } x = 2 + \text{multiples of } 4$$

At the other previous discontinuities we now have continuity.

For the Fourier cosine series we need an even extension

Note that the periodic extension IS continuous and the Fourier series gives the exact same sketch.

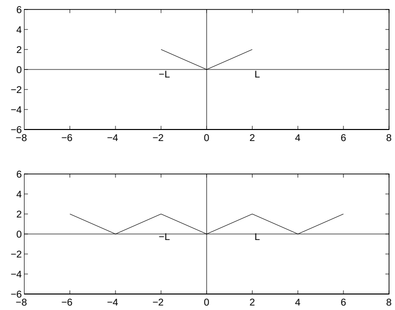


Figure 25: Sketch of the even extension and its periodic extension for 1d

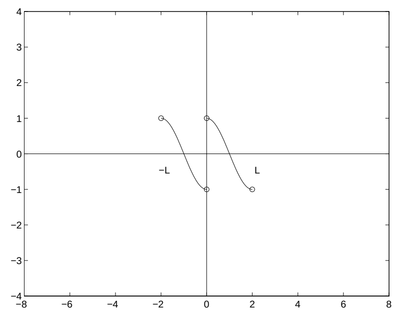


Figure 26: Sketch of the odd extension for 2

2.

$$\cos \frac{\pi x}{L} = \sum_{n=1}^{\infty} b_n \sin \frac{n \pi}{L} x$$

$$bn = \begin{cases} 0 & n \, \text{odd} \\ \dfrac{4n}{\pi \left(n^2 - 1\right)} & n \, \text{even} \end{cases}$$

Since we have a Fourier sine series, we need the odd extension of $f(x)$

Now extend by periodicity
At points of discontinuity the Fourier series give zero.

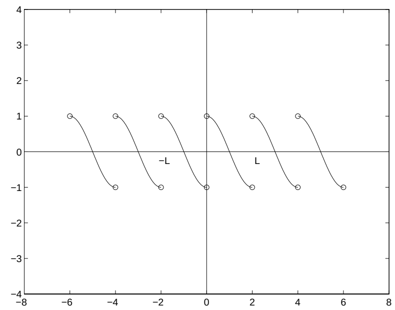


Figure 27: Sketch of the periodic extension of the odd extension for 2

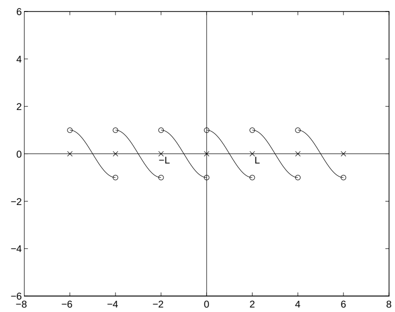


Figure 28: Sketch of the Fourier sine series for 2

First two terms of the Fourier sine series of $\cos \dfrac{\pi x}{L}$ are

$$= b_2 \, sin \frac{2\pi x}{L} + b_4 \sin \frac{4\pi x}{L}$$

$$= \frac{8}{3\pi} \sin \frac{2\pi}{L} x + \frac{16}{15\pi} \sin \frac{4\pi}{L} x$$

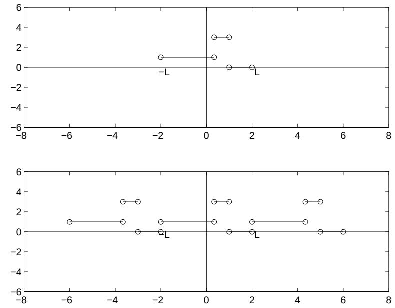


Figure 29: Sketch of $f(x)$ and its periodic extension for problem 3

3.

$$f(x) = \begin{cases} 1 & x < L/6 \\ 3 & L/6 < x < L/2 \\ 0 & x > L/2 \end{cases}$$

Fourier cosine series coefficients:

$$a_0 = \frac{2}{L} \left\{ \int_0^{L/6} dx + \int_{L/6}^{L/2} 3\, dx \right\} = \frac{2}{L} \left\{ \frac{L}{6} + 3 \left( \frac{L}{2} - \frac{L}{6} \right) \right\}$$

$$= \frac{2}{L} \left\{ \frac{L}{6} + 3\, \frac{2L}{6} \right\} = \frac{2}{L} \cdot \frac{7}{6} L = \frac{7}{3}$$

$$\begin{aligned}
a_n &= \tfrac{2}{L} \left\{ \int_0^{L/6} \cos \tfrac{n\pi}{L} x \, dx + 3 \int_{L/6}^{L/2} \cos \tfrac{n\pi}{L} x \, dx \right\} \\
&= \tfrac{2}{L} \left\{ \frac{\sin \frac{n\pi}{L} x}{\frac{n\pi}{L}} \Big|_0^{L/6} + 3 \, \frac{\sin \frac{n\pi}{L} x}{\frac{n\pi}{L}} \Big|_{L/6}^{L/2} \right\} \\
&= \tfrac{2}{L} \left\{ \frac{\sin \frac{n\pi}{6}}{\frac{n\pi}{L}} + 3 \, \frac{\sin \frac{n\pi}{2} - \sin \frac{n\pi}{6}}{\frac{n\pi}{6}} \right\} \\
&= \tfrac{2}{L} \, \tfrac{L}{n\pi} \left\{ 3 \underbrace{\sin \frac{n\pi}{2}}_{0 \text{ for n even}} - 2 \sin \tfrac{n\pi}{6} \right\} = \tfrac{2}{n\pi} \left\{ 3 \sin \tfrac{n\pi}{2} - 2 \sin \tfrac{n\pi}{6} \right\}
\end{aligned}$$

$\pm 1$ for $n$ odd

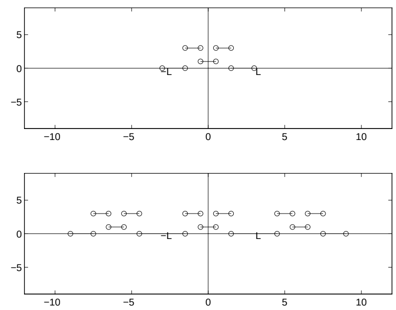

Figure 30: Sketch of the even extension of $f(x)$ and its periodic extension for problem 3

4.

$$x \,\epsilon\, [\,-L,\; L\,]$$

$$y \,\epsilon\, [\,a,\; b\,]$$

then $y = \frac{a+b}{2} + \frac{b-a}{2L}\, x$ $(*)$ is the transformation required (Note that if $x = -L$ then $y = a$ and if $x = L$ then $y = b$)

$g(y)$ is periodic of period $b - a$

$$g(y) = G(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\,\pi}{L}\, x + b_n \sin \frac{n\,\pi}{L}\, x)$$

$$a_n = \frac{1}{L} \int_{-L}^{L} G(x) \cos \frac{n\,\pi}{L}\, x\, dx$$

solving (*) for $x$ yields

$$x = \frac{2L}{b-a} \left[ y - \frac{a+b}{2} \right] \quad \Rightarrow \quad dx = \frac{2L}{b-a}\, dy$$

$$a_n = \frac{1}{L} \int_a^b g\,(y) \cos \left( \frac{n\,\pi}{L}\, \frac{2L}{b-a} \left( y - \frac{a+b}{2} \right) \right) \frac{2L}{b-a}\, dy$$

Therefore

$$a_n = \frac{2}{b-a} \int_a^b g\,(y) \cos \left[ \frac{2n\,\pi}{b-a} \left( y - \frac{a+b}{2} \right) \right] dy$$

Similarly for $b_n$

$$b_n = \frac{2}{b-a} \int_a^b g\,(y) \sin \left[ \frac{2n\,\pi}{b-a} \left( y - \frac{a+b}{2} \right) \right] dy$$

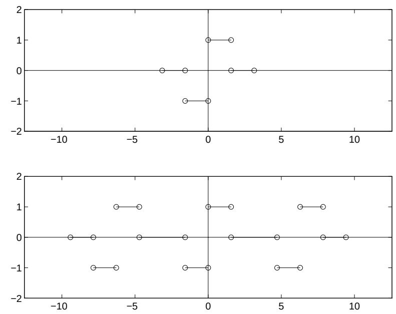


Figure 31: Sketch of the periodic extension of the odd extension of $f(x)$ (problem 5)

5.

$$f(x) = \begin{cases} 1 & 0 < x < \pi/2 \\ 0 & \pi/2 < x < \pi \end{cases}$$

Expand in series of sin $nx$

$$f(x) \sim \sum_{n=1}^{\pi} b_n \sin nx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi/2} 1 \cdot \sin nx \, dx$$

$$\uparrow$$

$f(x) =$ zero on the rest

$$= \frac{2}{\pi} \left( -\frac{1}{n} \cos nx \right) \Big|_0^{\pi/2}$$

$$= -\frac{2}{n\pi} \underbrace{\cos \frac{n\pi}{2}}_{\uparrow} + \frac{2}{n\pi}$$

this takes the values 0, $\pm 1$ depending on n!!!

## 3.7 Full solution of Several Problems

### Problems

1. Solve the heat equation

$$u_t = ku_{xx}, \qquad 0 < x < L, \qquad t > 0,$$

subject to the boundary conditions

$$u(0, t) = u(L, t) = 0.$$

Solve the problem subject to the initial value:

a. $u(x, 0) = 6 \sin \frac{9\pi}{L} x$.

b. $u(x, 0) = 2 \cos \frac{3\pi}{L} x$.

2. Solve the heat equation

$$u_t = ku_{xx}, \qquad 0 < x < L, \qquad t > 0,$$

subject to

$$u_x(0, t) = 0, \qquad t > 0$$
$$u_x(L, t) = 0, \qquad t > 0$$

a. $u(x, 0) = \begin{cases} 0 & x < \frac{L}{2} \\ 1 & x > \frac{L}{2} \end{cases}$

b. $u(x, 0) = 6 + 4 \cos \frac{3\pi}{L} x$.

3. Solve the eigenvalue problem

$$\phi'' = -\lambda \phi$$

subject to

$$\phi(0) = \phi(2\pi)$$
$$\phi'(0) = \phi'(2\pi)$$

4. Solve Laplace's equation inside a wedge of radius $a$ and angle $\alpha$,

$$\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

subject to

$$u(a, \theta) = f(\theta),$$
$$u(r, 0) = u_\theta(r, \alpha) = 0.$$

5. Solve Laplace's equation inside a rectangle $0 \le x \le L$, $0 \le y \le H$ subject to

a. $u_x(0, y) = u_x(L, y) = u(x, 0) = 0, \quad u(x, H) = f(x).$

b. $u(0, y) = g(y), \quad u(L, y) = u_y(x, 0) = u(x, H) = 0.$

c. $u(0, y) = u(L, y) = 0, \quad u(x, 0) - u_y(x, 0) = 0, \quad u(x, H) = f(x).$

6. Solve Laplace's equation <u>outside</u> a circular disk of radius $a$, subject to

a. $u(a, \theta) = \ln 2 + 4\cos 3\theta.$

b. $u(a, \theta) = f(\theta).$

7. Solve Laplace's equation inside the quarter circle of radius 1, subject to

a. $u_\theta(r, 0) = u(r, \pi/2) = 0, \qquad u(1, \theta) = f(\theta).$

b. $u_\theta(r, 0) = u_\theta(r, \pi/2) = 0, \qquad u_r(1, \theta) = g(\theta).$

c. $u(r, 0) = u(r, \pi/2) = 0, \qquad u_r(1, \theta) = 1.$

8. Solve Laplace's equation inside a circular annulus ($a < r < b$), subject to

a. $u(a, \theta) = f(\theta), \qquad u(b, \theta) = g(\theta).$

b. $u_r(a, \theta) = f(\theta), \qquad u_r(b, \theta) = g(\theta).$

9. Solve Laplace's equation inside a semi-infinite strip ($0 < x < \infty, \quad 0 < y < H$) subject to

$u_y(x, 0) = 0, \qquad u_y(x, H) = 0, \qquad u(0, y) = f(y).$

10. Consider the heat equation

$$u_t = u_{xx} + q(x, t), \qquad 0 < x < L,$$

subject to the boundary conditions

$$u(0, t) = u(L, t) = 0.$$

Assume that $q(x, t)$ is a piecewise smooth function of $x$ for each positive $t$. Also assume that $u$ and $u_x$ are continuous functions of $x$ and $u_{xx}$ and $u_t$ are piecewise smooth. Thus

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi}{L} x.$$

Write the ordinary differential equation satisfied by $b_n(t)$.

11. Solve the following inhomogeneous problem

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + e^{-t} + e^{-2t} \cos \frac{3\pi}{L} x,$$

subject to

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0,$$
$$u(x,0) = f(x).$$

Hint : Look for a solution as a Fourier cosine series. Assume $k \neq \frac{2L^2}{9\pi^2}$.

12. Solve the wave equation by the method of separation of variables

$$u_{tt} - c^2 u_{xx} = 0, \qquad 0 < x < L,$$

$$u(0,t) = 0,$$
$$u(L,t) = 0,$$
$$u(x,0) = f(x),$$
$$u_t(x,0) = g(x).$$

13. Solve the heat equation

$$u_t = 2u_{xx}, \qquad 0 < x < L,$$

subject to the boundary conditions

$$u(0,t) = u_x(L,t) = 0,$$

and the initial condition

$$u(x,0) = \sin \frac{3}{2}\frac{\pi}{L}x.$$

14. Solve the heat equation

$$\frac{\partial u}{\partial t} = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$$

inside a disk of radius $a$ subject to the boundary condition

$$\frac{\partial u}{\partial r}(a,\theta,t) = 0,$$

and the initial condition

$$u(r,\theta,0) = f(r,\theta)$$

where $f(r,\theta)$ is a given function.

15. Determine which of the following equations are separable:

| | | | |
|---|---|---|---|
| (a) | $u_{xx} + u_{yy} = 1$ | (b) | $u_{xy} + u_{yy} = u$ |
| (c) | $x^2 y u_{xx} + y^4 u_{yy} = 4u$ | (d) | $u_t + u u_x = 0$ |
| (e) | $u_{tt} + f(t) u_t = u_{xx}$ | (f) | $\frac{x^2}{y^2} u_{xxx} = u_y$ |

16. (a) Solve the one dimensional heat equation in a bar

$$u_t = k u_{xx} \qquad 0 < x < L$$

which is insulated at either end, given the initial temperature distribution

$$u(x,0) = f(x)$$

(b) What is the equilibrium temperature of the bar? and explain physically why your answer makes sense.

17. Solve the 1-D heat equation

$$u_t = k u_{xx} \qquad 0 < x < L$$

subject to the nonhomogeneous boundary conditions

$$u(0) = 1 \qquad u_x(L) = 1$$

with an initial temperature distribution $u(x,0) = 0$. (Hint: First solve for the equilibrium temperature distribution $v(x)$ which satisfies the steady state heat equation with the prescribed boundary conditions. Once $v$ is found, write $u(x,t) = v(x) + w(x,t)$ where $w(x,t)$ is the *transient* response. Substitue this $u$ back into the PDE to produce a new PDE for $w$ which now has homogeneous boundary conditions.

18. Solve Laplace's equation,

$$\nabla^2 u = 0, \ 0 \le x \le \pi, \ 0 \le y \le \pi$$

subject to the boundary conditions

$$u(x,0) = \sin x + 2\sin 2x$$

$$u(\pi, y) = 0$$
$$u(x, \pi) = 0$$
$$u(0, y) = 0$$

19. Repeat the above problem with

$$u(x,0) = -\pi^2 x^2 + 2\pi x^3 - x^4$$

1 a. $u_t = ku_{xx}$

$u(0, t) = 0$

$u(L, t) = 0$

$u(x, 0) = 6 \sin \frac{9\pi x}{L}$

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-k(\frac{n\pi}{L})^2 t}$$

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \equiv 6 \sin \frac{9\pi x}{L}$$

$\Rightarrow$ the only term from the sum that can survive is for $n = 9$ with $B_9 = 6$ $B_n = 0$ for $n \neq 9$

$$\boxed{\Rightarrow u(x, t) = 6 \sin \frac{9\pi x}{L} e^{-k(\frac{9\pi}{L})^2 t}}$$

b. $u(x, 0) = 2 \cos \frac{3\pi x}{L}$

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-k(\frac{n\pi}{L})^2 t}$$

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} = 2 \cos \frac{3\pi x}{L}$$

$$\Rightarrow \quad B_n = \frac{2}{L} \int_0^L 2 \cos \frac{3\pi x}{L} \sin \frac{n\pi x}{L} dx$$

compute the integral for $n = 1, 2, \ldots$ to get $B_n$.

To compute the coefficients, we need the integral

$$\int_0^L \cos\frac{3\pi}{L}x\,\sin\frac{n\pi}{L}x\,dx$$

Using the trigonometric identity

$$\sin a\,\cos b = \frac{1}{2}\left(\sin(a+b) + \sin(a-b)\right)$$

we have

$$\frac{1}{2}\int_0^L \left(\sin\frac{(n+3)\pi}{L}x + \sin\frac{(n-3)\pi}{L}x\right)dx$$

Now for $n \neq 3$ the integral is

$$-\frac{1}{2}\frac{\cos\frac{(n+3)\pi}{L}x}{\frac{(n+3)\pi}{L}}\Big|_0^L - \frac{1}{2}\frac{\cos\frac{(n-3)\pi}{L}x}{\frac{(n-3)\pi}{L}}\Big|_0^L$$

or when recalling that $\cos m\pi = (-1)^m$

$$-\frac{L}{2\pi(n+3)}\left[(-1)^{n+3} - 1\right] - \frac{L}{2\pi(n-3)}\left[(-1)^{n-3} - 1\right], \qquad \text{for } n \neq 3$$

Note that for $n$ odd, the coefficient is zero.

For $n = 3$ the integral is

$$\int_0^L \cos\frac{3\pi}{L}x\,\sin\frac{3\pi}{L}x\,dx = \frac{1}{2}\int_0^L \sin\frac{6\pi}{L}x\,dx$$

which is

$$-\frac{1}{2}\frac{L}{6\pi}\cos\frac{6\pi}{L}x\,\Big|_0^L = 0$$

2. $u_t = k u_{xx}$

$u_x(0,\, t) = 0$

$u_x(L,\, t) = 0$

$u(x,\, 0) = f(x)$

$u(x,\, t) = X(x)\, T(t)$

$\dot{T} x = k X''\, T$

$\dfrac{\dot{T}}{kT} = \dfrac{X''}{x} = -\lambda$

$\dot{T} + \lambda\, k\, T = 0$

$$\left\{ \begin{array}{l} X'' + \lambda\, X = 0 \\ \\ X'(0) = 0 \\ \\ X'(L) = 0 \end{array} \right\} \Rightarrow \begin{array}{l} X_n = A_n \cos \dfrac{n\,\pi\, x}{L}, \qquad n = 1, 2, \ldots \\ \lambda_n = \left(\dfrac{n\,\pi}{L}\right)^2, \qquad n = 1, 2, \ldots \\ \lambda_0 = 0 \qquad X_0 = A_0 \end{array}$$

$\dot{T}_n + \left(\dfrac{n\,\pi}{L}\right)^2 k T_n = 0$

$T_n = B_n\, e^{-(\frac{n\,\pi}{L})^2\, kt}$

$$u(x,\, t) = \underbrace{A_0\, B_0}_{=a_0} + \sum_{n=1}^{\infty} A_n \cos \frac{n\,\pi\, x}{L}\, B_n\, e^{-(\frac{n\,\pi}{L})^2\, kt}$$

$$u(x,\, t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\,\pi\, x}{L}\, e^{-(\frac{n\,\pi}{L})^2\, kt}$$

a.

$$f(x) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases}$$

$$u(x,\,0) = f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\,\pi}{L}\,x$$

$$a_0 = \frac{2}{2L}\int_{L/2}^{L} dx = \frac{1}{L}\left(L - \frac{L}{2}\right) = \frac{1}{2}$$

$$a_n = \frac{2}{L}\int_{L/2}^{L} \cos \frac{n\,\pi}{L}\,x\,dx = \frac{2}{L}\,\frac{L}{n\,\pi}\,\sin \frac{n\,\pi}{L}\,x\,\Big|_{L/2}^{L} = -\frac{2}{n\,\pi}\sin\frac{n\,\pi}{2}$$

$$\boxed{u(x,\,t) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(-\frac{2}{n\,\pi}\,\sin\,\frac{n\,\pi}{2}\right)\cos\frac{n\,\pi}{L}\,x\,e^{-k(\frac{n\,\pi}{L})^2\,t}}$$

b.

$$f(x) = 6 + 4\,\cos\frac{3\pi}{L}\,x$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\,\frac{n\pi}{L}\,x$$

$$a_0 = 6$$

$$a_3 = 4 \qquad a_n = 0 \qquad n \neq 3$$

$$\boxed{u(x,\,t) = 6 + 4\,\cos\tfrac{3\pi}{L}\,x\,e^{-k(\frac{3\,\pi}{L})^2\,t}}$$

3.

$$\phi'' + \lambda \phi = 0$$

$$\phi(0) = \phi(2\pi)$$

$$\phi'(0) = \phi'(2\pi)$$

$\underline{\lambda > 0}$ $\quad \phi = A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x$

$$\phi' = -A\sqrt{\lambda} \sin \sqrt{\lambda} x + B\sqrt{\lambda} \cos \sqrt{\lambda} x$$

$$\phi(0) = \phi(2\pi) \Rightarrow A = A \cos 2\pi\sqrt{\lambda} + B \sin 2\pi\sqrt{\lambda}$$

$$\phi'(0) = \phi'(2\pi) \Rightarrow B\sqrt{\lambda} = -A\sqrt{\lambda} \sin 2\pi\sqrt{\lambda} + B\sqrt{\lambda} \cos 2\pi\sqrt{\lambda}$$

$$A(1 - \cos 2\pi\sqrt{\lambda}) - B \sin 2\pi\sqrt{\lambda} = 0$$

$$A\sqrt{\lambda} \sin 2\pi\sqrt{\lambda} + B\sqrt{\lambda}(1 - \cos 2\pi\sqrt{\lambda}) = 0$$

A system of 2 homogeneous equations. To get a nontrivial solution one must have the determinant $= 0$.

$$\begin{vmatrix} 1 - \cos 2\pi\sqrt{\lambda} & -\sin 2\pi\sqrt{\lambda} \\ \sqrt{\lambda} \sin 2\pi\sqrt{\lambda} & \sqrt{\lambda}(1 - \cos 2\pi\sqrt{\lambda}) \end{vmatrix} = 0$$

$$\sqrt{\lambda}\,(1 - \cos 2\pi\sqrt{\lambda})^2 + \sqrt{\lambda} \sin^2 2\pi\sqrt{\lambda} = 0$$

$$\sqrt{\lambda}\,\{1 - 2\cos 2\pi\sqrt{\lambda} + \underbrace{\cos^2 2\pi\sqrt{\lambda} + \sin^2 2\pi\sqrt{\lambda}}_{1}\} = 0$$

$$2\sqrt{\lambda}\{1 - \cos 2\pi\sqrt{\lambda}\} = 0 \quad \Rightarrow \quad \sqrt{\lambda} = 0 \quad \text{or} \quad \cos 2\pi\sqrt{\lambda} = 1$$

$$2\pi\sqrt{\lambda} = 2\pi n \quad n = 1, 2, \ldots$$

Since $\lambda$ should be positive $\boxed{\lambda = n^2}$ $n = 1, 2, \ldots$

$$\lambda_n = n^2 \qquad \phi_n = A_n \cos nx + B_n \sin nx$$

$\underline{\lambda = 0}$ $\quad \phi = Ax + B$

$\phi' = A$

$$\phi(0) = \phi(2\pi) \quad \Rightarrow \quad B = 2\pi A + B \quad \Rightarrow \quad A = 0$$

$$\phi'(0) = \phi'(2\pi) \quad \Rightarrow \quad A = A$$

$\Rightarrow \quad \underline{\lambda = 0} \qquad \underline{\phi = B}$

$\underline{\lambda < 0}$ $\quad \phi = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$

$$\phi(0) = \phi(2\pi) \quad \Rightarrow \quad A + B = Ae^{\sqrt{-\lambda}2\pi} + Be^{-2\pi\sqrt{-\lambda}}$$

$$\phi'(0) = \phi'(2\pi) \quad \Rightarrow \quad \sqrt{-\lambda}A - \sqrt{-\lambda}B = \sqrt{-\lambda}Ae^{\sqrt{-\lambda}2\pi} - \sqrt{-\lambda}\,Be^{-2\pi\sqrt{-\lambda}}$$

$$A\,[1 - e^{2\pi\sqrt{-\lambda}}] + B\,[1 - e^{-2\pi\sqrt{-\lambda}}] = 0$$

$$\sqrt{-\lambda}\,A\left[1 - e^{2\pi\sqrt{-\lambda}}\right] - B\sqrt{-\lambda}\left[1 - e^{-2\pi\sqrt{-\lambda}}\right] = 0$$

$$\begin{vmatrix} 1 - e^{2\pi\sqrt{-\lambda}} & 1 - e^{-2\pi\sqrt{-\lambda}} \\ \sqrt{-\lambda}\,(1 - e^{2\pi\sqrt{-\lambda}}) & -\sqrt{-\lambda}\,(1 - e^{-2\pi\sqrt{-\lambda}}) \end{vmatrix} = 0$$

$$-\sqrt{-\lambda}\,(1 - e^{2\pi\sqrt{-\lambda}})\,(1 - e^{-2\pi\sqrt{-\lambda}}) - \sqrt{-\lambda}(1 - e^{2\pi\sqrt{-\lambda}})\,(1 - e^{-2\pi\sqrt{-\lambda}}) = 0$$

$$-2\sqrt{-\lambda}(1 - e^{2\pi\sqrt{-\lambda}})\,(1 - e^{-2\pi\sqrt{-\lambda}}) = 0$$

$1 - e^{2\pi\sqrt{-\lambda}} = 0 \qquad$ or $\qquad 1 - e^{-2\pi\sqrt{-\lambda}} = 0$

$e^{2\pi\sqrt{-\lambda}} = 1 \qquad\qquad e^{-2\pi\sqrt{-\lambda}} = 1$

Take ln of both sides

$2\pi\sqrt{-\lambda} = 0 \qquad\qquad -2\pi\sqrt{-\lambda} = 0$

$\sqrt{-\lambda} = 0 \qquad\qquad \sqrt{-\lambda} = 0$

not possible          not possible

Thus trivial solution if $\lambda < 0$

4. $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$u(a, \theta) = f(\theta)$$

$$u(r, 0) = u_\theta(r, \alpha) = 0$$

$$u(r, \theta) = R(r)\,\Theta(\theta)$$

$$\Theta\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{r^2}R\frac{\partial^2 \Theta}{\partial \theta^2} = 0$$

multiply by $\dfrac{r^2}{R\Theta}$

$$\frac{r}{R}(r\,R')' = -\frac{\Theta''}{\Theta} = \lambda$$

$$\begin{cases} \Theta'' + \lambda\,\Theta = 0 & \rightarrow \quad \Theta(0) = \Theta'(\alpha) = 0 \\ r\,(r\,R')' - \lambda\,R = 0 & \rightarrow \quad |\,R(0)\,| < \infty \end{cases}$$

$\Theta'' + \mu\,\Theta = 0$ $\quad$ $r(r\,R')' - \mu\,R = 0$

$\Theta\,(0) = 0$ $\quad$ $|\,R(0)\,| < \infty$

$\Theta'\,(\alpha) = 0$ $\quad$ $R_n = r^{(n-\frac{1}{2})\frac{\pi}{\alpha}}$

$\Downarrow$ $\quad$ only positive exponent

$\Theta_n = \sin\,(n - 1/2)\,\dfrac{\pi}{\alpha}\,\theta$ $\quad$ because of boundedness

$\mu_n = \left[(n - 1/2)\,\dfrac{\pi}{\alpha}\right]^2$

$n = 1,\, 2,\, \ldots$

$$\boxed{u\,(r,\,\theta) = \sum_{n=1}^{\infty} a_n\, r^{(n-1/2)\pi/\alpha}\,\sin\,\frac{n-1/2}{\alpha}\,\pi\,\theta}$$

$$f(\theta) = \sum_{n=1}^{\infty} a_n\, a^{(n-1/2)\pi/\alpha}\,\sin\,\frac{(n - 1/2)\pi}{\alpha}\,\theta$$

$$\boxed{a_n = \frac{\int_0^\alpha f(\theta)\,\sin\,(n - 1/2)\,\frac{\pi}{\alpha}\,\theta\,d\,\theta}{a^{(n-1/2)\pi/\alpha}\,\int_0^\alpha \sin^2\,(n - 1/2)\,\frac{\pi}{\alpha}\,\theta\,d\,\theta}}$$

5. $u_{xx} + u_{yy} = 0$

$u_x(0, y) = 0$

$u_x(L, y) = 0$

$u(x, 0) = 0$

$u(x, H) = f(x)$

$u(x, y) = X(x)Y(y) \quad \Longrightarrow \quad \dfrac{X''}{X} = -\dfrac{Y''}{Y} = -\lambda$

$u_x(0, y) = 0 \quad \Rightarrow \quad X'(0) = 0$

$u_x(L, y) = 0 \quad \Rightarrow \quad X'(L) = 0$

$u(x, 0) = 0 \quad \Rightarrow \quad Y(0) = 0$

$$\Rightarrow \quad \begin{array}{ll} X'' + \lambda X = 0 & Y'' - \lambda Y = 0 \\ X'(0) = 0 & Y(0) = 0 \\ X'(L) = 0 & \end{array}$$

$\Downarrow$ Table at the end of Chapter 4

$\lambda_n = \left(\dfrac{n\pi}{L}\right)^2 \qquad n = 0, 1, 2, \ldots$

$x_n = \cos \dfrac{n\pi}{L} x \qquad \Rightarrow \quad Y_n'' - \left(\dfrac{n\pi}{L}\right)^2 Y_n = 0 \qquad n = 0, 1, 2, \ldots$

If $n = 0 \Rightarrow Y_0'' = 0 \Rightarrow Y_0 = A_0 y + B_0$

$Y_0(0) = 0 \Rightarrow B_0 = 0$

$\Rightarrow \underline{Y_0(y) = A_0 y}$

If $n \neq 0 \Rightarrow Y_n = A_n e^{\left(\frac{n\pi}{L}\right)^2 y} + B_n e^{-\left(\frac{n\pi}{L}\right)^2 y}$

or

$Y_n = C_n \sinh \left(\dfrac{n\pi}{L} y + D_n\right)$

$Y_n(0) = 0 \Rightarrow D_n = 0$

$\Rightarrow \underline{Y_n = C_n \sinh \dfrac{n\pi}{L} y}$

$$u(x,\,y) \;=\; \underbrace{\alpha_0\,A_0}_{\frac{a_0}{2}}\,y\,\cdot\,1 \,+\, \sum_{n=1}^{\infty} \underbrace{\alpha_n\,C_n}_{a_n}\,\cos\,\frac{n\,\pi}{L}\,x\,\sinh\,\frac{n\,\pi}{L}\,y$$

$$u(x,\,H) \;=\; \frac{a_0}{2} \,+\, \sum_{n=1}^{\infty} a_n\,\sinh\,\frac{n\,\pi}{L}\,H\,\cos\,\frac{n\,\pi}{L}\,x \;\equiv\; f(x)$$

This is the Fourier cosine series of $f(x)$

$$a_0\,H \;=\; \frac{2}{L}\int_0^L f(x)\,dx$$

$$a_n\,\sinh\,\frac{n\,\pi}{L}\,H \;=\; \frac{2}{L}\int_0^L f(x)\,\cos\,\frac{n\,\pi}{L}\,x\,dx$$

$$\Rightarrow \quad a_0 \;=\; \frac{2}{H\,L}\int_0^L f(x)\,dx$$

$$a_n \;=\; \frac{2}{L\,\sinh\,\frac{n\,\pi}{L}\,H}\int_0^L f(x)\,\cos\,\frac{n\,\pi}{L}\,x\,dx$$

and:

$$u(x,\,y) \;=\; \frac{a_0}{2}\,y \,+\, \sum_{n=1}^{\infty} a_n\,\sinh\,\frac{n\,\pi}{L}\,y\,\cos\,\frac{n\,\pi}{L}\,x$$

5 b. $u_{xx} + u_{yy} = 0$

$u(0, y) = g(y)$

$u(L, y) = 0$

$u_y(x, 0) = 0$

$u(x, H) = 0$

$X'' - \lambda X = 0 \qquad Y'' + \lambda Y = 0$

$X(L) = 0 \qquad Y'(0) = 0$

$Y(H) = 0$

Using the summary of Chapter 4 we have

$$Y_n(y) = \cos \frac{(n + \frac{1}{2})\pi}{H} y, \qquad n = 0, 1, \dots$$

$$\lambda_n = \left[ \frac{\left(n + \frac{1}{2}\right)\pi}{H} \right]^2 \qquad n = 0, 1, \dots$$

Now use these eigenvalues in the $x$ equation: $X_n'' - \left[\frac{(n+\frac{1}{2})\pi}{H}\right]^2 X_n = 0 \qquad n = 0, 1, 2, \dots$

Solve:

$X_n = c_n \sinh\left(\left(n + \frac{1}{2}\right) \frac{\pi}{H} x + D_n\right)$

Use the boundary condition: $X_n(L) = 0$

$X_n(L) = c_n \sinh\left(\left(n + \frac{1}{2}\right) \frac{\pi L}{H} + D_n\right) = 0 \Rightarrow \frac{\left(n + \frac{1}{2}\right)\pi L}{H} + D_n = 0$

$\Rightarrow X_n = c_n \sinh\left(\frac{(n+\frac{1}{2})\pi}{H} (x - L)\right)$

$$\boxed{\Rightarrow u(x, y) = \sum_{n=0}^{\infty} a_n \sinh\left[\frac{(n + \frac{1}{2})\pi}{H}(x - L)\right] \cos\left(n + \frac{1}{2}\right)\frac{\pi}{H} y}$$

To find the coefficients $a_n$, we use the inhomogeneous boundary condition:

$$u(0, y) = g(y) = \sum_{n=0}^{\infty} a_n \sinh\left(\frac{(n + \frac{1}{2})\pi}{H}(-L)\right) \cos\left(n + \frac{1}{2}\right)\frac{\pi}{H} y$$

This is a Fourier cosine series expansion of $g(y)$, thus the coefficients are:

$$-\sinh \frac{(n + \frac{1}{2})\pi L}{H} a_n = \frac{2}{H} \int_0^H g(y) \cos\left[\left(n + \frac{1}{2}\right)\frac{\pi}{H} y\right] dy$$

$$\boxed{a_n = \frac{2}{-H \sinh \frac{(n+\frac{1}{2})\pi L}{H}} \int_0^H g(y) \cos\left[\left(n + \frac{1}{2}\right)\frac{\pi}{H} y\right] dy}$$

5. c. $$u(0,\,y) \,=\, 0 \quad\Rightarrow\quad X(0) \,=\, 0$$

$$u(L,\,y) \,=\, 0 \quad\Rightarrow\quad X(L) \,=\, 0$$

$$u(x,\,0) \,-\, u_y\,(x,\,0) \,=\, 0 \quad\Rightarrow\quad Y(0) \,-\, Y'(0) \,=\, 0$$

$X'' \,+\, \lambda X \,=\, 0$

$X(0) \,=\, 0$

$X(L) \,=\, 0$

$\Downarrow$

$\lambda_n \,=\, \left(\frac{n\,\pi}{L}\right)^2 \quad n \,=\, 1,\,2,\ldots$

$X_n \,=\, \sin\,\frac{n\,\pi}{L}\,x$

$Y_n'' \,-\, \left(\frac{n\,\pi}{L}\right)^2\,Y_n \,=\, 0$

$Y_n(0) \,-\, Y_n'(0) \,=\, 0$

$Y_n \,=\, A_n\,\cosh\,\frac{n\,\pi}{L}\,y \,+\, B_n\,\sinh\,\frac{n\,\pi}{L}\,y$

$Y_n' \,=\, \frac{n\,\pi}{L}\,\left\{A_n\,\sinh\,\frac{n\,\pi}{L}\,y \,+\, B_n\,\cosh\,\frac{n\,\pi}{L}\,y\right\}$

Substitute in the boundary condition.

$$\left(A_n \,-\, \tfrac{n\,\pi}{L}\,B_n\right)\underbrace{\cosh\,0}_{\neq 0} + \left(B_n \,-\, \tfrac{n\,\pi}{L}\,A_n\right)\underbrace{\sinh\,0}_{=0} \,=\, 0$$

$$\Rightarrow\quad A_n \,=\, \tfrac{n\,\pi}{L}\,B_n$$

$$Y_n \,=\, B_n\left[\frac{n\,\pi}{L}\,\cosh\,\frac{n\,\pi}{L}\,y \,+\, \sinh\,\frac{n\,\pi}{L}\,y\right]$$

$$\boxed{u(x,\,y) \,=\, \sum_{n=1}^{\infty} b_n\,\sin\,\frac{n\,\pi}{L}\,x\,\left[\frac{n\,\pi}{L}\,\cosh\,\frac{n\,\pi}{L}\,y \,+\, \sinh\,\frac{n\,\pi}{L}\,y\right]}$$

Use the boundary condition $u(x,\,H) \,=\, f(x)$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n\,\sin\,\frac{n\,\pi}{L}\,x\,\left[\frac{n\,\pi}{L}\,\cosh\,\frac{n\,\pi}{L}\,H \,+\, \sinh\,\frac{n\,\pi}{L}\,H\right]$$

This is a Fourier sine series of $f(x)$, thus the coefficients $b_n$ are given by

$$b_n\,\left[\frac{n\,\pi}{L}\,\cosh\,\frac{n\,\pi}{L}\,H \,+\, \sinh\,\frac{n\,\pi}{L}\,H\right] \,=\, \frac{2}{L}\,\int_0^L\,f(x)\,\sin\,\frac{n\,\pi}{L}\,x\,dx$$

Solve for $b_n$

$$\boxed{b_n \,=\, \frac{2}{\left[\frac{n\,\pi}{L}\,\cosh\,\frac{n\,\pi}{L}\,H \,+\, \sinh\,\frac{n\,\pi}{L}\,H\right]L}\int_0^L\,f(x)\,\sin\,\frac{n\,\pi}{L}\,x\,dx}$$

6 a. $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$ outside circle

$u(a, \theta) = \ln 2 + 4 \cos 3\theta$

$u(r, \theta) = R(r)\Theta(\theta)$

$r^2 R'' + rR' - \lambda R = 0$ $\qquad$ $\Theta'' + \lambda\Theta = 0$

$\Theta(0) = \Theta(2\pi)$

$\underline{\lambda = 0}\, R_0 = \alpha_0 + \beta_0 \ln r$ $\qquad$ $\Theta'(0) = \Theta'(2\pi)$

$\underline{\lambda = n^2}\, R_n = \alpha_n r^n + \beta_n r^{-n}$ $\qquad$ $\Downarrow$

Since we are solving $\qquad$ $\underline{\lambda < 0}$ trivial solution

outside the circle $\qquad$ $\underline{\lambda = 0}$ $\Theta_0(\theta) = 1$

$\ln r \to \infty$ as $r \to \infty$ $\qquad$ $\underline{\lambda > 0}$ $\lambda_n = n^2$

$r^n \to \infty$ as $r \to \infty$ $\qquad$ $\Theta_n = A_n \cos n\theta + B_n \sin n\theta$

thus $R_0 = \alpha_0$

$R_n = \beta_n r^{-n}$

$$u(r, \theta) = \underbrace{a_0\,\alpha_0 \cdot 1}_{= a_0/2} + \sum_{n=1}^{\infty} a_n (A_n \cos n\theta + B_n \sin n\theta)\, \beta_n r^{-n}$$

$$u(r, \theta) = a_0/2 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)\, r^{-n}$$

Use the boundary condition:

$$u(a, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n a^{-n} \cos n\theta + b_n a^{-n} \sin n\theta) = \ln 2 + 4 \cos 3\theta$$

$\Rightarrow b_n = 0 \quad \forall n$

$\frac{a_0}{2} = \ln 2$

$a_n a^{-n} = 4 \quad n = 3 \quad \Rightarrow \quad a_3 = 4a^3$

$a_n a^{-n} = 0 \quad n \neq 3 \quad \Rightarrow \quad a_n = 0 \quad n \neq 3$

$\Rightarrow \boxed{u(r, \theta) = \ln 2 + 4a^3 r^{-3} \cos 3\theta}$

6 b. The only difference between this problem and the previous one is the boundary condition

$$u(a,\,\theta) \;=\; f(\theta) \;=\; \frac{a_0}{2} \;+\; \sum_{n=1}^{\infty} (a_n\,a^{-n}\,\cos\,n\,\theta \;+\; b_n\,a^{-n}\,\sin\,n\,\theta)$$

$\Rightarrow\; a_0,\; a_n\,a^{-n},\; b_n\,a^{-n}$ are coefficients of Fourier series of $f$

$$a_0 \;=\; \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\,d\,\theta$$

$$a_n\,a^{-n} \;=\; \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\,\cos\,n\,\theta\,d\,\theta$$

$$b_n\,a^{-n} \;=\; \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\,\sin\,n\,\theta\,d\,\theta$$

Divide the last two equations by $a^{-n}$ to get the coefficients $a_n$ and $b_n$.

7 a. $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$

$u_\theta (r, 0) = 0$

$u (r, \pi/2) = 0$

$u (1, \theta) = f(\theta)$

$r^2 R'' + r R' - \lambda R = 0$

$R_n = c_n r^{2n-1} + D_n r^{2n-1}$

boundedness implies $R_n = c_n r^{2n-1}$

$\Theta'' + \lambda \Theta = 0$ no periodicity !!

$\Theta' (0) = 0$

$\Theta' (\pi/2) = 0$

If $\underline{\lambda < 0}$ trivial

$\underline{\lambda = 0}\, \Theta_0 = A_0 \theta + B_0$

$\Theta_0' = A_0 \quad \Theta_0' (0) = 0 \quad \Rightarrow \quad A_0 = 0$

$\Theta_0 (\pi/2) = 0 \quad \Rightarrow \quad B_0 = 0$

trivial

$\underline{\lambda > 0}\, \Theta = A \cos \sqrt{\lambda} \theta + B \sin \sqrt{\lambda} \theta$

$\Theta' = -\sqrt{\lambda} A \sin \sqrt{\lambda}\theta + B \sqrt{\lambda} \cos \sqrt{\lambda} \theta$

$\Theta'(0) = 0 \Rightarrow \quad B = 0$

$\Theta (\pi/2) = 0 \Rightarrow A \cos \sqrt{\lambda} \pi/2 = 0$

$\sqrt{\lambda} \pi/2 = \left(n - \frac{1}{2}\right) \pi \quad n = 1, 2, \cdots$

$\sqrt{\lambda} = 2 \left(n - \frac{1}{2}\right) = 2n - 1$

$\boxed{\lambda_n = (2n - 1)^2}$

$\Theta_n = \cos (2n - 1)\, \theta \quad , \quad n = 1, 2, \cdots$

Therefore the solution is

$$u = \sum_{n=1}^{\infty} a_n r^{2n-1} \cos (2n - 1) \theta$$

Use the boundary condition

$$u(1,\,\theta) \;=\; \sum_{n=1}^{\infty} a_n\, \cos\,(2n\,-\,1)\,\theta \;=\; f(\theta)$$

This is a Fourier cosine series of $f(\theta)$, thus the coefficients are given by

$$a_n \;=\; \frac{2}{\pi/2}\int_0^{\pi/2} f(\theta)\,\cos\,(2n\,-\,1)\,\theta\,d\,\theta$$

Remark: Since there is no constant term in this Fourier cosine series, we should have

$$a_0 \;=\; \frac{2}{\frac{\pi}{2}}\int_0^{\pi/2} f(\theta)\,d\theta \;=\; 0$$

That means that the boundary condition on the curved part of the domain is not arbitrary but must satisfy

$$\int_0^{\pi/2} f(\theta)\,d\theta \;=\; 0$$

7 b. $u_\theta(r, 0) = 0$

$u_\theta(r, \pi/2) = 0$

$u_r(1, \theta) = g(\theta)$

Use 7 a to get the 2 ODEs

$\Theta'' + \lambda\Theta = 0$ $\qquad r^2 R'' + r R' - \lambda R = 0$

$\Theta'(0) = \Theta'(\pi/2) = 0$

$\Downarrow$

$\lambda_n = \left(\frac{n\pi}{\frac{\pi}{2}}\right)^2 = (2n)^2 \quad n = 0, 1, 2, \ldots$

$\Theta_n = \cos 2n\theta, \qquad n = 0, 1, 2, \ldots$

Now substitute the eigenvalues in the $R$ equation
$r^2 R'' + r R' - (2n)^2 R = 0$
The solution is

$$R_0 = C_0 \ln r + D_0, \qquad n = 0$$
$$R_n = C_n r^{-2n} + D_n r^{2n}, \qquad n = 1, 2, \ldots$$

Since $\ln r$ and $r^{-2n}$ blow up as $r \to 0$ we have $C_0 = C_n = 0$. Thus

$$\boxed{u(r,\theta) = a_0 + \sum_{n=1}^{\infty} a_n r^{2n} \cos 2n\theta}$$

Apply the inhomogeneous boundary condition

$$u_r(r,\theta) = \sum_{n=1}^{\infty} 2n\, a_n r^{2n-1} \cos 2n\theta$$

And at $r = 1$

$$u_r(1,\theta) = \sum_{n=1}^{\infty} 2n\, a_n \cos 2n\theta = g(\theta)$$

This is a Fourier cosine series for $g(\theta)$ and thus

$$2n\, a_n = \frac{\int_0^{\pi/2} g(\theta) \cos 2n\theta\, d\theta}{\int_0^{\pi/2} \cos^2 2n\theta\, d\theta}$$

$$\boxed{a_n = \frac{\int_0^{\pi/2} g(\theta) \cos 2n\theta\, d\theta}{2n \int_0^{\pi/2} \cos^2 2n\theta\, d\theta} \qquad n = 1, 2, \ldots}$$

Note: $a_0$ is still arbitrary. Thus the solution is <u>not</u> unique. Physically we require $\displaystyle\int_0^{\pi/2} g(\theta)\, d\theta = 0$ which is to say that $a_0 = 0$.

7 c. $u(r, 0) = 0$

$u(r, \pi/2) = 0$

$u_r(1, \theta) = 1$

Use 7 a to get the 2 ODEs

$$\Theta'' + \lambda \Theta = 0 \qquad\qquad r^2 R'' + r R' - \lambda R = 0$$

$$\Theta(0) = \Theta(\pi/2) = 0$$

$$\Downarrow$$

$$\lambda_n = \left(\frac{n\pi}{\frac{\pi}{2}}\right)^2 = (2n)^2 \quad n = 1, 2, \ldots$$

$$\Theta_n = \sin 2n\,\theta, \qquad n = 1, 2, \ldots$$

Now substitute the eigenvalues in the $R$ equation
$r^2 R'' + r R' - (2n)^2 R = 0$
The solution is

$$R_n = C_n r^{-2n} + D_n r^{2n}, \qquad n = 1, 2, \ldots$$

Since $r^{-2n}$ blow up as $r \to 0$ we have $C_n = 0$. Thus

$$\boxed{u(r, \theta) = \sum_{n=1}^{\infty} a_n r^{2n} \sin 2n\theta}$$

Apply the inhomogeneous boundary condition

$$u_r(r, \theta) = \sum_{n=1}^{\infty} 2n\, a_n r^{2n-1} \sin 2n\theta$$

And at $r = 1$

$$u_r(1, \theta) = \sum_{n=1}^{\infty} 2n\, a_n \sin 2n\theta = 1$$

This is a Fourier sine series for the constant function 1 and thus

$$2n\, a_n = \frac{\int_0^{\pi/2} 1 \cdot \sin 2n\theta\, d\theta}{\int_0^{\pi/2} \sin^2 2n\theta\, d\theta}$$

$$a_n = \frac{\int_0^{\pi/2} 1 \cdot \sin 2n\theta\, d\theta}{2n \int_0^{\pi/2} \sin^2 2n\theta\, d\theta} = \frac{\frac{1-(-1)^n}{2n}}{2n\frac{\pi}{2}} = \frac{1-(-1)^n}{n^2\pi}$$

$$\boxed{a_n = \frac{1-(-1)^n)}{n^2\pi}}$$

8a.
$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$
$$u(a, \theta) = f(\theta)$$
$$u(b, \theta) = g(\theta)$$

$$r^2 R'' + rR' - \lambda R = 0 \qquad \Theta'' + \lambda\Theta = 0$$
$$\Theta(0) = \Theta(2\pi)$$
$$\Theta'(0) = \Theta'(2\pi)$$

The eigenvalues and eigenfunctions can be found in the summary of chapter 4

$$\lambda_0 = 0 \qquad \Theta_0 = 1 \qquad \text{for } n = 0$$
$$\lambda_n = n^2 \qquad \Theta_n = \cos n\theta \text{ and } \sin n\theta \qquad \text{for } n = 1, 2, \ldots$$

Use these eigenvalues in the $R$ equation and we get the following solutions:

$$R_0 = A_0 + B_0 \ln r \qquad n = 0$$
$$R_n = A_n r^n + B_n r^{-n} \qquad n = 1, 2, \ldots$$

Since $r = 0$ is outside the domain and $r$ is finite, we have no reason to throw away any of the 4 parameters $A_0, A_n, B_0, B_n$.

Thus the solution

$$u(r, \theta) = \underbrace{(A_0 + B_0 \ln r)}_{R_0} \cdot \underbrace{1}_{\Theta_0} \cdot a_0 + \sum_{n=1}^{\infty} \underbrace{(A_n r^n + B_n r^{-n})}_{R_n} \underbrace{(a_n \cos n\theta + b_n \sin n\theta)}_{\Theta_n}$$

Use the 2 inhomogeneous boundary conditions

$$f(\theta) = u(a, \theta) = \underbrace{A_0 a_0 + B_0 a_0 \ln a}_{\alpha_0} + \sum_{n=1}^{\infty} \underbrace{(A_n a^n + B_n a^{-n}) a_n}_{\alpha_n} \cos n\theta$$
$$+ \sum_{n=1}^{\infty} \underbrace{(A_n a^n + B_n a^{-n}) b_n}_{\beta_n} \sin n\theta$$

$$g(\theta) = u(b, \theta) = \underbrace{A_0 a_0 + B_0 a_0 \ln b}_{\gamma_0} + \sum_{n=1}^{\infty} \underbrace{(A_n b^n + B_n b^{-n}) a_n}_{\gamma_n} \cos n\theta$$

$$+ \sum_{n=1}^{\infty} \underbrace{(A_n\, b^n \;+\; B_n\, b^{-n})\, b_n}_{\delta_n} \sin\, n\,\theta$$

These are Fourier series of $f(\theta)$ and $g(\theta)$ thus the coefficients $\alpha_0$, $\alpha_n$, $\beta_n$ for $f$ and the coefficients $\gamma_0$, $\gamma_n$, $\delta_n$ for $g$ can be written as follows

$$\begin{aligned}
\alpha_0 &= \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta \\
\alpha_n &= \frac{1}{\pi} \int_0^{2\pi} f(\theta)\, \cos n\theta\, d\theta \\
\beta_n &= \frac{1}{\pi} \int_0^{2\pi} f(\theta)\, \sin n\theta\, d\theta \\
\gamma_0 &= \frac{1}{2\pi} \int_0^{2\pi} g(\theta)\, d\theta \\
\gamma_n &= \frac{1}{\pi} \int_0^{2\pi} g(\theta)\, \cos n\theta\, d\theta \\
\delta_n &= \frac{1}{\pi} \int_0^{2\pi} g(\theta)\, \sin n\theta\, d\theta
\end{aligned}$$

On the other hand these coefficients are related to the unknowns $A_0, a_0, B_0, b_0, A_n, a_n, B_n$ and $b_n$ via the three systems of 2 equations each

$$\left.\begin{aligned}
\alpha_0 &= A_0\, a_0 \;+\; B_0\, a_0\, \ln\, a \\
\gamma_0 &= A_0\, a_0 \;+\; B_0\, a_0\, \ln\, b
\end{aligned}\right\} \quad \text{solve for } A_0\, a_0\,,\; B_0\, a_0$$

$$\left.\begin{aligned}
\alpha_n &= (A_n\, a^n \;+\; B_n\, a^{-n})\, a_n \\
\gamma_n &= (A_n\, b^n \;+\; B_n\, b^{-n})\, a_n
\end{aligned}\right\} \quad \text{solve for } A_n\, a_n, B_n\, a_n$$

$$\left.\begin{aligned}
\beta_n &= (A_n\, a^n \;+\; B_n\, a^{-n})\, b_n \\
\delta_n &= (A_n\, b^n \;+\; B_n\, b^{-n})\, b_n
\end{aligned}\right\} \quad \text{solve for } A_n\, b_n, B_n\, b_n$$

Notice that we only need the products $A_0a_0$, $B_0b_0$,$A_na_n$, $B_na_n$, $A_nb_n$, and $B_nb_n$.

$$B_0a_0 \;=\; \frac{\gamma_0 \;-\; \alpha_0}{\ln b \;-\; \ln a}$$

$$A_0 a_0 = \frac{\alpha_0 \ln b - \gamma_0 \ln a}{\ln b - \ln a}$$

$$B_n a_n = \frac{\alpha_n b^n - \gamma_n a^n}{b^n a^{-n} - a^n b^{-n}}$$

$$A_n a_n = \frac{\gamma_n b^n - \alpha_n a^n}{b^{2n} - a^{2n}}$$

In a similar fashion

$$B_n b_n = \frac{\beta_n b^n - \delta_n a^n}{b^n a^{-n} - a^n b^{-n}}$$

$$A_n b_n = \frac{\delta_n b^n - \beta_n a^n}{b^{2n} - a^{2n}}$$

8. b Similar to 8.a

$$u(r,\,\theta)\;=\;(A_0\;+\;B_0\;\ln\,r)\,a_0\;+\;\sum_{n=1}^{\infty}\,(A_n\,r^n\;+\;B_n\,r^{-n})\,(a_n\;\cos\,n\,\theta\;+\;b_n\;\sin\,n\,\theta)$$

To use the boundary conditions:

$$u_r(a,\,\theta)\;=\;f(\theta)$$

$$u_r(b,\,\theta)\;=\;g(\theta)$$

We need to differentiate $u$ with respect to $r$

$$u_r(r,\,\theta)\;=\;\frac{B_0}{r}\,a_0\;+\;\sum_{n=1}^{\infty}\,(n\,A_n\,r^{n-1}\;-\;n\,B_n\,r^{-n-1})\,(a_n\;\cos\,n\,\theta\;+\;b_n\;\sin\,n\,\theta)$$

Substitute $r\;=\;a$

$$u_r(a,\,\theta)\;=\;\frac{B_0}{a}\,a_0\;+\;\sum_{n=1}^{\infty}\,(n\,A_n\,a^{n-1}\;-\;n\,B_n\,a^{-n-1})\,(a_n\;\cos\,n\,\theta\;+\;b_n\;\sin\,n\,\theta)$$

This is a Fourier series expansion of $f(\theta)$ thus the coefficients are

$$\frac{B_0}{a}a_0\;=\;\frac{1}{2\pi}\int_0^{2\pi}\;f(\theta)\,d\theta\;\equiv\;\alpha_0$$

$$(nA_na^{n-1}\;-\;nB_na^{-n-1})a_n\;=\;\frac{1}{\pi}\int_0^{2\pi}\;f(\theta)\;\cos\;n\theta\,d\theta\;\equiv\;\alpha_n$$

$$(nA_na^{n-1}\;-\;nB_na^{-n-1})b_n\;=\;\frac{1}{\pi}\int_0^{2\pi}\;f(\theta)\;\sin\;n\theta\,d\theta\;\equiv\;\beta_n$$

Now substitute $r\;=\;b$

$$u_r(b,\,\theta)\;=\;\frac{B_0}{b}\,a_0\;+\;\sum_{n=1}^{\infty}\,(n\,A_n\,b^{n-1}\;-\;n\,B_n\,b^{-n-1})\,(a_n\;\cos\,n\,\theta\;+\;b_n\;\sin\,n\,\theta)$$

This is a Fourier series expansion of $g(\theta)$ thus the coefficients are

$$\frac{B_0}{b}a_0\;=\;\frac{1}{2\pi}\int_0^{2\pi}\;g(\theta)\,d\theta\;\equiv\;\gamma_0$$

$$(nA_nb^{n-1}\;-\;nB_nb^{-n-1})a_n\;=\;\frac{1}{\pi}\int_0^{2\pi}\;g(\theta)\;\cos\;n\theta\,d\theta\;\equiv\;\gamma_n$$

$$(nA_n b^{n-1} - nB_n b^{-n-1}) b_n = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \sin n\theta \, d\theta \equiv \delta_n$$

Solve for $A_n a_n$, $B_n a_n$:

$$(nA_n a^{n-1} - nB_n a^{-n-1}) a_n = \alpha_n$$

$$(nA_n b^{n-1} - nB_n b^{-n-1}) a_n = \gamma_n$$

We have

$$A_n a_n = \frac{\alpha_n \, b^{-n-1} - \gamma_n \, a^{-n-1}}{n(a^{n-1} \, b^{-n-1} - b^{n-1} \, a^{-n-1})}$$

$$B_n a_n = \frac{\alpha_n \, b^{n-1} - \gamma_n \, a^{n-1}}{n(a^{n-1} \, b^{-n-1} - b^{n-1} \, a^{-n-1})}$$

Solve for $A_n b_n$, $B_n b_n$:

$$(nA_n a^{n-1} - nB_n a^{-n-1}) b_n = \beta_n$$

$$(nA_n b^{n-1} - nB_n b^{-n-1}) b_n = \delta_n$$

We have

$$A_n b_n = \frac{\beta_n \, b^{-n-1} - \delta_n \, a^{-n-1}}{n(a^{n-1} \, b^{-n-1} - b^{n-1} \, a^{-n-1})}$$

$$B_n b_n = \frac{\beta_n \, b^{n-1} - \delta_n \, a^{n-1}}{n(a^{n-1} \, b^{-n-1} - b^{n-1} \, a^{-n-1})}$$

There are two equations for $B_0 a_0$:

$$B_0 a_0 = b\gamma_0$$

$$B_0 a_0 = a\alpha_0$$

This means that $f$ and $g$ are not independent, but

$$a\alpha_0 = b\gamma_0$$

which means that

$$a \int_0^{2\pi} f(\theta) \, d\theta = b \int_0^{2\pi} g(\theta) \, d\theta$$

Note also that there is <u>no</u> condition on $A_0 a_0$.

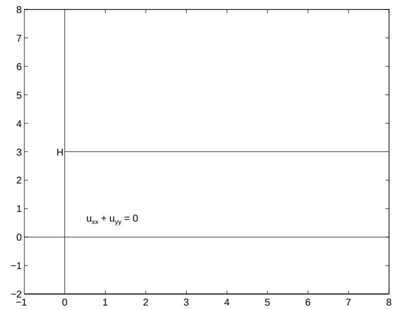


Figure 32: Sketch of domain

9.

$u_y(x,\, 0) \,=\, 0$

$u_y(x,\, H) \,=\, 0$

$u(0,\, y) \,=\, f(y)$

$X'' - \lambda X = 0$ $\qquad$ $Y'' + \lambda Y = 0$
solution should $\qquad$ $Y'(0) = 0$
be bounded $\qquad$ $Y'(H) = 0$
when $x \to \infty$ $\qquad$ copy from table in Chapter 4 summary

$$\lambda_n = \left(\frac{n\pi}{H}\right)^2 \qquad n = 0,\, 1,\, 2,\, \cdots$$

$$Y_n = \cos \frac{n\pi}{H} y$$

$X_n'' - \left(\frac{n\pi}{H}\right)^2 X_n = 0 \qquad n = 1,\, 2,\, \cdots$

$X_n = A_n e^{\frac{n\pi}{H} x} + B_n e^{-\frac{n\pi}{H} x}$

to get bounded solution $A_n = 0$

For $n = 0$

$X_0'' = 0$

$X_0 = A_0 x + B_0$
for boundedness $A_0 = 0$

$$u = B_0 \cdot 1 + \sum_{n=1}^{\infty} B_n e^{-\frac{n\pi}{H} x} \cos \frac{n\pi}{H} y$$

$$u(0,\, y) = f(y) = B_0 + \sum_{n=1}^{\infty} B_n \cos \frac{n\pi}{H} y$$

Fourier cosine series of $f(y)$.

10. $u_t = u_{xx} + q(x, t) \qquad 0 < x < L$

subject to BC $\quad u(0, t) = u(L, t) = 0$

Assume: $q(x, t)$ piecewise smooth for each positive t.

$u$ and $u_x$ continuous

$u_{xx}$ and $u_t$ piecewise smooth.

Thus,

$$u(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi}{L} x$$

(a). Write the ODE satisfied by $b_n(t)$, and

(b). Solve this heat equation.

<u>STEPS:</u>

1. Compute $q_n(t)$, the known heat source coefficient
2. Plug $u$ and $q$ series expansions into PDE.
3. Solve for $b_n(t)$ - the homogeneous and particular solutions, $b_n^H(t)$ and $b_n^P(t)$
4. Apply initial condition, $b_n(0)$, to find coefficient $A_n$ in the $b_n(t)$ solution. Assume $u(x, 0) = f(x)$

1.

$$q(x, t) = \sum_{n=1}^{\infty} q_n(t) \sin \frac{n\pi}{L} x$$

$$q_n(t) = \frac{2}{L} \int_0^L q(x, t) \sin \frac{n\pi}{L} x \, dx$$

2.

$$u_t = \sum_{n=1}^{\infty} b_n'(t) \sin \frac{n\pi}{L} x$$

$$u_{xx} = \sum_{n=1}^{\infty} b_n(t) \left[ -\left(\frac{n\pi}{L}\right)^2 \right] \sin \frac{n\pi}{L} x$$

$$\sum_{n=1}^{\infty} b_n'(t) \sin \frac{n\pi}{L} x = \sum_{n=1}^{\infty} b_n'(t) \left[ -\left(\frac{n\pi}{L}\right)^2 \right] \sin \frac{n\pi}{L} x + \sum_{n=1}^{\infty} q_n(t) \sin$$

We have a Fourier Sine series on left and Fourier Sine series on right, so the coefficients must be the same; i.e.,

(a) $\boxed{b_n'(t) = -\left(\frac{n\pi}{L}\right)^2 b_n(t) + q_n(t)}$ $\Rightarrow$ A first order ODE for $b_n(t)$.

III. Solve $b_n'(t) = -\left(\frac{n\pi}{L}\right)^2 b_n(t) + q_n(t)$

Solution Form: $b_n(t) = A_n\, b_n^H(t) + b_n^P(t)$

Homogeneous Solution: $b_n^H(t) = e^{-\left(\frac{n\pi}{L}\right)^2 t}$

Particular Solution: $b_n^P(t) = e^{\left(\frac{n\pi}{L}\right)^2 t} \int_0^t e^{-\left(\frac{n\pi}{L}\right)^2 \tau} q_n(\tau)\, d\tau$

$$\boxed{b_n(t) = A_n\, e^{-\left(\frac{n\pi}{L}\right)^2 t} + e^{\left(\frac{n\pi}{L}\right)^2 t} \int_0^t e^{-\left(\frac{n\pi}{L}\right)^2 \tau} q_n(\tau)\, d\tau}$$

(Step IV is an extra step, not required in homework problem.)

<u>IV</u>. Find $A_n$ from initial condition. $u(x, 0) = f(x) = \sum_{n=1}^{\infty} b_n(0) \sin \frac{n\pi}{L} x$

$$b_n(0) = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x\, dx$$

$$b_n(0) = A_n + e^0 \int_0^0 e^{-\left(\frac{n\pi}{L}\right)^2 \tau} q_n(\tau)\, dt$$

$$b_n(0) = A_n + 1 \cdot 0 = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x\, dx$$

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x\, dx$$

$$b_n(t) = \frac{2}{L} \int_0^L \left( f(x) \sin \frac{n\pi}{L} x\, dx \right) \left( e^{-\left(\frac{n\pi}{L}\right)^2 t} \right) + e^{\left(\frac{n\pi}{L}\right)^2 t} \int_0^t e^{-\left(\frac{n\pi}{L}\right)^2 \tau} q_n(\tau)\, d\tau$$

Plug this into $u(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi}{L} x$

11. $u_t = k\, u_{xx} + \underbrace{e^{-2t} \cos \frac{3\,\pi}{L}}_{q\,(x,\,t)} x$

$u_x\,(0,\, t) = 0$

$u_x\,(L,\, t) = 0$

$u(x,\, 0) = f(x)$

The boundary conditions imply

$$u(x,\, t) = \sum_{n=0}^{\infty} b_n(t) \cos \frac{n\,\pi}{L} x$$

$$\text{Let} \quad q(x,\, t) = \sum_{n=0}^{\infty} q_n(t) \cos \frac{n\,\pi}{L} x \quad \Rightarrow \quad q_0(t) = e^{-t}$$

$$q_3(t) = e^{-2t} \quad \text{the rest are zero !}$$

Thus

$$\dot{b}_n = -k \left(\frac{n\,\pi}{L}\right)^2 b_n + q_n \qquad n = 0,\, 1,\, \cdots$$

$$\underline{n = 0} \qquad \dot{b}_0 = q_0 = e^{-t} \quad \Rightarrow \quad \underline{b_0 = -\, e^{-t}}$$

$$\left.\begin{array}{l} \dot{b}_1 + k \left(\frac{n\,\pi}{L}\right)^2 b_1 = q_1 = 0 \\ \\ \dot{b}_2 + k \left(\frac{2n\,\pi}{L}\right)^2 b_2 = 0 \end{array}\right\} \text{homogeneous}$$

$$\dot{b}_3 + k \left(\frac{3n\,\pi}{L}\right)^2 b_3 = e^{-2t}$$

rest are homogeneous.

One can solve each equation to obtain all $b_n$.

$$\dot{b}_n + k \left(\frac{n\,\pi}{L}\right)^2 b_n = 0 \quad \Rightarrow \quad \boxed{b_n = C_n e^{-k\left(\frac{n\,\pi}{L}\right)^2 t} \quad n = 1,\, 2,\, 4,\, 5,\, \cdots \quad \text{note: } n \neq 3}$$

$$\dot{b}_3 + k \left(\frac{3\,\pi}{L}\right)^2 b_3 = e^{-2t} \quad \text{Solution of homogeneous is}$$

$$b_3 = C_3\, e^{-k\left(\frac{3\,\pi}{L}\right)^2 t}$$

For particular solution try $\quad b_3 = C\, e^{-2t}$

$$\Rightarrow \quad -2C\,e^{-2t} + k\left(\frac{3\,\pi}{L}\right)^2 = C\,e^{-2t} = e^{-2t}$$

$$\left[-2 + k\left(\frac{3\,\pi}{L}\right)^2\right] C = 1$$

$$C = \frac{1}{k\left(\frac{3\,\pi}{L}\right)^2 - 2}$$

denominator is not zero as assumed in the problem.

$$\Rightarrow \quad \boxed{b_3 = C_3\,e^{k\left(\frac{3\,\pi}{L}\right)^2 t} + \frac{1}{k\left(\frac{3\,\pi}{L}\right)^2 - 2}\,e^{-2t}}$$

12. $u_{tt} - c^2 u_{xx} = 0 \qquad 0 < x < L$

$$u(0,t) = u(L,\,t) = 0$$

$$u(x,\,0) = f(x)$$

$$u_t(x,\,0) = g(x)$$

$$XT'' - c^2 X'' T = 0$$

$$\frac{T''}{c^2T} = \frac{X''}{X} = -\lambda$$

$$X'' + \lambda X = 0 \qquad\qquad T'' + \lambda c^2 T = 0$$

$$X(0) = X(L) = 0$$

$$X_n = \sin\frac{n\pi}{L}x \qquad\qquad T_n'' + \left(\frac{n\pi}{L}\right)^2 c^2 T_n = 0$$

$$n = 1,\,2,\,\cdots$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad\qquad T_n = \alpha_n \cos\frac{n\pi c}{L}t + \beta_n \sin\frac{n\pi c}{L}t$$

$$\boxed{u(x,\,t) = \sum_{n=1}^{\infty}\left\{\alpha_n \cos\frac{n\pi c}{L}t + \beta_n \sin\frac{n\pi c}{L}t\right\}\sin\frac{n\pi}{L}x}$$

$$u(x,\,0) = f(x) = \sum_{n=1}^{\infty}\alpha_n \sin\frac{n\pi}{L}x$$

$$u_t(x,\,0) = g(x) = \sum_{n=1}^{\infty}\frac{n\pi c}{L}\beta_n \sin\frac{n\pi}{L}x$$

$$\boxed{\alpha_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx}$$

$$\frac{n\pi c}{L}\beta_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

$$\boxed{\beta_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx}$$

13. $u_t = 2u_{xx}$

$u(0, t) = 0$

$u_x(L, t) = 0$

$u(x, 0) = \sin \frac{3}{2} \frac{\pi}{L} x$

$u = XT$

$X\dot{T} = 2X''T$

$$\frac{\dot{T}}{2T} = \frac{X''}{X} = -\lambda \qquad X'' + \lambda X = 0 \qquad \dot{T} + 2\lambda T = 0$$

$$X(0) = 0$$

$$X'(L) = 0$$

$$X_n = \sin\left(n + \frac{1}{2}\right) \frac{\pi}{L} x \qquad n = 0, 1, \cdots$$

$$\lambda_n = \left[\left(n + \frac{1}{2}\right) \frac{\pi}{L}\right]^2$$

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-2\left[\left((n + \frac{1}{2}\right) \frac{\pi}{L}\right]^2 t} \sin\left(n + \frac{1}{2}\right) \frac{\pi}{L} x$$

At $t = 0$

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(n + \frac{1}{2}\right) \frac{\pi}{L} x$$

But also

$u(x, 0) = \sin \frac{3}{2} \frac{\pi}{L} x$

Therefore

$$a_1 = 1, \quad a_n = 0 \quad n > 1$$

$$\boxed{u(x,t) = e^{-2\left(\frac{3}{2}\frac{\pi}{L}\right)^2 t} \sin \frac{3}{2} \frac{\pi}{L} x}$$

14. $u_t = k\left[\frac{1}{r}(r u_r)_r + \frac{1}{r^2} u_{\theta\theta}\right]$

$u_r(a, \theta, t) = 0$ inside a disk

$u(r, \theta, 0) = f(r, \theta)$

$\Theta T' R = kT\left[\Theta \frac{1}{r}(r R')' + \frac{1}{r^2} R \Theta''\right]$

$$\frac{T'}{kT} = \frac{\frac{1}{r}(rR')'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda$$

$T' + \lambda k T = 0$ $\quad \frac{\frac{1}{r}(rR')'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda$ multiply through by $r^2$

$$\frac{r(rR')'}{R} + \lambda r^2 = -\frac{\Theta''}{\Theta} = \mu$$

$\Theta'' + \mu \Theta = 0;$ $\qquad r(rR')' + \lambda r^2 R - \mu R = 0$

$\Theta(0) = \Theta(2\pi)$ $\qquad |R(0)| < \infty$

$\Theta'(0) = \Theta'(2\pi)$ $\qquad R'(a) = 0$

$\Downarrow$ $\qquad \downarrow$

$\mu_n = n^2$ $\qquad R = J_n(\sqrt{\lambda} r)$

$n = 1, 2, \cdots$

$$\Theta_n = \begin{cases} \sin n\theta \\ \cos n\theta \end{cases} \qquad \boxed{J_n'(\sqrt{\lambda} a) = 0 \quad \text{gives } \lambda_{nm}}$$

$\mu_0 = 0$ $\qquad \Theta_0 = 1$

$T_{nm}' + \lambda_{nm} k T_{nm} = 0 \rightarrow T_{nm} = e^{-\lambda_{nm} k t}$

$$u(r, \theta, t) = \sum_{m=1}^{\infty} \underbrace{\left[\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)\right]}_{\Theta_n} \underbrace{J_n\left(\sqrt{\lambda_{nm}} r\right)}_{R_{nm}} \underbrace{e^{-k\lambda_{nm} t}}_{T_{nm}}$$

$$f(r, \theta) = \sum_{m=1}^{\infty}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)\right] J_n\left(\sqrt{\lambda_{nm}} r\right)$$

Fourier-Bessel expansion of f.

See (4.5 later)

15. a.

$$u_{xx} + u_{yy} = 1$$

Try $u(x, y) = X(x)Y(y)$

$$X''Y + XY'' = 1$$

This is NOT separable. But the homogeneous equation is.

b.

$$u_{xx} + u_{yy} = u$$

Try $u(x, y) = X(x)Y(y)$

$$X''Y + XY'' = XY$$

Divide by $XY$

$$\frac{X''}{X} = 1 - \frac{Y''}{Y}$$

The equation is separable.

c.

$$x^2 y u_{xx} + y^4 u_{yy} = 4u$$

Try $u(x, y) = X(x)Y(y)$

$$x^2 y X''Y + y^4 XY'' = 4XY$$

Divide by $yXY$

$$x^2 \frac{X''}{X} = \frac{4}{y} - y^3 \frac{Y''}{Y}$$

The equation is separable.

d.

$$u_t + uu_x = 0$$

Try $u(x, t) = X(x)T(t)$

$$\dot{T}X + XX'T^2 = 0$$

Divide by $XT^2$

$$\frac{\dot{T}}{T^2} = -X'$$

The equation is separable.

e.

$$u_{tt} + f(t)u_t = u_{xx}$$

Try $u(x,t) = X(x)T(t)$

$$\ddot{T}X + f(t)X\dot{T} = X''T$$

Divide by $XT$

$$\frac{\ddot{T}}{T} + f(t)\frac{\dot{T}}{T} = \frac{X''}{X}$$

The equation is separable.

f.

$$\frac{x^2}{y^2}u_{xxx} = u_y$$

Try $u(x,y) = X(x)Y(y)$

$$x^2X'''Y = y^2XY'$$

Divide by $XY$

$$x^2\frac{X'''}{X} = y^2\frac{Y'}{Y}$$

The equation is separable.

16. $u_t = ku_{xx}$

$u_x(0, t) = 0$

$u_x(L, t) = 0$

$u(x, 0) = f(x)$

$$u(x, t) = \sum_{n=0}^{\infty} u_n \cos \frac{n \pi x}{L} e^{-k(\frac{n \pi}{L})^2 t}$$

$$u(x, 0) = \sum_{n=0}^{\infty} u_n \cos \frac{n \pi x}{L} \equiv f(x)$$

$u_n$ are the coefficients of expanding $f(x)$ in terms of Fourier cosine series.

$$u_0 = \frac{1}{L} \int_0^L f(x) dx$$

$$u_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi}{L} x dx$$

b. The equilibrium is when $u_{xx} = 0$ subject to the same boundary conditions. The solution is then obtained by integration with respect to $x$

$u_x = K$ and $K = 0$ because of the boundary conditions. Now integrate again to get $u = C$. This means that the temperature is constant. There is no other condition to fix this constant. The problem doen't have a unique solution.

Note that if we let $t$ to go to $\infty$, part a tells us that $u \to u_0$ (since all other terms contain a decaying exponential in time). The value of $u_0$ found there to be the average of the function $f(x)$. Thus we should take the constant $C$ to be the average initial temperature

$$C = \frac{1}{L} \int_0^L f(x) dx$$

17.

$$u_t = ku_{xx} \qquad 0 < x < L$$

Boundary conditions (inhomogeneous)

$$u(0,t) = 1$$

$$u_x(L,t) = 1$$

Initial condition

$$u(x,0) = 0$$

Let $w(x,t)$ satisfies the inhomogeneous boundary conditions

$$w(0,t) = 1$$

$$w_x(L,t) = 1$$

For example, we can take a linear function in $x$

$$w(x,t) = \alpha x + \beta$$

Using the boundary conditions we get

$$\alpha = \beta = 1$$

and so

$$w(x,t) = x + 1$$

Now let $v(x,t) = u(x,t) - w(x,t)$, then clearly $v$ will satisfy homogeneous boundary conditions, and the PDE becomes:

$$v_t + w_t = k(v_{xx} + w_{xx})$$

Since $w_t = w_{xx} = 0$, we can write the PDE

$$v_t = kv_{xx}$$

The initial condition is $v(x,0) + w(x,0) = 0$ or

$$v(x,0) = -x - 1$$

So the problem is now:

$$v_t = kv_{xx}$$

$$v(0,t) = 0$$

$$v_x(L,t) = 0$$

$$v(x,0) = -x - 1$$

The eigenfunctions and eigenvalues are

$$\sin \frac{n+1/2}{L}\pi x$$

$$\left(\frac{n+1/2}{L}\pi\right)^2, \qquad n = 1, 2, \ldots$$

Thus, we can expand

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin \frac{n+1/2}{L}\pi x$$

At $t = 0$ we have

$$-x - 1 = v(x,0) = \sum_{n=1}^{\infty} v_n(0) \sin \frac{n+1/2}{L}\pi x$$

so

$$v_n(0) = -\frac{2}{L} \int_0^L (x+1) \sin \frac{n+1/2}{L}\pi x dx$$

Substitute the expansion is the PDE and equate coefficients

$$\dot{v}_n(t) + k \left(\frac{n+1/2}{L}\pi\right)^2 v_n(t) = 0$$

$$v_n(0) = -\frac{2}{L} \int_0^L (x+1) \sin \frac{n+1/2}{L}\pi x dx$$

The solution is then

$$v_n(t) = v_n(0) e^{-k\left(\frac{n+1/2}{L}\pi\right)^2 t}$$

and

$$v(x,t) = \sum_{n=1}^{\infty} v_n(0) e^{-k\left(\frac{n+1/2}{L}\pi\right)^2 t} \sin \frac{n+1/2}{L}\pi x$$

and

$$u(x,t) = x + 1 + \sum_{n=1}^{\infty} v_n(0) e^{-k\left(\frac{n+1/2}{L}\pi\right)^2 t} \sin \frac{n+1/2}{L}\pi x$$

where $v_n(0)$ are given above.

18.

$$\nabla^2 u = 0, \qquad 0 \le x \le \pi,\ 0 \le y \le \pi$$

$$u(x,0) = \sin x + 2\sin 2x$$

$$u(x,\pi) = 0$$

$$u(0,y) = u(\pi,y) = 0$$

Separation of variables leads to

$$X'' + \lambda X = 0$$

and

$$Y'' - \lambda Y = 0$$

The last boundary condition dictates

$$X(0) = X(\pi) = 0$$

and we can solve the ODE for $X$

$$\lambda_n = n^2,\ n = 1, 2, \ldots$$

$$X_n = \sin nx,\ n = 1, 2, \ldots$$

Thus the ODE for $Y$ becomes

$$Y_n'' - n^2 Y_n = 0$$

with a bounday condition coming from next to last condition

$$Y_n(\pi) = 0$$

Ths solution is

$$Y_n(y) = \sinh n(y - \pi)$$

Thus we have

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sinh n(y-\pi) \sin nx$$

Now use the only inhomogeneous boundary condition

$$\sin x + 2\sin 2x = u(x,0) = \sum_{n=1}^{\infty} a_n \sinh n(-\pi) \sin nx$$

since the hyperbolic sine is an odd function

$$\sin x + 2\sin 2x \ = \ -\sum_{n=1}^{\infty} a_n \sinh n\pi \sin nx$$

Comparing coefficients we see that $\sin x$ has a coefficient of 1 and therefore

$$-a_1 \sinh \pi = 1$$

$$-a_2 \sinh 2\pi = 2$$

The rest of the coefficients are zero

$$a_n = 0,\ n \geq 2$$

Therefore the solution is

$$u(x, y) = -\frac{1}{\sinh \pi} \sin x - \frac{2}{\sinh 2\pi} \sin 2x$$

19.

$$\nabla^2 u = 0, \qquad 0 \le x \le \pi,\ 0 \le y \le \pi$$
$$u(x,0) = -\pi^2 x^2 + 2\pi x^3 - x^4$$
$$u(x,\pi) = 0$$
$$u(0,y) = u(\pi,y) = 0$$

The only difference is in the inhomogeneous condition, thus the general solution is the same

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sinh n(y-\pi) \sin nx$$

Now use the only inhomogeneous boundary condition

$$-\pi^2 x^2 + 2\pi x^3 - x^4 = u(x,0) \; = \; -\sum_{n=1}^{\infty} a_n \sinh n\pi \sin nx$$

The coefficients $-a_n \sinh n\pi$ are those of the Fourier sine expansion of $-\pi^2 x^2 + 2\pi x^3 - x^4$, i.e.

$$-a_n \sinh n\pi \; = \; \frac{2}{\pi} \int_0^{\pi} \left(-\pi^2 x^2 + 2\pi x^3 - x^4\right) \sin nx dx$$

Now we integrate (using integration by parts to reduce the powers of $x$)

$$\begin{aligned}
\int x^4 \sin nx dx \quad &= \quad -\frac{x^4}{n} \cos nx + \frac{4}{n} \int x^3 \cos nx dx \\
&= \quad -\frac{x^4}{n} \cos nx + \frac{4}{n} \left[ \frac{x^3}{n} \sin nx - \frac{3}{n} \int x^2 \sin nx dx \right] \\
&= \quad -\frac{x^4}{n} \cos nx + \frac{4x^3}{n^2} \sin nx - \frac{12}{n^2} \int x^2 \sin nx dx \\
\int x^3 \sin nx dx \quad &= \quad -\frac{x^3}{n} \cos nx + \frac{3}{n} \int x^2 \cos nx dx \\
&= \quad -\frac{x^3}{n} \cos nx + \frac{3}{n} \left[ \frac{x^2}{n} \sin nx - \frac{2}{n} \int x \sin nx dx \right] \\
&= \quad -\frac{x^3}{n} \cos nx + \frac{3x^2}{n^2} \sin nx - \frac{6}{n^2} \int x \sin nx dx
\end{aligned}$$

Now to our integral

$$
\begin{aligned}
\int_0^\pi \left(-\pi^2x^2 + 2\pi x^3 - x^4\right)\sin nx dx &= -\pi^2 \int_0^\pi x^2 \sin nx dx \\
&+ 2\pi \left[ -\frac{x^3}{n}\cos nx\Big|_0^\pi + \frac{3x^2}{n^2}\sin nx\Big|_0^\pi \right. \\
&- \left. \frac{6}{n^2}\int_0^\pi x \sin nx dx \right] \\
&\quad - \left[ -\frac{x^4}{n}\cos nx\Big|_0^\pi + \frac{4x^3}{n^2}\sin nx\Big|_0^\pi \right. \\
&- \left. \frac{12}{n^2}\int_0^\pi x^2 \sin nx dx \right]
\end{aligned}
$$

The sine function vanishes at the end points and $\cos 0 = 1$ and $\cos n\pi = (-1)^n$ so

$$
\begin{aligned}
\int_0^\pi \left(-\pi^2x^2 + 2\pi x^3 - x^4\right)\sin nx dx &= \left(-\pi^2 + \frac{12}{n^2}\right)\int_0^\pi x^2 \sin nx dx \\
&- \frac{12\pi}{n^2}\int_0^\pi x \sin nx dx - \frac{2\pi}{n}\pi^3(-1)^n + \frac{\pi^4}{n}(-1)^n \\
&= \left(-\pi^2 + \frac{12}{n^2}\right)\left(-\frac{\pi^2}{n}(-1)^n\right) \\
&+ \frac{2}{n}\left(-\pi^2 + \frac{12}{n^2}\right)\int_0^\pi x \cos nx dx \\
&- \frac{12\pi}{n^2}\int_0^\pi x \sin nx dx
\end{aligned}
$$

After evaluating the integrals, we get

$$
-a_n \sinh n\pi = \frac{2}{\pi}\left(\frac{24}{n^5} - \frac{2\pi^2}{n^2}\right)\begin{cases} 0 & \text{for } n \text{ even} \\ -2 & \text{for } n \text{ odd} \end{cases}
$$

Thus for $n$ odd, we have

$$
a_n = \frac{8}{n^2\pi \sinh n\pi}\left(\frac{12}{n^3} - \pi^2\right)
$$

The solution is then

$$
u(x,y) = \sum_{n=1,3,\ldots} \frac{8}{n^2\pi \sinh n\pi}\left(\frac{12}{n^3} - \pi^2\right)\sinh n(y-\pi)\sin nx
$$

# 4 PDEs in Higher Dimensions

## 4.1 Introduction

## 4.2 Heat Flow in a Rectangular Domain

**Problems**

1. Solve the heat equation

$$u_t(x, y, t) = k\left(u_{xx}(x, y, t) + u_{yy}(x, y, t)\right),$$

on the rectangle $0 < x < L, 0 < y < H$ subject to the initial condition

$$u(x, y, 0) = f(x, y),$$

and the boundary conditions

a.

$$u(0, y, t) = u_x(L, y, t) = 0,$$
$$u(x, 0, t) = u(x, H, t) = 0.$$

b.

$$u_x(0, y, t) = u(L, y, t) = 0,$$
$$u_y(x, 0, t) = u_y(x, H, t) = 0.$$

c.

$$u(0, y, t) = u(L, y, t) = 0,$$
$$u(x, 0, t) = u_y(x, H, t) = 0.$$

2. Solve the heat equation on a rectangular box

$$0 < x < L, 0 < y < H, 0 < z < W,$$

$$u_t(x, y, z, t) = k(u_{xx} + u_{yy} + u_{zz}),$$

subject to the boundary conditions

$$u(0, y, z, t) = u(L, y, z, t) = 0,$$

$$u(x, 0, z, t) = u(x, H, z, t) = 0,$$
$$u(x, y, 0, t) = u(x, y, W, t) = 0,$$

and the initial condition

$$u(x, y, z, 0) = f(x, y, z).$$

1. $u_t = k(u_{xx} + u_{yy})$

$u(x, y, 0) = f(x, y)$

$u = X(x)Y(y)T(t)$

$xY\dot{T} = kYX''T + kXTY''$

$$\frac{\dot{T}}{kT} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda$$

$$\dot{T} + \lambda kT = 0 \qquad \frac{X''}{X} = -\frac{Y''}{Y} - \lambda = -\mu$$

$$X'' + \mu X = 0 \qquad Y'' + (\lambda - \mu)Y = 0$$

a. $X(0) = X'(L) = 0$

$Y(0) = Y(H) = 0$

$$\Rightarrow X_n = \sin \frac{\left(n - \frac{1}{2}\right)\pi}{L} x \qquad \mu_n = \left[\frac{\left(n - \frac{1}{2}\right)\pi}{L}\right]^2 \qquad n = 1, 2, \cdots$$

$$\Rightarrow Y_{nm} = \sin \frac{m\pi}{H} y \qquad \lambda_{nm} - \mu_n = \left(\frac{m\pi}{H}\right)^2 \qquad n = 1, 2, \cdots \qquad m = 1, 2, \cdots$$

$$\lambda_{nm} = \left[\frac{\left(n - \frac{1}{2}\right)\pi}{L}\right]^2 + \left(\frac{m\pi}{H}\right)^2 \qquad n, m = 1, 2, \cdots$$

$T_{nm} = e^{-\lambda_{nm}kt}$

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} a_{nm}\, e^{-\lambda_{nm}kt} \sin \left(n - \frac{1}{2}\right) \frac{\pi}{L} x \sin \frac{m\pi}{H} y$$

$$f(x, y) = u(x, y, 0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \sin \left(n - \frac{1}{2}\right) \frac{\pi}{L} x \sin \frac{m\pi}{H} y$$

$$a_{nm} = \frac{\int_0^H \int_0^L f(x, y) \sin \left(n - \frac{1}{2}\right) \frac{\pi}{L} x \sin \frac{m\pi}{H} y\, dx\, dy}{\int_0^H \int_0^L \sin^2 \left(n - \frac{1}{2}\right) \frac{\pi}{L} x \sin^2 \frac{m\pi}{H} y\, dx\, dy}$$

1b. $X'(0) = X(L) = 0$

$Y'(0) = Y'(H) = 0$

$$X_n = \cos \frac{\left(n - \frac{1}{2}\right) \pi}{L} x \qquad \mu_n = \left[ \frac{\left(n - \frac{1}{2}\right) \pi}{L} \right]^2 \qquad n = 1, 2, \cdots$$

$$Y_{nm} = \cos \frac{m \pi}{H} x \qquad \lambda_{nm} = \left[ \frac{\left(n - \frac{1}{2}\right) \pi}{L} \right]^2 + \left( \frac{m \pi}{H} \right)^2 \qquad n = 1, 2, \cdots \qquad m =$$
$0, 1, 2, \cdots$

$$u(x,y,t) = \sum_{n=1}^{\infty} \frac{1}{2} a_{n0} \, e^{-k \lambda_{n0} t} \cos \frac{\left(n - \frac{1}{2}\right) \pi}{L} x + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{nm} \, e^{-k \lambda_{nm} t} \cos \frac{\left(n - \frac{1}{2}\right) \pi}{L} x \cos \frac{m \pi}{H} y$$

$$a_{nm} = \frac{\int_0^H \int_0^L f(x, y) \cos \left(n - \frac{1}{2}\right) \frac{\pi}{L} x \cos \frac{m \pi}{H} y \, dx \, dy}{\int_0^H \int_0^L \cos^2 \left(n - \frac{1}{2}\right) \frac{\pi}{L} x \cos^2 \frac{m \pi}{H} y \, dx \, dy} \qquad n = 1, 2, \cdots \qquad m = 0, 1, 2, \cdots$$

1c. $X(0) = X(L) = 0$

$Y(0) = Y'(H) = 0$

$$X_n = \sin\frac{n\pi}{L}x \qquad \mu_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, \cdots$$

$$Y_{nm} = \sin\frac{\left(m - \frac{1}{2}\right)\pi}{H}y \qquad \lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left[\frac{\left(m - \frac{1}{2}\right)\pi}{H}\right]^2 \qquad m, n = 1, 2, \cdots$$

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{nn}\, e^{-k\lambda_{nm}t}\,\sin\frac{n\pi}{L}x\,\sin\frac{\left(m - \frac{1}{2}\right)\pi}{H}y$$

$$a_{nm} = \frac{\int_0^L\int_0^H f(x, y)\,\sin\frac{n\pi}{L}x\,\sin\frac{\left(m - \frac{1}{2}\right)\pi}{H}y\,dy\,dx}{\int_0^L\int_0^H \sin^2\frac{n\pi}{L}x\,\sin^2\frac{\left(m - \frac{1}{2}\right)\pi}{H}y\,dy\,dx}$$

2. $u_t = k(u_{xx} + u_{yy} + u_{zz})$

$$\dot{T}XYZ = kT(X''YZ + XY''Z + XYZ'')$$

$$\frac{\dot{T}}{kT} = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -\lambda$$

$$\dot{T} + \lambda kT = 0 \qquad \frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} - \lambda = -\mu$$

$$X'' + \mu X = 0 \qquad \frac{Y''}{Y} = -\frac{Z''}{Z} - \lambda + \mu = -\nu$$
$$X(0) = X(L) = 0$$

$$Y'' + \nu Y = 0$$
$$Y(0) = Y(H) = 0$$

$$Z'' + (\lambda - \mu - \nu)Z = 0$$
$$Z(0) = Z(W) = 0$$

$$X_n = \sin\left(\frac{n\pi}{L}\right)x \qquad \mu_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, \cdots$$

$$Y_m = \sin\frac{m\pi}{H}y \qquad \nu_m = \left(\frac{m\pi}{H}\right)^2 \qquad m = 1, 2, \cdots$$

$$Z_{nm\ell} = \sin\frac{\ell\pi}{W}z \qquad \lambda_{nm\ell} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 + \left(\frac{\ell\pi}{W}\right)^2 \qquad \ell = 1, 2, \cdots$$

$$u(x,y,z,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{\ell=1}^{\infty} a_{nm\ell}\, e^{-k\lambda_{nm\ell}t} \sin\frac{n\pi}{L}x \sin\frac{m\pi}{H}y \sin\frac{\ell\pi}{W}z$$

$$a_{nm\ell} = \frac{\int_0^L \int_0^H \int_0^W f(x, y, z) \sin\frac{n\pi}{L}x \sin\frac{m\pi}{H}y \sin\frac{\ell\pi}{W}z\,dz\,dy\,dx}{\int_0^L \int_0^H \int_0^W \sin^2\frac{n\pi}{L}x \sin^2\frac{m\pi}{H}y \sin^2\frac{\ell\pi}{W}z\,dz\,dy\,dx}$$

## 4.3 Vibrations of a rectangular Membrane

### Problems

1. Solve the wave equation

$$u_{tt}(x,y,t) = c^2 \left(u_{xx}(x,y,t) + u_{yy}(x,y,t)\right),$$

on the rectangle $0 < x < L, 0 < y < H$ subject to the initial conditions

$$u(x,y,0) = f(x,y),$$

$$u_t(x,y,0) = g(x,y),$$

and the boundary conditions

a.

$$u(0,y,t) = u_x(L,y,t) = 0,$$
$$u(x,0,t) = u(x,H,t) = 0.$$

b.

$$u(0,y,t) = u(L,y,t) = 0,$$
$$u(x,0,t) = u(x,H,t) = 0.$$

c.

$$u_x(0,y,t) = u(L,y,t) = 0,$$
$$u_y(x,0,t) = u_y(x,H,t) = 0.$$

2. Solve the wave equation on a rectangular box

$$0 < x < L, 0 < y < H, 0 < z < W,$$

$$u_{tt}(x,y,z,t) = c^2(u_{xx} + u_{yy} + u_{zz}),$$

subject to the boundary conditions

$$u(0,y,z,t) = u(L,y,z,t) = 0,$$

$$u(x,0,z,t) = u(x,H,z,t) = 0,$$
$$u(x,y,0,t) = u(x,y,W,t) = 0,$$

and the initial conditions

$$u(x,y,z,0) = f(x,y,z),$$
$$u_t(x,y,z,0) = g(x,y,z).$$

3. Solve the wave equation on an isosceles right-angle triangle with side of length $a$

$$u_{tt}(x,y,t) = c^2(u_{xx} + u_{yy}),$$

subject to the boundary conditions

$$u(x,0,t) = u(0,y,t) = 0,$$

$$u(x,y,t) = 0, \qquad \text{on the line} \qquad x + y = a$$

and the initial conditions

$$u(x,y,0) = f(x,y),$$

$$u_t(x,y,0) = g(x,y).$$

1. $u_{tt} = c^2 (u_{xx} + u_{yy})$

$$\ddot{T}XY = c^2 T (X''Y + XY'')$$

$$\frac{\ddot{T}}{c^2 T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda$$

$$\ddot{T} + \lambda c^2 T = 0 \qquad \frac{X''}{X} = -\frac{Y''}{Y} - \lambda = -\mu$$

$$X'' + \mu X = 0 \qquad Y'' + (\lambda - \mu)Y = 0$$

a. $X(0) = X'(L) = 0$

$Y(0) = Y(H) = 0$

as in previous section

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ a_{nm} \cos c\sqrt{\lambda_{nm}}\, t + b_{nm} \sin c\sqrt{\lambda_{nm}}\, t \right\} \sin \frac{\left(n - \frac{1}{2}\right)\pi}{L} x \sin \frac{m\pi}{H} y$$

Initial Conditions

$$f(x,\, y) = u(x,y,0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \sin \frac{\left(n - \frac{1}{2}\right)\pi}{L} x \sin \frac{m\pi}{H} y \qquad \text{yields } a_{nm}$$

$$a_{nm} = \frac{\int_0^H \int_0^L f(x,\, y) \sin \left(n - \frac{1}{2}\right) \frac{\pi}{L} x \sin \frac{m\pi}{H} y \, dx\, dy}{\int_0^H \int_0^L \sin^2 \left(n - \frac{1}{2}\right) \frac{\pi}{L} x \sin^2 \frac{m\pi}{H} y \, dx\, dy}$$

$$g(x,\, y) = u_t\,(x,\, y,\, 0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c\sqrt{\lambda_{nm}}\, b_{nm} \sin (n - \frac{1}{2}) \frac{\pi}{L} x \sin \frac{m\pi}{H} y$$

$$b_{nm} = \frac{\int_0^L \int_0^H g(x,\, y) \sin \frac{\left(n - \frac{1}{2}\right)\pi}{L} x \sin \frac{m\pi}{H} y \, dy\, dx}{c\sqrt{\lambda_{nm}} \int_0^L \int_0^H \sin^2 (n - \frac{1}{2}) \frac{\pi}{L} x \sin^2 \frac{m\pi}{H} y \, dy\, dx}$$

b.

$$X(0) = X(L) = 0$$

$$Y(0) = Y(H) = 0$$

$$X_n = \sin \tfrac{n\pi}{L} x \qquad \mu_n = \left(\tfrac{n\pi}{L}\right)^2 \qquad n = 1, 2, \cdots$$

$$Y_{nm} = \sin \tfrac{m\pi}{H} y \qquad \lambda_{nm} = \left(\tfrac{n\pi}{L}\right)^2 + \left(\tfrac{m\pi}{H}\right)^2 \qquad m = 1, 2, \cdots$$

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ a_{nm} \cos c\sqrt{\lambda_{nm}}\, t + b_{nm} \sin c\sqrt{\lambda_{nm}}\, t \right\} \sin \frac{n\pi}{L} x \sin \frac{m\pi}{H} y$$

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \sin \frac{n\pi}{L} x \sin \frac{m\pi}{H} y$$

$$g(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c\sqrt{\lambda_{nm}}\, b_{nm} \sin \frac{n\pi}{L} x \sin \frac{m\pi}{H} y$$

$a_{nm}$ , $b_{nm}$ in a similar fashion to part a.

$$a_{nm} = \frac{\int_0^H \int_0^L f(x, y) \sin \frac{n\pi}{L} x \sin \frac{m\pi}{H} y \, dx \, dy}{\int_0^H \int_0^L \sin^2 \frac{n\pi}{L} x \sin^2 \frac{m\pi}{H} y \, dx \, dy}$$

$$b_{nm} = \frac{\int_0^L \int_0^H g(x, y) \sin \frac{n\pi}{L} x \sin \frac{m\pi}{H} y \, dy \, dx}{c\sqrt{\lambda_{nm}} \int_0^L \int_0^H \sin^2 \frac{n\pi}{L} x \sin^2 \frac{m\pi}{H} y \, dy \, dx}$$

c. see 1b in 7.1

$$X_n = \cos \frac{\left(n - \frac{1}{2}\right)\pi}{L} x \qquad \mu_n = \left[\frac{\left(n - \frac{1}{2}\right)\pi}{L}\right]^2 \qquad n = 1, 2, \cdots$$

$$Y_{nm} = \cos \frac{m\pi}{H} y \qquad \lambda_{nm} = \left[\frac{\left(n - \frac{1}{2}\right)\pi}{L}\right]^2 + \left(\frac{m\pi}{H}\right)^2 \qquad m = 0, 1, 2, \cdots$$

$$u(x, y, t) = \sum_{n=1}^{\infty} \left\{ a_{n0} \cos c\sqrt{\lambda_{n0}}\, t + b_{n0} \sin c\sqrt{\lambda_{n0}}\, t \right\} \cos \frac{\left(n - \frac{1}{2}\right)\pi}{L} x$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ a_{nm} \cos c\sqrt{\lambda_{nm}}\, t + b_{nm} \sin c\sqrt{\lambda_{nm}}\, t \right\} \cos \frac{\left(n - \frac{1}{2}\right)\pi}{L} x \cos \frac{m\pi}{H} y$$

$f(x, y)$ yields $a_{n0},\ a_{nm}$

$g(x, y)$ yields $b_{n0},\ b_{nm}$

2. Since boundary conditions are the same as in 2 section 7.1

$$u(x,y,z,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{\ell=1}^{\infty} \left\{ a_{nm\ell} \cos c\sqrt{\lambda_{nm\ell}}\, t + b_{nm\ell} \sin c\sqrt{\lambda_{nm\ell}}\, t \right\} \sin \frac{n\pi}{L} x \sin \frac{m\pi}{H} y \sin \frac{\ell\pi}{W} z$$

$f(x, y, z)$ yields $a_{nm\ell}$

$$a_{nm\ell} = \frac{\int_0^L \int_0^H \int_0^W f(x, y, z) \sin \frac{n\pi}{L} x \sin \frac{m\pi}{H} y \sin \frac{\ell\pi}{W} z \, dz\, dy\, dx}{\int_0^L \int_0^H \int_0^w \sin^2 \frac{n\pi}{L} x \sin^2 \frac{m\pi}{H} y \sin^2 \frac{\ell\pi}{W} z \, dz\, dy\, dx}$$

$g(x, y, z)$ yields $b_{nm\ell}$

$$b_{nm\ell} = \frac{\int_0^L \int_0^H \int_0^W g(x, y, z) \sin \frac{n\pi}{L} x \sin \frac{m\pi}{H} y \sin \frac{\ell\pi}{W} z \, dz\, dy\, dx}{c\sqrt{\lambda_{nm\ell}} \int_0^L \int_0^H \int_0^W \sin^2 \frac{n\pi}{L} x \sin^2 \frac{m\pi}{H} y \sin^2 \frac{\ell\pi}{W} z \, dz\, dy\, dx}$$

3.

See the solution of Helmholtz equation (problem 2 in section 7.4)

$$\psi_{nm}(x,y) \;=\; \sin\frac{\pi}{a}(m+n)x\,\sin\frac{\pi}{a}ny \;-\; (-1)^m\,\sin\frac{\pi}{a}(m+n)y\,\sin\frac{\pi}{a}nx$$

$$\lambda_{nm} \;=\; \frac{\pi}{a}\sqrt{(m+n)^2\;+\;n^2} \qquad n,m=1,2,\cdots$$

The solution is similar to 1b

$$u(x,y,t) \;=\; \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\{a_{nm}\cos c\sqrt{\lambda_{nm}}\,t \;+\; b_{nm}\,\sin\,c\sqrt{\lambda_{nm}}\,t\right\}\psi_{nm}(x,y)$$

$$f(x,\,y) \;=\; \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\,\psi_{nm}(x,y)$$

$$g(x,\,y) \;=\; \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c\sqrt{\lambda_{nm}}\,b_{nm}\,\psi_{nm}(x,y)$$

$a_{nm}$ , $b_{nm}$ in a similar fashion to 1a.

$$a_{nm} \;=\; \frac{\int_0^a\int_0^a f(x,\,y)\,\psi_{nm}(x,y)\,dx\,dy}{\int_0^a\int_0^a \psi_{nm}^2(x,y)\,dx\,dy}$$

$$b_{nm} \;=\; \frac{\int_0^a\int_0^a g(x,\,y)\psi_{nm}(x,y)\,dy\,dx}{c\sqrt{\lambda_{nm}}\int_0^a\int_0^a \psi_{nm}^2(x,y)\,dy\,dx}$$

## 4.4 Helmholtz Equation

**Problems**

1. Solve

$$\nabla^2\phi + \lambda\phi = 0 \qquad\qquad [0,1]\times[0,1/4]$$

subject to

$$\phi(0,y) = 0$$
$$\phi_x(1,y) = 0$$
$$\phi(x,0) = 0$$
$$\phi_y(x,1/4) = 0.$$

Show that the results of the theorem are true.

2. Solve Helmholtz equation on an isosceles right-angle triangle with side of length $a$

$$u_{xx} + u_{yy} + \lambda u = 0,$$

subject to the boundary conditions

$$u(x,0,t) = u(0,y,t) = 0,$$

$$u(x,y,t) = 0, \qquad \text{on the line} \qquad x + y = a.$$

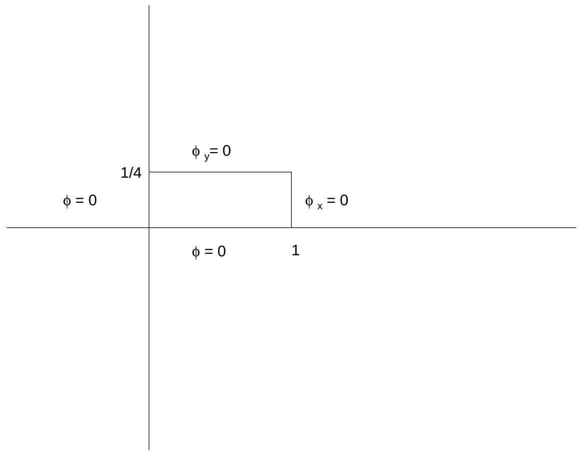


Figure 33: Domain fro problem 1 of 7.4

1.

$$\varphi(x, y) = XY$$

$$X''Y + XY'' + \lambda XY = 0$$

$$\frac{X''}{X} = -\lambda - \frac{Y''}{Y} = -\mu$$

$$X'' + \mu X = 0 \qquad Y'' + (\lambda - \mu) Y = 0$$

$$X(0) = X'(1) = 0 \qquad Y(0) = Y'(1/4) = 0$$

$$\Downarrow \qquad \Downarrow$$

$$X_n = \sin(n - \tfrac{1}{2})\pi x \qquad Y_{nm} = \sin(m - \tfrac{1}{2}) 4\pi y$$

$$\mu_n = \left[(n - \tfrac{1}{2})\pi\right]^2 \qquad \lambda_{nm} = \left[(n - \tfrac{1}{2})\pi\right]^2 + [(4m - 2)\pi]^2$$

$$n = 1, 2, \cdots \qquad m = 1, 2, \cdots$$

$$\varphi_{nm} = \sin(n - \tfrac{1}{2})\pi x \; \sin(4m - 2)\pi y$$

$$\lambda_{nm} = [(n - \tfrac{1}{2})\pi]^2 + [4m - 2)\pi]^2 \qquad n, m = 1, 2, \cdots$$

Infinite number of eigenvalues

$\lambda_{11} = \frac{1}{4}\pi^2 + 4\pi^2$ is the smallest.

There is no largest since $\lambda_{nm} \to \infty$ as $n$, $m$ increase

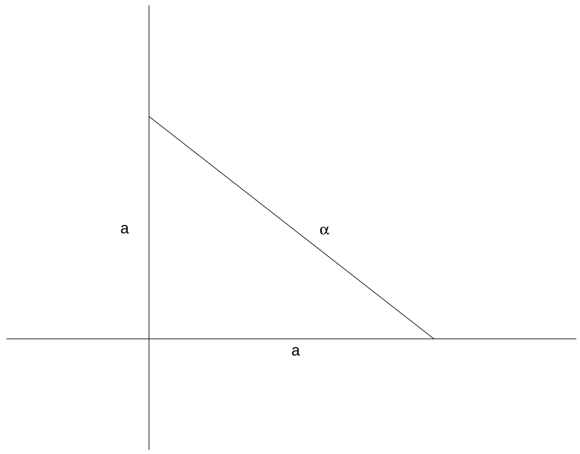


Figure 34: Domain for problem 2 of 7.4

2.

The analysis is more involved when the equation is NOT separable in coordinates suitable for the boundary. Only two nonseparable cases have been solved in detail, one for a boundary which is an isosceles right triangle.

The function

$$\sin \frac{\mu \pi x}{a} \sin \frac{\nu \pi}{a} y$$

is zero along the $x$ and $y$ part of the boundary but is not zero along the diagonal side. However, the combination

$$\sin \tfrac{\mu \pi}{a} x \sin \tfrac{\nu \pi}{a} y \mp \sin \tfrac{\mu \pi}{a} y \sin \tfrac{\nu \pi}{a} x$$

is zero along the diagonal if $\mu$ and $\nu$ are integers. (The + sign is taken when $|\mu - \nu|$ is even and the $-$ sign when $|\mu - \nu|$ is odd).

The eigenfunctions

$$\psi_{mn}(x, y) = \sin \frac{\pi}{a}(m+n)x \sin \frac{\pi}{a} ny - (-1)^m \sin \frac{\pi}{a}(m+n) y \sin \frac{\pi}{a} nx$$

where $m$, $n$ are positive integers.

The only thing we have to show is the boundary condition on the line $x + y = a$. To show this, rotate by $\pi/4$

$$\Rightarrow x = \frac{1}{\sqrt{2}}(\xi - \eta)$$

$$y = \tfrac{1}{\sqrt{2}}(\xi + \eta)$$

$$\psi_{mn} = \begin{cases} \sin \dfrac{\pi}{\alpha}(m + 2n)\,\xi \, \sin \dfrac{\pi}{\alpha} m\,\eta \quad - \quad \sin \frac{\pi}{\alpha}(m + 2n)\,\eta \, \sin \frac{\pi}{\alpha} m\,\xi & m = 2, 4, \cdots \\ \\ \cos \dfrac{\pi}{\alpha}(m + 2n)\,\eta \, \cos \dfrac{\pi}{\alpha} m\,\xi \quad - \quad \cos \frac{\pi}{\alpha}(m + 2n)\,\xi \, \cos \frac{\pi}{\alpha} m\,\eta & m = 1, 3, \cdots \end{cases}$$

$\Rightarrow \psi_{mn} = 0$ for $\xi = \alpha/2$ which is $x + y = a$.

The eigenvalues are:

$$\lambda_{mn} = \left(\frac{\pi}{a}\right) \sqrt{(m + n)^2 + n^2}$$

## 4.5 Vibrating Circular Membrane

### Problems

1. Solve the heat equation

$$u_t(r,\theta,t) = k\nabla^2 u, \qquad 0 \le r < a, 0 < \theta < 2\pi, t > 0$$

subject to the boundary condition

$$u(a,\theta,t) = 0 \qquad \text{(zero temperature on the boundary)}$$

and the initial condition

$$u(r,\theta,0) = \alpha(r,\theta).$$

2. Solve the wave equation

$$u_{tt}(r,t) = c^2(u_{rr} + \frac{1}{r}u_r),$$
$$u_r(a,t) = 0,$$
$$u(r,0) = \alpha(r),$$
$$u_t(r,0) = 0.$$

Show the details.

3. Consult numerical analysis textbook to obtain the smallest eigenvalue of the above problem.

4. Solve the wave equation

$$u_{tt}(r,\theta,t) - c^2\nabla^2 u = 0, \qquad 0 \le r < a, 0 < \theta < 2\pi, t > 0$$

subject to the boundary condition

$$u_r(a,\theta,t) = 0$$

and the initial conditions

$$u(r,\theta,0) = 0,$$
$$u_t(r,\theta,0) = \beta(r)\cos 5\theta.$$

5. Solve the wave equation

$$u_{tt}(r,\theta,t) - c^2\nabla^2 u = 0, \qquad 0 \le r < a, 0 < \theta < \pi/2, t > 0$$

subject to the boundary conditions

$$u(a,\theta,t) = u(r,0,t) = u(r,\pi/2,t) = 0 \qquad \text{(zero displacement on the boundary)}$$

and the initial conditions

$$u(r,\theta,0) = \alpha(r,\theta),$$
$$u_t(r,\theta,0) = 0.$$

1. $u_t = k\,\nabla^2 u \qquad 0 \le r \le a$

$u(a,\,\theta,\,t) = 0 \qquad 0 \le \theta \le 2\pi$

$u(r,\,\theta,\,0) = \alpha(r,\,\theta) \qquad t > 0$

$$\dot T R\Theta = kT\left[\Theta\,\tfrac{1}{r}\,(r\,R')' + \tfrac{1}{r^2}\,R\,\Theta''\right]$$

$$\frac{\dot T}{kT} = \frac{1}{R\,r}\,(r\,R')' + \frac{1}{r^2}\,\frac{\Theta''}{\Theta} = -\lambda$$

$$\dot T + k\,\lambda\,T = 0 \qquad \frac{r}{R}\,(r\,R')' + \frac{\Theta''}{\Theta} = -\lambda r^2$$

$$\frac{r}{R}\,(r\,R')' + \lambda r^2 = -\frac{\Theta''}{\Theta} = \mu$$

$$\Theta'' + \mu\,\Theta = 0 \qquad r\,(r\,R')' + (\lambda r^2 - \mu)\,R = 0$$

$$\Theta(0) = \Theta(2\,\pi) \qquad |\,R(0)\,| < \infty$$

$$\Theta'(0) = \Theta'(2\,\pi) \qquad R(a) = 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\mu_m = m^2 \qquad \boxed{R_m = C_{1m}\,J_m\,(\sqrt{\lambda}\,r) \text{ to satisfy } |\,R(0)\,| < \infty}$$

$$\Theta_m = \begin{cases} \sin\,m\,\theta \\ \\ \cos\,m\,\theta \end{cases} \quad m = 1,\,2,\,\cdots$$

$$R_m(\sqrt{\lambda}\,a) = C_m\,J_m\,(\sqrt{\lambda}\,a) = 0$$

$$\mu_0 = 0 \qquad\qquad \Downarrow$$

$\Theta_0 = 1 \qquad \lambda_{nm}$ are solutions of $\boxed{J_m\,(\sqrt{\lambda}\,a) = 0}$

$$u(r,\,\theta,\,t) \;=\; \sum_{m=1}^{\infty}\;\sum_{n=1}^{\infty}(a_{nm}\;\cos\,m\,\theta\;+\;b_{nm}\;\sin\,m\,\theta)\,e^{-k\,\lambda_{nm}t}\,J_m\,(\sqrt{\lambda_{nm}}\,r)$$

$$+\;\underbrace{\sum_{n=1}^{\infty}\;a_{n0}\;\cdot\;\underbrace{1}_{=\Theta_0}\;e^{-k\,\lambda_{n0}t}\,J_0\,(\sqrt{\lambda_{n0}}\,r)}_{m=0}$$

$$\alpha\,(r,\,\theta)\;=\;\sum_{n=1}^{\infty}\;a_{n0}\,J_0\,(\sqrt{\lambda_{n0}}\,r)\;+\;\sum_{m=1}^{\infty}\;\sum_{n=1}^{\infty}(a_{nm}\;\cos\,m\,\theta\;+\;b_{nm}\;\sin\,m\,\theta)\,J_m\,(\sqrt{\lambda_{nm}}\,r)$$

$$a_{n0}\;=\;\frac{\int_0^{2\pi}\;\int_0^a\;\alpha\,(r,\,\theta)\,J_0\,(\sqrt{\lambda_{n0}}\,r)\,rd\,rd\theta}{\int_0^{2\pi}\;\int_0^a\;J_0^2\,(\sqrt{\lambda_{n0}}\,r)\,rd\,rd\theta}$$

$$a_{nm}\;=\;\frac{\int_0^{2\pi}\;\int_0^a\;\alpha\,(r,\,\theta)\;\cos\;\,m\,\theta\,J_m\,(\sqrt{\lambda_{nm}}\,r)\,rd\,rd\theta}{\int_0^{2\pi}\;\int_0^a\;\cos^2\;m\,\theta\,J_m^2\,(\sqrt{\lambda_{nm}}\,r)\,rd\,rd\theta}$$

$$b_{nm}\;=\;\frac{\int_0^{2\pi}\;\int_0^a\;\alpha\,(r,\,\theta)\;\sin\;\,m\,\theta\,J_m\,(\sqrt{\lambda_{nm}}\,r)\,rd\,rd\theta}{\int_0^{2\pi}\;\int_0^a\;\sin^2\;m\,\theta\,J_m^2\,(\sqrt{\lambda_{nm}}\,r)\,rd\,rd\theta}$$

2. $u_{tt} - c^2 (u_{rr} + \frac{1}{r} u_r)$

$$u_r (a, t) = 0$$

$$u (r, 0) = \alpha (r)$$

$$u_t (r, 0) = 0$$

$$\ddot{T} R - c^2 (R'' + \frac{1}{r} R') T = 0$$

$$\frac{\ddot{T}}{c^2 T} = \frac{R'' + \frac{1}{r} R'}{R} = -\lambda$$

$$\ddot{T} + \lambda c^2 T = 0 \qquad \underbrace{R'' + \frac{1}{r} R'}_{} + \lambda R = 0$$

$$\frac{1}{r} (r R')' + \lambda R = 0$$

multiply by $r^2$

$$\left. \begin{array}{l} r (r R')' + \lambda r^2 R = 0 \\ | R(0) | < \infty \\ R'(a) = 0 \end{array} \right\}$$

This is Bessel's equation with $\mu = 0$

$\Rightarrow R_n (r) = J_0 (\sqrt{\lambda_n} r)$

where $\sqrt{\lambda_n} J_0' (\sqrt{\lambda_n} a) = 0$

gives the eigenvalues $\lambda_n$

$$u(r, t) = \sum_{n=1}^{\infty} \left\{ a_n \cos \sqrt{\lambda_n} ct + b_n \sin c \sqrt{\lambda_n} t \right\} J_0 (\sqrt{\lambda_n} r)$$

$$\alpha (r) = \sum_{n=1}^{\infty} a_n J_0 (\sqrt{\lambda_n} r)$$

This yields $a_n$. $\Rightarrow a_n = \dfrac{\int_0^a \alpha (r) J_0 (\sqrt{\lambda_n} r) r dr}{\int_0^a J_0^2 (\sqrt{\lambda_n} r) r dr}$

$$0 = u_t (r, 0) = \sum_{n=1}^{\infty} c \sqrt{\lambda_n} b_n J_0 (\sqrt{\lambda_n} r) \Rightarrow \underline{b_n = 0}$$

4. $u_{tt} - c^2 \nabla^2 u = 0$

$u_r(a, \theta, t) = 0$

$u(r, \theta, 0) = 0$

$u_t(r, \theta, 0) = \beta(r) \cos 5\theta$

$\Downarrow$

$\ddot{T} + \lambda c^2 T = 0$ $\qquad \Theta'' + \mu \Theta = 0$ $\qquad r(r R')' + (\lambda r^2 - \mu) R = 0$

$T(0) = 0$ $\qquad \Theta(0) = \Theta(2\pi)$ $\qquad |R(0)| < \infty$

$\Theta'(0) = \Theta'(2\pi)$ $\qquad R'(a) = 0$

$T = a \cos c\sqrt{\lambda_{nm}}\, t + b \sin c\sqrt{\lambda_{nm}}\, t$

$\Downarrow$

$\mu_0 = 0 \quad \Theta_0 = 1$ $\qquad R = J_n(\sqrt{\lambda}\, r)$

Since $T(0) = 0$

$R'(a) = J_n'(\sqrt{\lambda}\, a) \cdot \sqrt{\lambda} = 0$

$$\mu_n = n^2 \quad \Theta_m = \begin{cases} \cos n\theta \\ \sin n\theta \end{cases}$$

$\Downarrow$ $\qquad \Downarrow$

$T = \sin c\sqrt{\lambda_{nm}}\, t$

$\lambda_{n0} = 0$

or

$J_n'(\sqrt{\lambda_{nm}}\, a) = 0$

$m = 1, 2, \cdots$

for each $n = 0, 1, 2, \cdots$

$$u(r,\theta,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \{a_{nm} \cos n\theta + b_{nm} \sin n\theta\} \left\{ J_n(\sqrt{\lambda_{nm}}\, r) \right\} \sin c\sqrt{\lambda_{nm}}\, t$$

$$u_t(r, \theta, 0) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \{a_{nm} \cos n\theta + b_{nm} \sin n\theta\} \; J_n(\sqrt{\lambda_{nm}\, r})\, c\sqrt{\lambda_{nm}} \underbrace{\cos c\sqrt{\lambda_{nm}}\, t}_{=1 \text{ at } t=0}$$

Since $u_t(r, \theta, 0) = \beta(r) \cos 5\theta$ all $\sin n\theta$ term should vanish i.e. $b_n = 0$ and all $a_n = 0$ except $a_5$ $(n = 5)$

$$\beta(r) \cos 5\theta = \sum_{m=0}^{\infty} a_{5m} \cos 5\theta\, J_5(\sqrt{\lambda_{5m}\, r})\, c\sqrt{\lambda_{5m}}$$

This is a generalized Fourier series for $\beta(r)$

$$a_{5m}\, c\, \sqrt{\lambda_{5m}} \;=\; \frac{\int_0^a \;\beta\,(r)\, J_5\,(\sqrt{\lambda_{5m}\, r)\, rdr}}{\int_0^a \; J_5^2\,(\sqrt{\lambda_{5m}\, r)\, rdr}}$$

$$u(r,\,\theta,\,t) \;=\; \sum_{m=0}^{\infty} \, a_{5m} \,\cos\, 5\,\theta\, J_5\,(\sqrt{\lambda_{5m}\, r})\,\sin\, c\,\sqrt{\lambda_{5m}}\, t$$

where $\lambda_{5m}$ can be found from

$$\sqrt{\lambda_{5m}}\, J_5'\,(\sqrt{\lambda_{5m}\, a}) \;=\; 0$$

and $a_{5m}$ from

$$a_{5m} \;=\; \frac{\int_0^a \;\beta\,(r)\, J_5\,(\sqrt{\lambda_{5m}\, r)\, rdr}}{c\,\sqrt{\lambda_{5m}}\,\int_0^a \; J_5^2\,(\sqrt{\lambda_{5m}\, r)\, rdr}}$$

5. $u_{tt} - c^2 \nabla^2 u = 0$

$u(a, \theta, t) = 0$

$u(r, 0, t) = u(r, \pi/2, t) = 0$

$\ddot{T} + \lambda c^2 T = 0$

$\Theta'' + \mu \Theta = 0$

$\Theta(0) = \Theta(\pi/2) = 0$

$\Downarrow$

$\mu_n = (2n)^2$

$\Theta_n = \sin 2n\theta$

$n = 1, 2, \cdots$

$r(rR')' + (\lambda r^2 - \mu) R = 0$

$|R(0)| < \infty$

$R(a) = 0$

$\Downarrow$

$R(r) = J_{2n}(\sqrt{\lambda_{2n\,m}}\, r)$

$$\boxed{J_{2n}(\sqrt{\lambda_{2n,\,m}}\, a) = 0 \quad m = 1, 2, \cdots}$$

$$u(r, \theta, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} J_{2n}(\sqrt{\lambda_{2n,\,m}}\, r) \sin 2n\theta \underbrace{\cos c\sqrt{\lambda_{2n,\,m}}\, t}_{since\, u_t(r,\theta,0)=0}$$

$$g(r, \theta) = u(r, \theta, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} J_{2n}(\sqrt{\lambda_{2n,\,m}}\, r) \sin 2n\theta$$

$$a_{mn} = \frac{\int_0^a \int_0^{\pi/2} J_{2n}(\sqrt{\lambda_{2n,\,m}}\, r)\, g(r, \theta) \sin 2n\theta\, r\, d\theta\, dr}{\int_0^{\pi/2} \int_0^a J_{2n}^2(\sqrt{\lambda_{2n,\,m}}\, r) \sin^2 2n\theta\, r\, dr\, d\theta}$$

## 4.6 Laplace's Equation in a Circular Cylinder

### Problems

1. Solve Laplace's equation

$$\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0, \qquad 0 \le r < a, 0 < \theta < 2\pi, 0 < z < H$$

subject to each of the boundary conditions

a.
$$u(r,\theta,0) = \alpha(r,\theta)$$
$$u(r,\theta,H) = u(a,\theta,z) = 0$$

b.
$$u(r,\theta,0) = u(r,\theta,H) = 0$$
$$u_r(a,\theta,z) = \gamma(\theta,z)$$

c.
$$u_z(r,\theta,0) = \alpha(r,\theta)$$
$$u(r,\theta,H) = u(a,\theta,z) = 0$$

d.
$$u(r,\theta,0) = u_z(r,\theta,H) = 0$$
$$u_r(a,\theta,z) = \gamma(z)$$

2. Solve Laplace's equation

$$\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0, \qquad 0 \le r < a, 0 < \theta < \pi, 0 < z < H$$

subject to the boundary conditions

$$u(r,\theta,0) = 0,$$
$$u_z(r,\theta,H) = 0,$$
$$u(r,0,z) = u(r,\pi,z) = 0,$$
$$u(a,\theta,z) = \beta(\theta,z).$$

3. Find the solution to the following steady state heat conduction problem in a box

$$\nabla^2 u = 0, \qquad 0 \le x < L, 0 < y < L, 0 < z < W,$$

subject to the boundary conditions

$$\frac{\partial u}{\partial x} = 0, \qquad x = 0, x = L,$$
$$\frac{\partial u}{\partial y} = 0, \qquad y = 0, y = L,$$
$$u(x, y, W) = 0,$$
$$u(x, y, 0) = 4 \cos \frac{3\pi}{L} x \cos \frac{4\pi}{L} y.$$

4. Find the solution to the following steady state heat conduction problem in a box

$$\nabla^2 u = 0, \qquad 0 \le x < L, 0 < y < L, 0 < z < W,$$

subject to the boundary conditions

$$\frac{\partial u}{\partial x} = 0, \qquad x = 0, x = L,$$
$$\frac{\partial u}{\partial y} = 0, \qquad y = 0, y = L,$$
$$u_z(x, y, W) = 0,$$
$$u_z(x, y, 0) = 4 \cos \frac{3\pi}{L} x \cos \frac{4\pi}{L} y.$$

5. Solve the heat equation inside a cylinder

$$\frac{\partial u}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}, \qquad 0 \le r < a,\ 0 < \theta < 2\pi,\ 0 < z < H$$

subject to the boundary conditions

$$u(r, \theta, 0) = u(r, \theta, H) = 0,$$
$$u(a, \theta, z, t) = 0,$$

and the initial condition

$$u(r, \theta, z, 0) = f(r, \theta, z).$$

1. $\dfrac{1}{r}(ru_r)_r + \dfrac{1}{r^2}u_{\theta\theta} + u_{zz} = 0$

(a)

$$\Theta'' + \mu\,\Theta = 0 \qquad Z'' - \lambda Z = 0 \qquad r(r\,R')' + (\lambda r^2 - \mu)\,R = 0$$

$$\Theta(0) = \Theta(2\pi) \qquad Z(H) = 0 \qquad |R(0)| < \infty$$

$$\Theta'(0) = \Theta'(2\pi) \qquad R(a) = 0$$

$$\Downarrow \qquad \Downarrow$$

$\mu_0 = 0$ $\qquad$ $R_{nm} = J_m(\sqrt{\lambda_{nm}}\,r)$

$\Theta_0 = 1$ $\qquad$ satisfies boundedness

$\mu_m = m^2$

$$\Theta_m = \begin{cases} \sin m\theta \\ \\ \cos m\theta \end{cases}$$

$J_m(\sqrt{\lambda_{nm}}\,a) = 0$

yields eigenvalues

$m = 1, 2, \cdots$ $\qquad \Downarrow \qquad$ $n = 1, 2, \cdots$

$\underline{\lambda > 0!}$

$$Z_{nm} = sinh\sqrt{\lambda_{nm}}\,(z - H)$$

vanishes at $z = H$

$$u(r,\theta,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(a_{nm}\cos m\theta + b_{nm}\sin m\theta)\sinh\sqrt{\lambda_{nm}}\,(z - H)\,J_m(\sqrt{\lambda_{nm}}\,r)$$

$\uparrow$

This is zero for $m = 0$

$$\alpha(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(a_{nm}\cos m\theta + b_{nm}\sin m\theta)\underbrace{\sinh\sqrt{\lambda_{nm}}\,(-H)}_{\text{this is a constant}}\,J_m(\sqrt{\lambda_{nm}}\,r)$$

$$a_{nm} = \frac{\int_0^a\int_0^{2\pi}\alpha(r,\theta)\cos m\theta\,J_m(\sqrt{\lambda_{nm}}\,r)\,rd\theta\,dr}{\sinh\sqrt{\lambda_{nm}}\,(-H)\int_0^a\int_0^{2\pi}\cos^2 m\theta\,J_m^2(\sqrt{\lambda_{nm}}\,r)\,rd\theta\,dr}$$

$$b_{nm} = \frac{\int_0^a\int_0^{2\pi}\alpha(r,\theta)\sin m\theta\,J_m(\sqrt{\lambda_{nm}}\,r)\,rd\theta\,dr}{\sinh\sqrt{\lambda_{nm}}\,(-H)\int_0^a\int_0^{2\pi}\sin^2 m\theta\,J_m^2(\sqrt{\lambda_{nm}}\,r)\,rd\theta\,dr}$$

1 b.

$$\Theta'' + \mu\,\Theta = 0 \qquad Z'' - \lambda\,Z = 0 \qquad r(r\,R')' + (\lambda\,r^2 - \mu)\,R = 0$$

$$Z(0) = Z(H) = 0 \qquad |\,R\,(0)\,| < \infty$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$\mu_m = m^2 \qquad Z_n = \sin\tfrac{n\,\pi}{H}\,z$$

$$\Theta_m = \begin{cases} \sin\,m\,\theta \\ \\ \cos\,m\,\theta \end{cases} \qquad \lambda_n = -\left(\tfrac{n\,\pi}{H}\right)^2$$

$$m = 1,\,2,\,\cdots$$

$$r(r\,R')' + \left(-\left(\frac{n\,\pi}{H}\right)^2\,r^2 - m^2\right)\,R$$

$\uparrow$ extra minus sign

$$\mu_0 = 0$$

$$\Theta_0 = 1$$

$$\Downarrow$$

$$R_{nm} = I_m\,\left(\tfrac{n\,\pi}{H}\,r\right)$$

$$u(r,\theta,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(a_{nm}\,\cos\,m\,\theta + b_{nm}\,\sin\,m\,\theta)\,\sin\,\frac{n\,\pi}{H}\,z\;I_m\,\left(\frac{n\,\pi}{H}\,r\right)$$

$$u_r\,(a,\,\theta,\,z) = \gamma\,(\theta,\,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(a_{nm}\,\cos\,m\,\theta + b_{nm}\,\sin\,m\,\theta)\,\sin\,\frac{n\,\pi}{H}\,z\cdot\underbrace{\frac{n\,\pi}{H}\,I'_m\,\left(\frac{n\,\pi}{H}\,a\right)}_{\text{constant}}$$

$$a_{nm}\,\frac{n\,\pi}{H}\,I'_m\,\left(\frac{n\,\pi}{H}\,a\right) = \frac{\int_0^{2\pi}\int_0^H\,\gamma\,(\theta,\,z)\,\cos\,m\,\theta\,\sin\,\frac{n\pi}{H}\,z\,dz\,d\theta}{\int_0^{2\pi}\int_0^H\,\cos^2\,m\,\theta\,\sin^2\,\frac{n\pi}{H}\,z\,dz\,d\theta}$$

$$a_{nm} = \frac{\int_0^{2\pi}\int_0^H\,\gamma\,(\theta,\,z)\,\cos\,m\,\theta\,\sin\,\frac{n\pi}{H}\,z\,dz\,d\theta}{\frac{n\,\pi}{H}\,I'_m\,\left(\frac{n\,\pi}{H}\,a\right)\int_0^{2\pi}\int_0^H\,\cos^2\,m\,\theta\,\sin^2\,\frac{n\pi}{H}\,z\,dz\,d\theta}$$

$$b_{nm} = \frac{\int_0^{2\pi}\int_0^H\,\gamma\,(\theta,\,z)\,\sin\,m\,\theta\,\sin\,\frac{n\pi}{H}\,z\,dz\,d\theta}{\frac{n\,\pi}{H}\,I'_m\,\left(\frac{n\,\pi}{H}\,a\right)\int_0^{2\pi}\int_0^H\,\sin^2\,m\,\theta\,\sin^2\,\frac{n\pi}{H}\,z\,dz\,d\theta}$$

1 c.

$$\Theta'' + \mu\Theta = 0 \qquad Z'' - \lambda Z = 0 \qquad r(r R')' + (\lambda r^2 - \mu) R = 0$$

$$Z(H) = 0 \qquad |R(0)| < \infty$$

$$R(a) = 0$$

Solution as in 1a exactly !

But

$$u_z(r, \theta, 0) = \alpha(r, \theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (a_{nm} \cos m\theta + b_{nm} \sin m\theta) J_m(\sqrt{\lambda_{nm}}\, r) \sqrt{\lambda_{nm}} \cosh \sqrt{\lambda_{nm}}(-H)$$

$$a_{nm} \sqrt{\lambda_{nm}} \cosh \sqrt{\lambda_{nm}}(-H) = \frac{\int_0^{2\pi} \int_0^a \alpha(r, \theta) \cos m\theta\, J_m(\sqrt{\lambda_{nm}}\, r)\, r d\, r d\theta}{\int_0^{2\pi} \int_0^a \cos^2 m\theta\, J_m^2(\sqrt{\lambda_{nm}}\, r)\, r d\, r d\theta}$$

$$a_{nm} = \frac{\int_0^{2\pi} \int_0^a \alpha(r, \theta) \cos m\theta\, J_m(\sqrt{\lambda_{nm}}\, r)\, r d\, r d\theta}{\sqrt{\lambda_{nm}} \cosh \sqrt{\lambda_{nm}}(-H) \int_0^{2\pi} \int_0^a \cos^2 m\theta\, J_m^2(\sqrt{\lambda_{nm}}\, r)\, r d\, r d\theta}$$

$$b_{nm} = \frac{\int_0^{2\pi} \int_0^a \alpha(r, \theta) \sin m\theta\, J_m(\sqrt{\lambda_{nm}}\, r)\, r d\, r d\theta}{\sqrt{\lambda_{nm}} \cosh \sqrt{\lambda_{nm}}(-H) \int_0^{2\pi} \int_0^a \sin^2 m\theta\, J_m^2(\sqrt{\lambda_{nm}}\, r)\, r d\, r d\theta}$$

1 d.

$\Theta'' + \mu\,\Theta = 0$ $\qquad$ $Z'' - \lambda\,Z = 0$ $\qquad$ $r(r\,R')' + (\lambda\,r^2 - \mu)\,R = 0$

$\Downarrow$ $\qquad$ $Z(0) = Z'(H) = 0$ $\qquad$ $|\,R\,(0)\,| < \infty$

$\Downarrow$ $\qquad$ $\Downarrow$

as before $\qquad$ $\lambda_n = -\left[\left(n - \frac{1}{2}\right)\frac{\pi}{H}\right]^2$

$Z_n = \sin\left(n - \frac{1}{2}\right)\frac{\pi}{H}\,z \qquad R_{nm} = I_m\left(\left(n - \frac{1}{2}\right)\frac{\pi}{H}\,r\right)$

$$u\,(r,\,\theta,\,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(a_{nm}\cos m\,\theta + b_{nm}\sin m\,\theta)\sin\left(n - \frac{1}{2}\right)\frac{\pi}{H}z\,I_m\left(\left(n - \frac{1}{2}\right)\frac{\pi}{H}\,r\right)$$

$$u_r(a,\theta,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(a_{nm}\cos m\,\theta + b_{nm}\sin m\,\theta)\sin\left(n - \frac{1}{2}\right)\frac{\pi}{H}\,z\cdot\left(n - \frac{1}{2}\right)\frac{\pi}{H}\;I'_m\left(\left(n - \frac{1}{2}\right)\frac{\pi}{H}\,a\right)$$

Since $u_r(a,\theta,z) = \gamma(z)$ is independent of $\theta$, we must have no terms with $\theta$ in the above expansion, that is $b_{nm} = 0$ for all $n, m$ and $a_{nm} = 0$ for all $n, m \geq 1$. Thus $a_{1\,0} \neq 0$

$$\gamma(z) = a_{1\,0}\sin\frac{\pi}{2H}z\,\frac{\pi}{2H}\,I'_0\left(\frac{\pi}{2H}a\right)$$

$$a_{1\,0} = \frac{\int_0^H \gamma\,(z)\,\sin\frac{\pi}{2H}\,z\,dz}{\frac{\pi}{2H}\,I'_0\left(\frac{\pi}{2H}a\right)\int_0^H \sin^2\frac{\pi}{2H}\,z\,dz}$$

And the solution is

$$u(r,\theta,z) = a_{1\,0}\sin\frac{\pi}{2H}z\,I_0\left(\frac{\pi}{2H}r\right)$$

2. $u(r,\theta,0) = 0 \quad = u_z(r,\theta,H)$

$u(r,0,z) = 0 \quad = u(r,\pi,z)$

$u(a,\theta,z) \quad = \beta(\theta,z)$

$u = R(r)\,\Theta(\theta)\,Z(z)$

$Z'' + \lambda Z = 0 \qquad \Theta'' + \mu\Theta = 0 \qquad r(r R')' + (\lambda r^2 - \mu) R = 0$

$Z(0) = 0 \qquad \Theta(0) = \Theta(\pi) = 0 \qquad |R(0)| < \infty$

$Z'(H) = 0 \qquad \Downarrow$

$\Downarrow$

$Z_n = \sin\left(n - \frac{1}{2}\right)\frac{\pi}{H}z \qquad \Theta_m = \sin m\theta \qquad r(r R')' - \left\{\left[\left(n - \frac{1}{2}\right)\frac{\pi}{H}\right]^2 r^2 + m^2\right\} R$

$\mu_m = m^2$

$\lambda_n = \left[\left(n - \frac{1}{2}\right)\frac{\pi}{H}\right]^2 \qquad m = 1, 2, \cdots \qquad R(r) = I_m\left(\left(n - \frac{1}{2}\right)\frac{\pi}{H}r\right)$

$n = 1, 2, \cdots$

$$u(r,\theta,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn} I_m\left(\left(n - \frac{1}{2}\right)\frac{\pi}{H}r\right)\sin m\theta \sin\left(n - \frac{1}{2}\right)\frac{\pi}{H}z$$

at $r = a$

$$\beta(\theta,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \underbrace{c_{mn} I_m\left(\left(n - \frac{1}{2}\right)\frac{\pi}{H}a\right)}_{\text{coefficient of expansion}}\sin m\theta \sin\left(n - \frac{1}{2}\right)\frac{\pi}{H}z$$

$$c_{mn} = \frac{\int_0^{\pi}\int_0^{H}\beta(\theta,z)\sin m\theta\sin\left(n - \frac{1}{2}\right)\frac{\pi}{H}z\,dz\,d\theta}{I_m\left(\left(n - \frac{1}{2}\right)\frac{\pi}{H}a\right)\int_0^{\pi}\int_0^{H}\sin^2 m\theta\sin^2\left(n - \frac{1}{2}\right)\frac{\pi}{H}z\,dz\,d\theta}$$

3. $u_{xx} + u_{yy} + u_{zz} = 0$

$$BC: \quad u_x(0, y, z) = 0$$
$$u_x(L, y, z) = 0$$
$$u_y(x, 0, z) = 0$$
$$u_y(x, L, z) = 0$$
$$u(x, y, 0) = 4 \cos \frac{3\pi}{L} x \cos \frac{4\pi}{L} y$$

$$u(x, y, z) = X(x)Y(y)Z(z)$$

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ BC: X'(0) = X'(L) = 0 \end{cases} \Rightarrow \boxed{\begin{array}{l} \lambda_n = \left(\frac{n\pi}{L}\right)^2 \\ X_n = \cos \frac{n\pi}{L} x \qquad n = 0, 1, 2 \cdots \end{array}}$$

$$\frac{Y''}{Y} = \lambda - \frac{Z''}{Z} = -\mu$$

$$\begin{cases} Y'' + \mu Y = 0 \\ BC: Y'(0) = Y'(L) = 0 \end{cases} \Rightarrow \boxed{\begin{array}{l} \mu_n = \left(\frac{m\pi}{L}\right)^2 \\ Y_m = \cos \frac{m\pi}{L} y \quad m = 0, 1, 2 \cdots \end{array}}$$

$$\frac{Z''}{Z} = \lambda + \mu$$

$$\begin{cases} Z'' - (\lambda + \mu) Z = 0 \\ BC: Z(W) = 0 \end{cases} \qquad Z'' - \left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2\right] Z = 0$$

$$Z_{nm} = \sinh \sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2} (z - W)$$

general solution

$$u(x,\,y,\,z) \;=\; \sum_{m=0}^{\infty}\,\sum_{n=0}^{\infty} A_{mn}\,\cos\frac{n\pi}{L}\,x\,\cos\frac{m\pi}{L}\,y\,\sinh\sqrt{\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{L}\right)^2}\,(z-W)$$

$$u(x,\,y,\,0) \;=\; \sum_{m=0}^{\infty}\,\sum_{n=0}^{\infty} -A_{mn}\,\cos\frac{n\pi}{L}\,x\,\cos\frac{m\pi}{L}\,y\,\sinh\sqrt{\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{L}\right)^2}\,W$$

But $u(x,y,0) \;=\; 4\,\cos\frac{3\pi}{L}\,x\,\cos\frac{4\pi}{L}\,y$

Comparing coefficients

$A_{mn} \;=\; 0 \qquad$ for $m \neq 4 \quad$ or $n \neq 3$

For $n \;=\; 3;\; m \;=\; 4 \qquad -\,A_{43}\,\sinh\sqrt{\frac{9\pi^2}{L^2}+\frac{16\pi^2}{L^2}}\,W \;=\; 4$

$$-\,A_{43}\,\sinh\frac{5\pi}{L}\,W \;=\; 4$$

$$A_{43} \;=\; -\,\frac{4}{\sinh\frac{5\pi}{L}\,W}$$

$$u(x,\,y,\,z) \;=\; -\,\frac{4}{\sinh\frac{5\pi}{L}\,W}\,\cos\frac{3\pi}{L}\,x\,\cos\frac{4\pi}{L}\,y\,\sinh\frac{5\pi}{L}\,(z-W)$$

4. $u_{xx} + u_{yy} + u_{zz} = 0$

$$u_x(0,\, y,\, z) = 0 \quad , \quad u_x(L,\, y,\, z) = 0$$

$$u_y(x,\, 0,\, z) = 0 \quad , \quad u_y(x,\, L,\, z) = 0$$

$$u(x,\, y,\, W) = 0$$

$$u(x,\, y,\, 0) = 4\cos\frac{3\pi}{L}x\cos\frac{4\pi}{L}y$$

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ BC:\, X'(0) = X'(L) = 0 \end{cases} \quad \Rightarrow \quad \boxed{\begin{array}{l} \lambda_n = \left(\frac{n\pi}{L}\right)^2 \\ X_n = \cos\frac{n\pi}{L}x \end{array}} \quad n = 0,\, 1,\, 2\cdots$$

$$\begin{cases} Y'' + \mu Y = 0 \\ BC:\, Y'(0) = Y'(L) = 0 \end{cases} \quad \Rightarrow \quad \boxed{\begin{array}{l} \mu_m = \left(\frac{m\pi}{L}\right)^2 \\ Y_m = \cos\frac{m\pi}{L}y \end{array}} \quad m = 0,\, 1,\, 2\cdots$$

$$Z'' - (\lambda + \mu)Z = 0$$

$$Z'(W) = 0$$

$$Z'' - \left[(\frac{n\pi}{L})^2 + (\frac{m\pi}{L})^2\right] Z = 0$$

$$\boxed{Z_{nm} = \cosh\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2}\,(z - W)}$$

$$u(x,\,y,\,z)\;=\;A_{00}\;+\;\sum_{m=1}^{\infty}\;A_{m0}\;\cos\;\frac{m\pi}{L}\,y\;\cosh\;\frac{m\pi}{L}\,(z\;-\;W)$$
$$+\;\sum_{n=1}^{\infty}\;A_{0n}\;\cos\;\frac{n\pi}{L}\,x\;\cos\;\cosh\;\frac{n\pi}{L}\,(z\;-\;W)$$
$$+\;\sum_{m=1}^{\infty}\;\sum_{n=1}^{\infty}\;A_{mn}\;\cos\;\frac{n\pi}{L}\,x\;\cos\;\frac{m\pi}{L}\,y\;\cosh\;\sqrt{\left(\frac{n\pi}{L}\right)^2\;+\;\left(\frac{m\pi}{L}\right)^2}\,(z\;-\;W)$$

$$u_z(x,y,z)\;=\;\sum_{m=1}^{\infty}\;\frac{m\pi}{L}\,A_{m0}\;\cos\;\frac{m\pi}{L}\,y\;\sinh\;\frac{m\pi}{L}\,(z\;-\;W)$$
$$+\;\sum_{n=1}^{\infty}\;\frac{n\pi}{L}\,A_{0n}\;\cos\;\frac{n\pi}{L}\,x\;\sinh\;\frac{n\pi}{L}\,(z\;-\;W)$$
$$+\;\sum_{m=1}^{\infty}\;\sum_{n=1}^{\infty}\;\sqrt{\left(\frac{n\pi}{L}\right)^2\;+\;\left(\frac{m\pi}{L}\right)^2}\,A_{mn}\;\cos\;\frac{n\pi}{L}\,x\;\cos\;\frac{m\pi}{L}\,y\;\sinh\;\sqrt{\left(\frac{n\pi}{L}\right)^2\;+\;\left(\frac{m\pi}{L}\right)^2}\,(z-W)$$

At $z\;=\;0$ $\qquad u_z(x,y,0)\;=\;-\;\sum_{m=1}^{\infty}\;A_{m0}\,\frac{m\pi}{L}\;\cos\;\frac{m\pi}{L}\,y\;\sinh\;\frac{m\pi}{L}\,W$

$$-\;\sum_{n=1}^{\infty}\;A_{0n}\,\frac{n\pi}{L}\;\cos\;\frac{n\pi}{L}\,x\;\sinh\;\frac{n\pi}{L}\,W$$
$$-\;\sum_{m=1}^{\infty}\;\sum_{n=1}^{\infty}\;\sqrt{\left(\frac{n\pi}{L}\right)^2\;+\;\left(\frac{m\pi}{L}\right)^2}\,A_{mn}\;\cos\;\frac{n\pi}{L}\,x\;\cos\;\frac{m\pi}{L}\,y\;\sinh\;\sqrt{\left(\frac{n\pi}{L}\right)^2\;+\;\left(\frac{m\pi}{L}\right)^2}\,W$$

But $u_z(x,y,0)\;=\;4\;\cos\frac{3\pi}{L}\,x\;\cos\;\frac{4\pi}{L}\,y$

Comparing coefficients $A_{mn}\;=\;0$ for $n\;\neq\;3$ or $m\;\neq\;4$

For $n\;=\;3$ and $m\;=\;4$ we have $4\;=\;-\frac{5\pi}{L}\,A_{43}\;\sinh\;\frac{5\pi}{L}\,W$

$$A_{43}\;=\;-\;\frac{4}{\frac{5\pi}{L}\;\sinh\;\frac{5\pi}{L}\,W}$$

Note that $A_{00}$ is NOT specified.

$$\boxed{u(x,\,y,\,z)\;=\;A_{00}\;-\;\frac{4}{\frac{5\pi}{L}\;\sinh\;\frac{5\pi}{L}\,W}\;\cos\;\frac{3\pi}{L}\,x\;\cos\;\frac{4\pi}{L}\,y\;\cosh\;\frac{5\pi}{L}\,(z\;-\;W)}$$

5. $u_t = \frac{1}{r}(r\,u_r)_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}$

BC: $u(r,\theta,0) = 0$

$u(r,\theta,H) = 0$

$u(a,\theta,z) = 0$

$u(r,\theta,z,t) = R(r)\Theta(\theta)Z(z)T(t)$

$$R\Theta ZT' = \frac{1}{r}\Theta\,Z\,T\,(r\,R')' + \frac{1}{r^2}RZT\Theta'' + R\Theta TZ''$$

$$\frac{T'}{T} = \frac{\frac{1}{r}(r\,R')'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z}$$

$$\frac{Z''}{Z} = \frac{T'}{T} - \frac{\frac{1}{r}(r\,R')'}{R} - \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda$$

$$\boxed{\begin{array}{l} Z'' + \lambda Z = 0 \\ BC:\; Z(0) = 0 \\ Z(H) = 0 \end{array}}$$

$\Downarrow$

$$Z_n = \sin\frac{n\pi}{H}z$$

$$\lambda_n = \left(\frac{n\pi}{H}\right)^2 \qquad n = 1, 2, \cdots$$

$$\frac{1}{r^2}\frac{\Theta''}{\Theta} = \frac{T'}{T} - \frac{1}{r}\frac{(r\,R')'}{R} + \left(\frac{n\pi}{H}\right)^2$$

$$\frac{\Theta''}{\Theta} = \frac{T'}{T}r^2 - \frac{r(r\,R')'}{R} + \left(\frac{n\pi}{H}\right)^2 r^2 = -\mu$$

$$\boxed{\begin{array}{l} \Theta'' + \mu\Theta = 0 \\ BC:\; \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{array}}$$

$\Downarrow$

$$\Theta_m = \begin{cases} \sin m\,\theta \\ \\ \cos m\,\theta \end{cases}$$

$$\mu_m = m^2 \qquad m = 0, 1, 2, \cdots$$

$$\frac{T'}{T} r^2 = \frac{r(r\,R')'}{R} - \left(\frac{n\pi}{H}\right)^2 r^2 - m^2$$

$$\frac{T'}{T} = \frac{\frac{1}{r}(r\,R')'}{R} - \left(\frac{n\pi}{H}\right)^2 - \frac{m^2}{r^2} = -\nu$$

$$\boxed{T' + \nu T = 0}$$

$$\frac{\frac{1}{r}(r\,R')'}{R} = -\nu + \left(\frac{n\pi}{H}\right)^2 + \frac{m^2}{r^2}$$

$$\frac{r(r\,R')'}{R} = -\nu r^2 + \left(\frac{n\pi}{H}\right)^2 r^2 + m^2$$

$$\boxed{\begin{array}{l} r(r\,R')' - (\nu - \left(\frac{n\pi}{H}\right)^2) r^2\,R - m^2\,R = 0 \\ \\ BC: \;|\,R\,(0)\,| < \infty \\ \\ R\,(a) = 0 \end{array}}$$

$$R_{nm\ell} = I_m \left(\sqrt{\nu_\ell - \left(\frac{n\pi}{H}\right)^2}\, r\right)$$

This solution satisfies the boundedness at the origin. The eigenvalues $\nu_\ell$ can be found by using the second boundary condition:

$$I_m \left(\sqrt{\nu_\ell - \left(\frac{n\pi}{H}\right)^2}\, a\right) = 0$$

Since the function $I_m(x)$ vanishes only at zero for any $m = 1, 2, \cdots$ ($I_0$ is never zero) then there is only one $\nu$ (for any n) satisfying

$$\sqrt{\nu - \left(\frac{n\pi}{H}\right)^2}\, a = 0 \qquad m = 1, 2, \cdots$$

$$\nu = \left(\frac{n\pi}{H}\right)^2$$

The solution fot $T$ is $T_{nm} = e^{-\left(\frac{n\pi}{H}\right)^2 t}$

The solution for $R$ is $I_m(0 \cdot r)$ which is identically zero. This means that $u(r, \theta, z, t) = 0$. Physically, this is NOT surprising, since the problem has NO sources (homogeneous boundary conditions and homogeneous PDE).

## 4.7 Laplace's equation in a sphere

### Problems

1. Solve Laplace's equation on the sphere

$$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{\cot\theta}{r^2}u_\theta + \frac{1}{r^2\sin^2\theta}u_{\varphi\varphi} = 0, \qquad 0 \le r < a,\ 0 < \theta < \pi,\ 0 < \varphi < 2\pi,$$

subject to the boundary condition

$$u_r(a, \theta, \varphi) = f(\theta).$$

2. Solve Laplace's equation on the half sphere

$$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{\cot\theta}{r^2}u_\theta + \frac{1}{r^2\sin^2\theta}u_{\varphi\varphi} = 0, \qquad 0 \le r < a,\ 0 < \theta < \pi,\ 0 < \varphi < \pi,$$

subject to the boundary conditions

$$u(a, \theta, \varphi) = f(\theta, \varphi),$$

$$u(r, \theta, 0) = u(r, \theta, \pi) = 0.$$

1. $$u\,(r,\,\theta,\,\varphi)\;=\;\sum_{n=0}^{\infty}\,A_{n\,0}\,r^n\,P_n\,(\cos\,\varphi)$$

$$+\;\sum_{n=0}^{\infty}\;\sum_{m=1}^{\infty}\,r^n\,P_n^m\,(\cos\,\varphi)\,(A_{nm}\,\cos\,m\varphi\;+\;B_{nm}\,\sin\,m\,\varphi) \tag{7.7.37}$$

$$u_r\,(a,\,\theta,\,\varphi)\;=\;f(\theta)\;=$$

$$=\;\sum_{n=0}^{\infty}\,n\,A_{n0}\,a^{n-1}\,P_n\,(\cos\,\theta)$$

$$+\;\sum_{n=0}^{\infty}\;\sum_{m=1}^{\infty}\,n\;a^{n-1}\,P_n^m\,(\cos\,\theta)\,(A_{nm}\,\cos\,m\varphi\;+\;B_{nm}\,\sin\,m\,\varphi)$$

$$A_{n0}\,n\,a^{n-1}\;=\;\frac{\int_0^{\pi}\;f(\theta)\,P_n\,(\cos\,\theta)\;\sin\,\theta\,d\theta}{\int_0^{\pi}\;P_n^2\,(\cos\,\theta)\;\sin\,\theta\,d\theta}$$

$$n\,a^{n-1}\,A_{nm}\;=\;\frac{\int_0^{\pi}\;\int_0^{2\pi}\;f(\theta)\,P_n^m\,(\cos\,\theta)\;\cos\,m\,\varphi\;\sin\,\theta\,d\varphi\,d\theta}{\int_0^{\pi}\;\int_0^{2\pi}\,[P_n^m\,(\cos\,\theta)\;\cos\,m\,\varphi]^2\;\sin\,\theta\,d\varphi\,d\theta}$$

$$n\,a^{n-1}\,B_{nm}\;=\;\frac{\int_0^{\pi}\;\int_0^{2\pi}\;f(\theta)\,P_n^m\,(\cos\,\theta)\;\sin\,m\,\varphi\;\sin\,\theta\,d\varphi\,d\theta}{\int_0^{\pi}\;\int_0^{2\pi}\,[P_n^m\,(\cos\,\theta)\;\sin\,m\,\varphi]^2\;\sin\,\theta\,d\varphi\,d\theta}$$

$$A_{nm}\;=\;\frac{\int_0^{\pi}\;\int_0^{2\pi}\;f(\theta)\,P_n^m\,(\cos\,\theta)\;\cos\,m\,\varphi\;\sin\,\theta\,d\varphi\,d\theta}{n\,a^{n-1}\;\int_0^{\pi}\;\int_0^{2\pi}\,[P_n^m\,(\cos\,\theta)\;\cos\,m\,\varphi]^2\;\sin\,\theta\,d\varphi\,d\theta}$$

$$B_{nm}\;=\;\frac{\int_0^{\pi}\;\int_0^{2\pi}\;f(\theta)\,P_n^m\,(\cos\,\theta)\;\sin\,m\,\varphi\;\sin\,\theta\,d\varphi\,d\theta}{n\,a^{n-1}\;\int_0^{\pi}\;\int_0^{2\pi}\,[P_n^m\,(\cos\,\theta)\;\sin\,m\,\varphi]^2\;\sin\,\theta\,d\varphi\,d\theta}$$

2. $\varphi'' + \mu\varphi = 0$

$$\varphi(0) = \varphi(\pi) = 0$$

$$\Downarrow$$

$$\mu_m = m^2$$

$$\varphi_m = \sin m\varphi$$

$$m = 1, 2, \cdots$$

$$\Rightarrow \quad u(r, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} r^n \underbrace{P_n^m(\cos\theta)}_{\Theta \text{ equation}} A_{nm} \sin m\varphi$$

$$\uparrow$$

R equation

$$u(a, \theta, \varphi) = f(\theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} a^n A_{nm} P_n^m(\cos\theta) \sin m\varphi$$

$$a^n A_{nm} = \frac{\int_0^{\pi} \int_0^{\pi} f(\theta, \varphi) P_n^m(\cos\theta) \sin m\varphi \underbrace{\sin\theta \, d\theta \, d\varphi}_{\text{area elem.}}}{\int_0^{\pi} \int_0^{\pi} (P_n^m(\cos\theta))^2 \sin^2 m\varphi \sin\theta \, d\theta \, d\varphi}$$

$$A_{nm} = \frac{\int_0^{\pi} \int_0^{\pi} f(\theta, \varphi) P_n^m(\cos\theta) \sin m\varphi \underbrace{\sin\theta \, d\theta \, d\varphi}_{\text{area elem.}}}{a^n \int_0^{\pi} \int_0^{\pi} (P_n^m(\cos\theta))^2 \sin^2 m\varphi \sin\theta \, d\theta \, d\varphi}$$

$$u(r, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} r^n P_n^m(\cos\theta) A_{nm} \sin m\varphi$$

3. The equation becomes

$$u_{\theta\theta} + \cot\theta u_\theta + \frac{1}{\sin^2\theta} u_{\varphi\varphi} = 0, \quad 0 < \theta < \pi, \; 0 < \varphi < 2\pi$$

Using separation of variables

$$u(\theta, \varphi) = \Theta(\theta)\Phi(\varphi)$$

$$\Theta''\Phi + \cot\theta\Theta'\Phi + \frac{1}{\sin^2\theta}\Theta\Phi'' = 0$$

Divide by $\Phi\Theta$ and multiply by $\sin^2\theta$ we have

$$\sin^2\theta\frac{\Theta''}{\Theta} + \cos\theta\sin\theta\frac{\Theta'}{\Theta} = -\frac{\Phi''}{\Phi} = \mu$$

Thus

$$\Phi'' + \mu\Phi = 0$$

$$\sin^2\theta\Theta'' + \sin\theta\cos\theta\Theta' - \mu\Theta = 0$$

Because of periodicity, the $\Phi$ equation has solutions

$$\Phi_m = \begin{cases} \sin \quad m\varphi & \qquad m = 1, 2, \ldots \\ \\ \cos \quad m\varphi & \end{cases}$$

$$\Phi_0 = 1$$

$$\mu_m = m^2 \qquad m = 0, 1, 2, \ldots$$

Substituting these $\mu's$ in the $\Theta$ equation, we get (7.7.21) with $\alpha_1 = 0$. The solution of the $\Theta$ equation is thus given by (7.7.27) - (7.7.28) with $\alpha_1 = 0$.

# 5 Separation of Variables-Nonhomogeneous Problems

## 5.1 Inhomogeneous Boundary Conditions

### Problems

1. For each of the following problems obtain the function $w(x,t)$ that satisfies the boundary conditions and obtain the PDE

a.

$$u_t(x,t) = ku_{xx}(x,t) + x, \qquad 0 < x < L$$
$$u_x(0,t) = 1,$$
$$u(L,t) = t.$$

b.

$$u_t(x,t) = ku_{xx}(x,t) + x, \qquad 0 < x < L$$
$$u(0,t) = 1,$$
$$u_x(L,t) = 1.$$

c.

$$u_t(x,t) = ku_{xx}(x,t) + x, \qquad 0 < x < L$$
$$u_x(0,t) = t,$$
$$u_x(L,t) = t^2.$$

2. Same as problem 1 for the wave equation

$$u_{tt} - c^2 u_{xx} = xt, \qquad 0 < x < L$$

subject to each of the boundary conditions

a.

$$u(0,t) = 1 \qquad u(L,t) = t$$

b.

$$u_x(0,t) = t \qquad u_x(L,t) = t^2$$

c.

$$u(0,t) = 0 \qquad u_x(L,t) = t$$

d.

$$u_x(0,t) = 0 \qquad u_x(L,t) = 1$$

1a. $u_x(0, t) = 1$

$u(L, t) = t$

$w(x, t) = A(t)x + B(t)$

$1 = w_x(0, t) = A(t) \quad \Rightarrow \quad A(t) = 1$

$t = w(L, t) = A(t)L + B(t) \quad \Rightarrow \quad B(t) = t - L$

$\Rightarrow \quad \underline{w(x, t) = x + t - L}$

$w_t = 1$
$w_{xx} = 0$
$v = u - w \quad \Rightarrow \quad u = v + w$

$\Rightarrow \quad \underline{v_t + 1 = kv_{xx} + x}$

b. $w = Ax + B$
$1 = w(0, t) = B(t)$
$1 = w_x(L, t) = A(t)$
$\underline{w = x + 1}$
$w_t = w_{xx} = 0$

$\underline{v_t = kv_{xx} + x}$

c. $w_x(0,\, t) \,=\, t$

$w_x(L,\, t) \,=\, t^2$

try $\quad w \,=\, A(t)\, x \,+\, B$

$w_x \,=\, A(t)$ and we can <u>not</u> satisfy the 2 conditions.

try $\quad w \,=\, A(t)\, x^2 \,+\, B(t)\, x$

$w_x \,=\, 2A(t)\, x \,+\, B(t)$

$t \,=\, w_x\,(0,\, t) \,=\, B(t)$

$t^2 \,=\, w_x(L,\, t) \,=\, 2A(t)L \,+\, \underbrace{B(t)}_{=t} \quad \Rightarrow \quad A(t) \,=\, \frac{t^2 - t}{2L}$

$$\underline{w \,=\, \frac{t^2 \,-\, t}{2L}\, x^2 \,+\, tx}$$

$$w_t \,=\, \frac{2t \,-\, 1}{2L}\, x^2 \,+\, x$$

$$w_{xx} \,=\, \frac{t^2 \,-\, t}{L}$$

$$v_t \,+\, \left(\tfrac{2t-1}{2L}\, x^2 \,+\, x\right) \,=\, k\,(v_{xx} \,+\, \tfrac{t^2-t}{L}) \,+\, x$$

$$v_t \,=\, kv_{xx} \,-\, \frac{2t \,-\, 1}{2L}\, x^2 \,-\, x \,+\, k\,\frac{t^2 \,-\, t}{L} \,+\, x$$

$$\underline{v_t \,=\, kv_{xx} \,-\, \frac{2t \,-\, 1}{2L}\, x^2 \,+\, k\,\frac{t^2 \,-\, t}{L}}$$

2. $u_{tt} - c^2 u_{xx} = xt$

a. $w(x, t) = \frac{t-1}{L} x + 1$

$w_{tt} = 0 \qquad w_{xx} = 0$

$\underline{v_{tt} - c^2 v_{xx} = xt}$

b. $w = \frac{t^2 - t}{2L} x^2 + tx$ as in 1c

$w_{tt} = \frac{1}{L} x^2$

$w_{xx} = \frac{t^2 - t}{L}$

$u = v + w$

$v_{tt} + \frac{1}{L} x^2 - c^2 \left( v_{xx} + \frac{t^2 - t}{L} \right) = xt$

$\underline{v_{tt} - c^2 v_{xx} = -\frac{1}{L} x^2 + c^2 \frac{t^2 - t}{L} + xt}$

c. $w(0, t) = 0 \qquad w_x(L, t) = t$

$w = Ax + B \qquad w_x = A$

$B = 0 \qquad A = t$

$\underline{w = tx}$

$w_{tt} = w_{xx} = 0$

$\underline{v_{tt} - c^2 v_{xx} = xt}$

d. $w_x(0,\, t) \,=\, 0$

$w_x(L,\, t) \,=\, 1$

Try $\quad w \,=\, A(t)\, x^2 \,+\, B(t)\, x \quad$ as in 1c

$$w_x \,=\, 2A(t)x \,+\, B(t)$$

$$0 \,=\, B(t) \qquad 1 \,=\, 2A(t)L \,+\, \underbrace{B(t)}_{=0}$$

$$A(t) \,=\, \frac{1}{2L}$$

$$\underline{w \,=\, \frac{1}{2L}\, x^2}$$

$$w_{tt} \,=\, 0 \qquad w_{xx} \,=\, \tfrac{1}{L}$$

$$v \,=\, u \,+\, w$$

$$v_{tt} \,-\, c^2 \left(v_{xx} \,+\, \tfrac{1}{L}\right) \,=\, x\, t$$

$$\underline{v_{tt} \,-\, c^2\, v_{xx} \,=\, +\frac{c^2}{L} \,+\, x\, t}$$

## 5.2 Method of Eigenfunction Expansions

### Problems

1. Solve the heat equation

$$u_t = ku_{xx} + x, \qquad 0 < x < L$$

subject to the initial condition

$$u(x, 0) = x(L - x)$$

and each of the boundary conditions

a.

$$u_x(0, t) = 1,$$
$$u(L, t) = t.$$

b.

$$u(0, t) = 1,$$
$$u_x(L, t) = 1.$$

c.

$$u_x(0, t) = t,$$
$$u_x(L, t) = t^2.$$

2. Solve the heat equation

$$u_t = u_{xx} + e^{-t}, \qquad 0 < x < \pi, \quad t > 0,$$

subject to the initial condition

$$u(x, 0) = \cos 2x, \qquad 0 < x < \pi,$$

and the boundary condition

$$u_x(0, t) = u_x(\pi, t) = 0.$$

1. $u_t = k u_{xx} + x$

$u(x,0) = x(L-x)$

a. $u_x(0,t) = 1$

$$\Rightarrow \quad w = x + t - L$$

$u(L,t) = t$

Solve $\quad v_t = k v_{xx} - 1 + x \qquad$ (see 1a last section)

$v_x(0,t) = 0 \qquad v(x,0) = x(L-x) - (x-L) = (x+1)(t-x)$

$v(L,t) = 0$

eigenvalues: $\qquad \left[\left(n - \frac{1}{2}\right)\frac{\pi}{L}\right]^2 \qquad n = 1, 2, \cdots$

eigenfunctions : $\qquad \cos\left(n - \frac{1}{2}\right)\frac{\pi}{L}x \qquad n = 1, 2, \cdots$

$$\boxed{v - \sum_{n=1}^{\infty} v_n(t) \cos\left(n - \frac{1}{2}\right)\frac{\pi}{L}x}$$

$$-1 + x = \sum_{n=1}^{\infty} s_n \cos\left(n - \frac{1}{2}\right)\frac{\pi}{L}x \quad \Rightarrow \quad \boxed{s_n = \frac{\int_0^L (-1+x)\cos\left(n-\frac{1}{2}\right)\frac{\pi}{L}x\,dx}{\int_0^L \cos^2\left(n-\frac{1}{2}\right)\frac{\pi}{L}x\,dx}}$$

$$\sum_{n=1}^{\infty} \dot{v}_n(t)\cos\left(n-\frac{1}{2}\right)\frac{\pi}{L}x = k\sum_{n=1}^{\infty}\left\{-\left[\left(n-\frac{1}{2}\right)\frac{\pi}{L}\right]^2\right\} v_n \cos\left(n-\frac{1}{2}\right)\frac{\pi}{L}x$$

$$+ \sum_{n=1}^{\infty} s_n \cos\left(n-\frac{1}{2}\right)\frac{\pi}{L}x$$

Compare coefficients

$\dot{v}_n(t) + k\left(\left(n-\frac{1}{2}\right)\frac{\pi}{L}\right)^2 v_n = s_n$

$$\boxed{v_n = v_n(0)e^{-[\left(n-\frac{1}{2}\right)\frac{\pi}{L}]^2 kt} + s_n \underbrace{\int_0^t e^{-[\left(n-\frac{1}{2}\right)\frac{\pi}{L}]^2 k(t-\tau)}\,d\tau}_{\text{see}(8.\ 2.\ 39)}}$$

$v_n(0) =$ coefficients of expansion of $(1+x)(L-x)$

$$\boxed{v_n(0) = \frac{\int_0^L (1+x)(L-x)\cos\left(n-\frac{1}{2}\right)\frac{\pi}{L}x\,dx}{\int_0^L \cos^2\left(n-\frac{1}{2}\right)\frac{\pi}{L}x\,dx}} \qquad \boxed{u = v + w}$$

1b. $u(0,\, t) \,=\, u_x\,(L,\, t) \,=\, 1$

$\Rightarrow w \,=\, x \,+\, 1$

$v_t \,=\, k\, v_{xx} \,+\, x$

$v\,(0,\, t) \,=\, 0$

$v_x\,(L,\, t) \,=\, 1$

$v\,(x,\, 0) \,=\, x\,(L \,-\, x) \,-\, (x \,+\, 1)$

eigenvalues: $\left[\left(n \,-\, \frac{1}{2}\right)\, \frac{\pi}{L}\right]^2 \qquad n \,=\, 1,\, 2,\, \cdots$

eigenfunctions: $\sin\, \left(n \,-\, \frac{1}{2}\right)\, \frac{\pi}{L}\, x \qquad n \,=\, 1,\, 2,\, \cdots$

$$v \,=\, \sum_{n=1}^{\infty} v_n(t)\, \sin\, \left(n \,-\, \frac{1}{2}\right)\, \frac{\pi}{L}\, x$$

$$x \,=\, \sum_{n=1}^{\infty} s_n\, \sin\, \left(n \,-\, \frac{1}{2}\right)\, \frac{\pi}{L}\, x \qquad s_n \,=\, \frac{\int_0^L\, x\, \sin\, \left(n \,-\, \frac{1}{2}\right)\, \frac{\pi}{L}\, x\, dx}{\int_0^L\, \sin^2\, \left(n \,-\, \frac{1}{2}\right)\, \frac{\pi}{L}\, x\, dx}$$

$$\dot{v}_n \,+\, k\, \left[\left(n \,-\, \frac{1}{2}\right)\, \frac{\pi}{L}\right]^2\, v_n \,=\, s_n$$

$$v_n(t) \,=\, v_n(0)\, e^{-[\left(n - \frac{1}{2}\right)\, \frac{\pi}{L}]^2\, kt} \,+\, s_n\, \frac{1 - e^{-[\left(n - \frac{1}{2}\right)\, \frac{\pi}{L}]^2 t}}{[\left(n - \frac{1}{2}\right)\, \frac{\pi}{L}]^2}$$

$$v_n(0) \,=\, \frac{\int_0^L [x\,(L \,-\, x) \,-\, (x \,+\, 1)]\, \sin\, \left(n \,-\, \frac{1}{2}\right)\, \frac{\pi}{L}\, x\, dx}{\int_0^L\, \sin^2\, \left(n \,-\, \frac{1}{2}\right)\, \frac{\pi}{L}\, x\, dx}$$

↑

Coefficients of expansion of initial condition for $v$

$\underline{u \,=\, v \,+\, w}$

1c. $u_x(0,\, t) \,=\, t$

$u_x(L,\, t) \,=\, t^2$

$w \,=\, \frac{t^2 - t}{2L}\, x^2 \,+\, tx$

$$v_t \,=\, k v_{xx} \,-\, \underbrace{\frac{2t \,-\, 1}{2L}\, x^2 \,-\, x \,+\, k\, \frac{t^2 \,-\, t}{L} \,+\, x}_{\text{this gives } s_n(t)}$$

$$v_x(0,\, t) \,=\, v_x(L,\, t) \,=\, 0 \qquad \Rightarrow \qquad \lambda_n \,=\, \left(\frac{n\,\pi}{L}\right)^2 \quad n \,=\, 0,\, 1,\, 2,\, \cdots$$

$$X_n(x) \,=\, \cos\, \frac{n\,\pi}{L}\, x \quad n \,=\, 0,\, 1,\, 2,\, \cdots$$

$$v(x,\, 0) \,=\, x\,(L \,-\, x) \,-\, \underbrace{w(x,\, 0)}_{=\,0} \,=\, x\,(L \,-\, x)$$

$$s_n(t) \,=\, \frac{\int_0^L \left\{ -\frac{2t-1}{2L}\, x^2 \,+\, k\, \frac{t^2 - t}{L} \right\} \cos\, \frac{n\,\pi}{L}\, x\, dx}{\int_0^L \cos^2\, \frac{n\,\pi}{L}\, x\, dx}$$

$$v_n(t) \,=\, v_n(0)\, e^{-k\left(\frac{n\,\pi}{L}\right)^2 t} \,+\, \int_0^t s_n(\tau)\, e^{-k\left(\frac{n\,\pi}{L}\right)^2 (t-\tau)}\, d\tau$$

$$v(x,\, t) \,=\, \frac{1}{2} v_0(t) \,+\, \sum_{n=1}^{\infty} v_n(t)\, \cos\, \frac{n\,\pi}{L}\, x$$

$$v_n(0) \,=\, \frac{\int_0^L x\,(L \,-\, x)\, \cos\, \frac{n\,\pi}{L}\, x\, dx}{\int_0^L \cos^2\, \frac{n\,\pi}{L}\, x\, dx}$$

$$u \,=\, v \,+\, \frac{t^2 \,-\, t}{2L}\, x^2 \,+\, tx$$

2. $u_t = u_{xx} + e^{-t} \qquad 0 < x < \pi , \quad t > 0$

$u(x, 0) = \cos 2x \qquad 0 < x < \pi$

$u_x(0, t) = u_x(\pi, t) = 0$

Since the boundary conditions are homogeneous we can immediately expand $u(x, t)$, the right hand side and the initial temperature distribution in terms of the eigenfunctions. These eigenfunctions are

$$\phi_n = \cos nx$$

$$\lambda = n^2$$

$$n = 0, 1, 2, \ldots$$

$$u(x, t) = \frac{1}{2}u_0(t) + \sum_{n=1}^{\infty} u_n(t) \cos nx$$

$$u(x, 0) = \frac{1}{2}u_0(0) + \sum_{n=1}^{\infty} u_n(0) \cos nx = \cos 2x$$

$\vdots$

by initial condition

$$\Rightarrow \begin{cases} u_n(0) = 0 & n \neq 2 \\ u_2(0) = 1 & \end{cases}$$

$$e^{-t} = \sum_{n=1}^{\infty} s_n(t) \cos nx + \frac{1}{2} s_0(t)$$

$$s_n(t) = \frac{\int_0^\pi e^{-t} \cos nx \, dx}{\int_0^\pi \cos^2 nx \, dx} = \frac{e^{-t} \int_0^\pi \cos nx \, dx}{\int_0^\pi \cos^2 nx \, dx}$$

for $n \neq 0$ the numerator is zero !!

For $n = 0$ both integrals yields the same value, thus

$$s_0(t) = e^{-t}$$

$$s_n(t) = 0 , \quad n \neq 0$$

Now substitute $u_t$, $u_{xx}$ from the expansions for u:

$$\frac{1}{2}\dot{u}_0(t) + \sum_{n=1}^{\infty} \dot{u}_n(t) \cos nx = \sum_{n=1}^{\infty} (-n^2) u_n(t) \cos nx + \frac{1}{2}s_0(t) + \sum_{n=1}^{\infty} s_n(t) \cos nx$$

For $n = 0$ $\quad \frac{1}{2}\,\dot{u}_0(t) = \frac{1}{2}\,e^{-t} = \frac{1}{2}\,s_0(t)$

$n \neq 0$ $\quad \dot{u}_n + n^2\,u_n = 0$

Solve the ODES

$$u_n = C_n\,e^{-n^2 t} \qquad \begin{array}{ll} u_n(0) = 0 \quad n \neq 2 & \Rightarrow \quad C_n = 0 \\ u_2(0) = 1 & \Rightarrow \quad C_2 = 1 \end{array}$$

$$u_0 = -\,e^{-t} + C_0 \qquad u_0(0) = 0 \quad \Rightarrow \quad C_0 - 1 = 0 \quad \Rightarrow \quad C_0 = 1$$

$$\boxed{u(x,\,t) = 1 - e^{-t} + e^{-4t}\cos 2x}$$

## 5.3 Forced Vibrations

### Problems

1. Consider a vibrating string with time dependent forcing

$$u_{tt} - c^2 u_{xx} = S(x,t), \qquad 0 < x < L$$

subject to the initial conditions

$$u(x,0) = f(x),$$
$$u_t(x,0) = 0,$$

and the boundary conditions

$$u(0,t) = u(L,t) = 0.$$

a. Solve the initial value problem.

b. Solve the initial value problem if $S(x,t) = \cos \omega t$. For what values of $\omega$ does resonance occur?

2. Consider the following damped wave equation

$$u_{tt} - c^2 u_{xx} + \beta u_t = \cos \omega t, \qquad 0 < x < \pi,$$

subject to the initial conditions

$$u(x,0) = f(x),$$
$$u_t(x,0) = 0,$$

and the boundary conditions

$$u(0,t) = u(\pi,t) = 0.$$

Solve the problem if $\beta$ is small ($0 < \beta < 2c$).

3. Solve the following

$$u_{tt} - c^2 u_{xx} = S(x,t), \qquad 0 < x < L$$

subject to the initial conditions

$$u(x,0) = f(x),$$
$$u_t(x,0) = 0,$$

and each of the following boundary conditions

a.

$$u(0,t) = A(t) \qquad u(L,t) = B(t)$$

b.

$$u(0,t) = 0 \qquad u_x(L,t) = 0$$

c.

$$u_x(0,t) = A(t) \qquad u(L,t) = 0.$$

4. Solve the wave equation

$$u_{tt} - c^2 u_{xx} = xt, \qquad 0 < x < L,$$

subject to the initial conditions

$$u(x,0) = \sin x$$
$$u_t(x,0) = 0$$

and each of the boundary conditions

a.

$$u(0,t) = 1,$$
$$u(L,t) = t.$$

b.

$$u_x(0,t) = t,$$
$$u_x(L,t) = t^2.$$

c.

$$u(0,t) = 0,$$
$$u_x(L,t) = t.$$

d.

$$u_x(0,t) = 0,$$
$$u_x(L,t) = 1.$$

5. Solve the wave equation

$$u_{tt} - u_{xx} = 1, \qquad 0 < x < L,$$

subject to the initial conditions

$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

and the boundary conditions

$$u(0,t) = 1,$$
$$u_x(L,t) = B(t).$$

1a. $u_{tt} - c^2 u_{xx} = S(x, t)$

$$u(x, 0) = f(x)$$

$$u_t(x, 0) = 0$$

$$u(0, t) = u(L, t) = 0$$

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t)\, \phi_n(x)$$

$$S(x, t) = \sum_{n=1}^{\infty} s_n(t)\, \phi_n(x)$$

$$\left.\begin{array}{lcl} \phi_n(x) & = & \sin \frac{n\pi}{L} x \\ \lambda_n & = & \left(\frac{n\pi}{L}\right)^2 \end{array}\right\} \qquad n = 1, 2, \cdots$$

$$s_n(t) = \frac{\int_0^L S(x, t) \sin \frac{n\pi}{L} x\, dx}{\int_0^L \sin^2 \frac{n\pi}{L} x\, dx}$$

$$\ddot{u}_n(t) + c^2 \left(\frac{n\pi}{L}\right)^2 u_n(t) = s_n(t)$$

$$u_n(t) = c_1 \cos \frac{n\pi}{L} ct + c_2 \sin \frac{n\pi}{L} ct + \int_0^t s_n(\tau) \frac{\sin c \frac{n\pi}{L} (t - \tau)}{c \frac{n\pi}{L}} d\tau$$

$$u_n(0) = c_1 = \frac{\int_0^L f(x) \sin \frac{n\pi}{L} x\, dx}{\int_0^L \sin^2 \frac{n\pi}{L} x\, dx} \text{ since } u(x, 0) = f(x)$$

$$\dot{u}_n(0) = c_2\, c\, \frac{n\pi}{L} = 0 \quad \text{since} \quad u_t(x, 0) = 0 \quad \Rightarrow \quad c_2 = 0$$

$$u(x, t) = \sum_{n=1}^{\infty} \left\{ c_1 \cos \frac{n\pi}{L} ct + \frac{L}{c\, n\pi} \int_0^t s_n(\tau) \sin c \frac{n\pi}{L} (t - \tau)\, d\tau \right\} \sin \frac{n\pi}{L} x$$

$c_1$ is given above.

b. If $S = \cos wt$

$$s_n(t) = \frac{\cos wt \int_0^L \sin \frac{n\pi}{L} x\, dx}{\int_0^L \sin^2 \frac{n\pi}{L} x\, dx} = A_n \cos wt$$

$$\text{where } A_n = \frac{\int_0^L \sin \frac{n\pi}{L} x\, dx}{\int_0^L \sin^2 \frac{n\pi}{L} x\, dx}$$

$$u(x, t) = \sum_{n=1}^{\infty} \left\{ c_1 \cos \frac{n\pi c}{L} t + \frac{L}{c\, n\, \pi} A_n \underbrace{\int_0^t \cos w\tau \sin \frac{n\pi c}{L} (t - \tau)\, d\tau}_{\text{This integral can be computed}} \right\} \sin \frac{n\pi}{L} x$$

c. Resonance occurs when

$$w = c \frac{n\pi}{L} \text{ for any } n = 1, 2, \cdots$$

2. $u_{tt} - c^2 u_{xx} + \beta u_t = \cos w t$

$$u(x,\,0) = f(x)$$

$$u_t(x,\,0) = 0$$

$$u(0,\,t) = u\,(\pi,\,t) = 0 \quad \Rightarrow \quad \phi_n = \sin nx$$

$$n = 1,\, 2,\, \cdots$$

$$\lambda_n = n^2$$

$$u(x,\,t) = \sum_{n=1}^{\infty} u_n(t)\,\sin nx$$

$$\cos w t = \sum_{n=1}^{\infty} s_n(t)\,\sin nx$$

$$s_n\,(t) = \frac{\cos w t \int_0^{\pi} \sin nx\,dx}{\int_0^{\pi} \sin^2 nx\,dx} = A_n \cos w t$$

$$\sum_{n=1}^{\infty} (\ddot{u}_n + c^2 n^2 u_n + \beta \dot{u}_n)\,\sin nx = \sum_{n=1}^{\infty} s_n(t)\,\sin nx$$

(*) $\ddot{u}_n + \beta \dot{u}_n + c^2 n^2 u_n = s_n(t) = A_n \cos wt$

For the homogeneous:

Let $u_n = e^{\mu t}$ $\quad (\mu^2 + \beta\,\mu + c^2 n^2) = 0 \quad \mu = \frac{-\beta \pm \sqrt{\beta^2 - 4c^2 n^2}}{2}$

For $\beta < 2c$, $\beta^2 - 4c^2 n^2 < 0 \quad \Rightarrow$ complex conjugate roots

$$u_n = \left(c_1 \cos \frac{\sqrt{4c^2 n^2 - \beta^2}}{2}\,t + c_2 \sin \frac{\sqrt{4c^2 n^2 - \beta^2}}{2}\right) e^{-(\beta/2)\,t}$$

↑
Solution for homogeneous.

Because of damping factor $e^{-(\beta/2)\,t}$ there should not be a problem of resonance. We must find a particular solution for inhomogeneous.

$$u_n^P = B_n \cos w t + C_n \sin w t$$

$$\dot{u}_n = -B_n w \sin w t + C_n w \cos w t$$

$$\ddot{u}_n = -B_n w^2 \cos w t - C_n w^2 \sin w t$$

Substitute in (*) and compare coefficients of cos $w\,t$

$-B_n\,w^2 \,+\, \beta\,C_n\,w \,+\, c^2\,n^2\,B_n \,=\, A_n$
Compare coefficients of sin $w\,t$

$-C_n\,w^2 \,-\, B_n\,w \,+\, C_n \,=\, 0$

$$\downarrow$$

$$B_n \,=\, \frac{C_n\,(1\,-\,w^2)}{w}$$

$\frac{C_n\,(1-w^2)}{w}\,(c^2\,n^2\,-\,w^2) \,+\, \beta\,C_n\,w \,=\, A_n$

$$C_n\left\{\underbrace{\beta\,w \,+\, \frac{c^2\,n^2\,-\,w^2}{w}\,(1\,-\,w^2)}_{=D_n}\right\} \,=\, A_n$$

$C_n \,=\, \frac{A_n}{D_n}$

$B_n \,=\, \frac{A_n\,(1-w^2)}{D_n\,w}$

$\boxed{u_n^P \,=\, \frac{A_n}{D_n}\,\frac{(1-w^2)}{w}\,\cos\,w\,t \,+\, \frac{A_n}{D_n}\,\sin\,w\,t}$ where

$$D_n \,=\, \beta\,w \,+\, \frac{c^2\,n^2\,-\,w^2}{w}\,(1\,-\,w^2) \qquad A_n \,=\, \frac{\int_0^\pi \sin\,n\,x\,dx}{\int_0^\pi \sin^2\,nx\,dx}$$

Therefore the general solution of the inhomogeneous is

$$\boxed{u_n \,=\, \left(c_1\,\cos\,\frac{\sqrt{4c^2\,n^2-\beta^2}}{2}\,t \,+\, c_2\,\sin\,\frac{\sqrt{4c^2\,n^2-\beta^2}}{2}\,t\right)\,e^{-(\beta/2)\,t)} \,+\, \frac{A_n}{D_n}\,\frac{1-w^2}{w}\,\cos\,w\,t \,+\, \frac{A_n}{D_n}\,\sin\,w\,t}$$

(**) $$\boxed{u(x,\,t) \,=\, \sum_{n=1}^{\infty}\,u_n(t)\,\sin\,nx}$$

$$u_t(x,\,t) \,=\, \sum_{n=1}^{\infty}\,\dot{u}_n(t)\,\sin\,nx$$

$$\dot{u}_n(t) \,=\, (c_1\,\cos\,rt \,+\, c_2\,\sin\,rt)\,\left(-\frac{\beta}{2}\right)\,e^{-\frac{\beta}{2}\,t} \,+\, (-rc_1\,\sin\,rt \,+\, rc_2\,\cos\,rt)\,e^{-\beta/2\,t}$$
$$-\,\frac{A_n}{D_n}\,(1\,-\,w^2)\,\sin\,w\,t \,+\, \frac{A_n}{D_n}\,w\,\cos\,w\,t$$

where $r = \frac{\sqrt{4c^2\,n^2-\beta^2}}{2}$

$$u_t(x,\,0) \;=\; 0 \;\Rightarrow \quad \sum_{n=1}^{\infty} \left\{ c_1 \left( -\frac{\beta}{2} \right) \,+\, r c_2 \,+\, \frac{A_n}{D_n}\, w \right\} \sin\, nx \;=\; 0$$

$$\Rightarrow \quad \boxed{-c_1 \tfrac{\beta}{2} \,+\, r c_2 \,+\, \tfrac{A_n}{D_n}\, w \;=\; 0} \qquad (\#)$$

$u(x,\,0) \;=\; f(x) \;\Rightarrow \quad u_n(0)$ are Fourier coefficients of f(x)

$$u_n(0) \;=\; \left( c_1 \,+\, \frac{A_n}{D_n}\, \frac{1 \,-\, w^2}{w} \right) \;=\; \frac{\int_0^{\pi}\, f(x)\, \sin\, nx\, dx}{\int_0^{\pi}\, \sin^2\, nx\, dx} \quad \Rightarrow \quad \text{we have } c_1$$

Use $c_1$ in (#) to get $c_2$ and the solution is in (**) with $u_n$ at top of page.

3. $u_{tt} - c^2 u_{xx} = S(x\,t)$

$$u(x,\,0) = f(x)$$

$$u_t(x,\,0) = 0$$

a. $u(0,\,t) = A(t)$

$$u(L,\,t) = B(t) \quad \Rightarrow \quad w = \alpha\, x + \beta$$

$$\beta = A(t)$$

$$\alpha L + \beta = B$$

$$\alpha = \tfrac{B - \beta}{L}$$

$$\boxed{w = \tfrac{B(t) - A(t)}{L}\, x + A(t)}$$

$$w_{xx} = 0 \qquad w_{tt} = \tfrac{\ddot{B} - \ddot{A}}{L}\, x + \ddot{A}$$

$$v = u - w$$

$$u = v + w$$

$$v_{tt} - c^2 v_{xx} = S(x,\,t) - w_{tt} \equiv \hat{S}(x,\,t)$$

$$v(x,\,0) = f(x) - \tfrac{B(0) - A(0)}{L}\, x - A(0) \equiv F(x)$$

$$v_t(x,\,0) = 0 - w_t\,(x,\,0) = 0 - \tfrac{\dot{B}(0) - \dot{A}(0)}{L}\, x - \dot{A}(0) \equiv G(x)$$

$$v(0,\,t) = 0$$

$$v(L,\,t) = 0$$

Solve the homogeneous

$$\lambda_n = \left(\tfrac{n\,\pi}{L}\right)^2$$

$$n = 1,\, 2,\, \cdots$$

$$\phi_n = \sin\, \tfrac{n\,\pi}{L}\, x$$

$$\boxed{v(x,\,t) = \sum_{n=1}^{\infty} v_n(t)\, \sin\, \frac{n\,\pi}{L}\, x}$$

$$\hat{S}(x,\,t) = \sum_{n=1}^{\infty} s_n(t)\, \sin\, \frac{n\,\pi}{L}\, x \quad , \quad \boxed{s_n(t) = \frac{\int_0^L \hat{S}(x,\,t)\, \sin\, \frac{n\,\pi}{L}\, x\, dx}{\int_0^L \sin^2\, \frac{n\,\pi}{L}\, x\, dx}}$$

$$\sum_{n=1}^{\infty} \left(\ddot{v}_n + c^2 \left(\frac{n\,\pi}{L}\right)^2 v_n\right) \sin\, \frac{n\,\pi}{L}\, x = \sum_{n=1}^{\infty} s_n\, \sin\, \frac{n\,\pi}{L}\, x$$

$\ddot{v}_n + \left(\frac{c\,n\,\pi}{L}\right)^2 v_n = s_n$

$v_n(0)$ coefficient of expanding $F(x)$

$\dot{v}_n(0)$ coefficient of expanding $G(x)$

$$\boxed{v_n = \underbrace{c_1}_{\downarrow} \cos \frac{c\,n\,\pi}{L} t + \underbrace{c_2}_{\downarrow} \sin \frac{c\,n\,\pi}{L} t + \int_0^t s_n(\tau) \frac{\sin \frac{c\,n\,\pi}{L}(t-\tau)}{\frac{c\,n\,\pi}{L}} d\tau}$$

$$v_n(0) \qquad \frac{\dot{v}_n(0)}{\frac{c\,n\,\pi}{L}} \qquad \text{(see 8.3.12-13)}$$

$$\boxed{u = v + w}$$

b. $u(0,\, t) \,=\, 0$

$u_x(L,\, t) \,=\, 0 \quad \Rightarrow \qquad$ Homogeneous. b.c.

$\lambda_n \,=\, \left(\left(n \,-\, \frac{1}{2}\right) \,\frac{\pi}{L}\right)^2$

$\phi_n \,=\, \sin \,\frac{(n-\frac{1}{2})\,\pi}{L}\, x$

$n \,=\, 1,\, 2,\, \cdots$

$$u \,=\, \sum_{n=1}^{\infty} u_n(t)\, \sin\, \frac{(n \,-\, \frac{1}{2})\,\pi}{L}\, x$$

$$S \,=\, \sum_{n=1}^{\infty} s_n(t)\, \sin\, \frac{(n \,-\, \frac{1}{2})\,\pi}{L}\, x$$

$$\ddot{u}_n \,+\, c^2 \left(\frac{\left(n-\frac{1}{2}\right)\pi}{L}\right)^2 u_n \,=\, s_n$$

$$u_n \,=\, c_1 \,\cos\, \frac{\left(n \,-\, \frac{1}{2}\right)\, \pi\, c}{L}\, t + c_2 \,\sin\, \frac{\left(n \,-\, \frac{1}{2}\right)\, \pi\, c}{L}\, t + \int_0^t s_n(\tau)\, \frac{\sin\,(n \,-\, 1/2)\, \frac{\pi}{L}\, c\, (t \,-\, \tau)}{\left(\left(n \,-\, \frac{1}{2}\right)\, \frac{\pi\, c}{L}\right)}\, d\tau$$

$\downarrow$

$u_n(0)$ coefficients of $f(x)$

$c_2 \,=\, 0$ (since $u_t \,=\, 0$)

c. $u_x(0,\, t) \,=\, A(t)$

$u(L,\, L) \,=\, 0$

$w \,=\, ax \,+\, b$

$w_x(0,\, t) \,=\, A(t) \,=\, a \qquad \Rightarrow \quad a \,=\, A(t)$

$w(L,\, t) \,=\, 0 \qquad = aL \,+\, b \qquad \Rightarrow \quad b \,=\, -aL \quad \Rightarrow b \,=\, -\,A(t)L$

$w \,=\, A(t)x \,-\, A(t)L \,=\, \underline{A(t)\,(x \,-\, L)}$

$$w_{xx} \,=\, 0 \qquad w_{tt} \,=\, \ddot{A}(x \,-\, L)$$

$v \,=\, u \,-\, w$

$u \,=\, v \,+\, w$

$v_{tt} \,-\, c^2\, v_{xx} \,=\, S(x,\, t) \,-\, \ddot{A}(t)\,(x \,-\, L) \,\equiv\, \hat{S}\,(x,\, t)$

$v\,(x,\, 0) \,=\, f(x) \,-\, A(0)(x \,-\, L) \,\equiv\, F(x)$

$v_t(x,\, 0) \,=\, 0 \,-\, \dot{A}(0)\,(x \,-\, L) \,\equiv G(x)$

continue as in b.

4a. $u_{tt} - c^2 u_{xx} = xt$

$u(x, 0) = \sin x$

$u_t(x, 0) = 0$

$$\left.\begin{array}{l} u(0, t) = 1 \\ \\ u(L, t) = t \end{array}\right\} \quad \Rightarrow \quad w(x, t) = \frac{t - 1}{L} x + 1 \,;\, w_t = \frac{x}{L}$$

$$w_{tt} = 0$$

$v_{tt} - c^2 v_{xx} = xt$

$v(0, t) = v(L, t) = 0 \qquad \Rightarrow \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \phi_n = \sin \frac{n\pi}{L} x \qquad n = 1, 2, \cdots$

$v(x, 0) = \sin x - \left(-\frac{x}{L} + 1\right)$

$v_t(x, 0) = 0 - \frac{x}{L}$

$$v(x, t) = \sum_{n=1}^{\infty} v_n(t) \sin \frac{n\pi}{L} x$$

$$xt = \sum_{n=1}^{\infty} s_n(t) \sin \frac{n\pi}{L} x \qquad \Rightarrow \qquad s_n(t) = \frac{\int_0^L xt \sin \frac{n\pi}{L} x \, dx}{\int_0^L \sin^2 \frac{n\pi}{L} x \, dx}$$

$$\sin x + \frac{x}{L} - 1 = \sum_{n=1}^{\infty} v_n(0) \sin \frac{n\pi}{L} x \qquad \Rightarrow \qquad v_n(0) = \frac{\int_0^L (\sin x + \frac{x}{L} - 1) \sin \frac{n\pi}{L} x \, dx}{\int_0^L \sin^2 \frac{n\pi}{L} x \, dx}$$

$$-\frac{x}{L} = \sum_{n=1}^{\infty} \dot{v}_n(0) \sin \frac{n\pi}{L} x \qquad \dot{v}_n(0) = \frac{- \int_0^L \frac{x}{L} \sin \frac{n\pi}{L} x \, dx}{\int_0^L \sin^2 \frac{n\pi}{L} x \, dx}$$

$$\ddot{v}_n + \left(\frac{n\pi}{L}\right)^2 c^2 v_n = s_n(t)$$

$$\to v_n = c_1 \cos c\sqrt{\lambda_n}\, t + c_2 \sin c\sqrt{\lambda_n}\, t + \int_0^t s_n(\tau) \frac{\sin c\sqrt{\lambda_n}\,(t - \tau)}{c\sqrt{\lambda_n}} \, d\tau$$

$v_n(0) = c_1$

$\dot{v}_n(0) = c_2 \, c \sqrt{\lambda_n}$

continue as in 3b.

b. $u_{tt} - c^2 u_{xx} = xt$

$u(x,\, 0) = \sin x$

$u_t(x,\, 0) = 0$

$$\left.\begin{aligned} u_x(0,\, t) &= t \\ u_x(L,\, t) &= t^2 \end{aligned}\right\} \quad \Rightarrow \quad w(x,\, t) = \frac{t^2 - t}{2L} x^2 + tx \qquad w_t = \frac{2t - 1}{2L} x^2 + x$$

$$w_{tt} = \frac{x^2}{L} \qquad w_{xx} = \frac{t^2 - t}{L}$$

Thus $\qquad w(x,0) = 0, \qquad w_t(x,0) = x - \dfrac{x^2}{2L}$

Let $v = u - w$

then

$$v_{tt} - c^2 v_{xx} = \underbrace{xt - \frac{x^2}{L} + c^2 \frac{t^2 - t}{L}}_{s(x,t)}$$

$v_x(0,\, t) = v_x(L,\, t) = 0 \qquad \Rightarrow \qquad \lambda_n = \left(\frac{n\,\pi}{L}\right)^2 \qquad \phi_n = \cos \frac{n\,\pi}{L} x \qquad n = 0,\, 1,\, 2,\, \cdots$

$v(x,\, 0) = \sin x \qquad$ since $w(x,0) = 0$

$v_t(x,\, 0) = 0 - x + \frac{x^2}{2L}$

$$v(x,\, t) = \frac{1}{2} v_0(t) + \sum_{n=1}^{\infty} v_n(t) \cos \frac{n\,\pi}{L} x$$

$$s(x,t) = \frac{1}{2} s_0(t) + \sum_{n=1}^{\infty} s_n(t) \cos \frac{n\,\pi}{L} x$$

$$\Rightarrow \quad s_n(t) = \frac{1}{L} \int_0^L \left( xt - \frac{x^2}{L} + c^2 \frac{t^2 - t}{L} \right) \cos \frac{n\,\pi}{L} x\, dx \qquad n = 0, 1, 2, \cdots$$

$$\sin x = \frac{1}{2} v_0(0) + \sum_{n=1}^{\infty} v_n(0) \cos \frac{n\,\pi}{L} x$$

$$\Rightarrow \quad v_n(0) = \frac{\int_0^L \sin x \cos \frac{n\,\pi}{L} x\, dx}{\int_0^L \cos^2 \frac{n\,\pi}{L} x\, dx} \qquad n = 0, 1, 2, \cdots$$

$$-x + \frac{x^2}{2L} = \frac{1}{2} \dot{v}_0(0) + \sum_{n=1}^{\infty} \dot{v}_n(0) \cos \frac{n\,\pi}{L} x$$

$$\Rightarrow \quad \dot{v}_n(0) \;=\; \frac{\int_0^L \left(-x \,+\, \frac{x^2}{2L}\right) \,\cos\, \frac{n\,\pi}{L}\, x\, dx}{\int_0^L \,\cos^2\, \frac{n\,\pi}{L}\, x\, dx} \qquad\qquad n = 0, 1, 2, \cdots$$

$$\ddot{v}_n \,+\, \left(\frac{n\,\pi}{L}\right)^2 c^2\, v_n \;=\; s_n(t)$$

$$\ddot{v}_0 \;=\; s_0(t)$$

The solution of the ODE for $n = 0$ is obtained by integration twice and using the initial conditions

$$v_0(t) \;=\; \int_0^t \left(\int_0^\xi s_0(\tau) d\tau\right) d\xi \;+\; v_0(0) \;+\; t\,\dot{v}_0(0)$$

$$v_n \;=\; C_n\,\cos\, c\,\frac{n\pi}{L}\,t \;+\; D_n\,\sin\, c\,\frac{n\pi}{L}\,t \;+\; \int_0^t s_n(\tau)\,\frac{\sin\, c\,\frac{n\pi}{L}\,(t \,-\, \tau)}{c\,\frac{n\pi}{L}}\,d\tau$$

$$v_n(0) \;=\; C_n$$

$$\dot{v}_n(0) \;=\; D_n\, c\, \frac{n\pi}{L}$$

$$\boxed{v_n(t) \;=\; v_n(0)\cos\frac{cn\pi}{L}t \;+\; \frac{L\dot{v}_n(0)}{cn\pi}\sin\frac{cn\pi}{L}t \;+\; \int_0^t s_n(\tau)\,\frac{\sin\,\frac{cn\pi}{L}\,(t \,-\, \tau)}{\frac{cn\pi}{L}}\,d\tau}$$

Now that we have all the coefficients in the expansion of $v$, recall that $u \,=\, v \,+\, w$.

c. $u_{tt} - c^2 u_{xx} = xt$

$u(x, 0) = \sin x$

$u_t(x, 0) = 0$

$$\left.\begin{array}{l} u(0, t) = 0 \\ u_x(L, t) = t \end{array}\right\} \quad \Rightarrow \quad w(x, t) = xt\,; \qquad w_t = x$$

$$\begin{array}{ll} w_{tt} = 0 & w_{xx} = 0 \\ w(x,0) = 0 & w_t(x,0) = x \end{array}$$

$v_{tt} - c^2 v_{xx} = xt$

$v(0, t) = v_t(L, t) = 0$

$\Rightarrow \qquad \lambda_n = \left(\frac{(n-1/2)\,\pi}{L}\right)^2 \qquad \phi_n = \sin \frac{(n-1/2)\,\pi}{L}\, x \qquad n = 1, 2, \cdots$

$v(x, 0) = \sin x$

$v_t(x, 0) = -x$

$$v(x, t) = \sum_{n=1}^{\infty} v_n(t) \sin \frac{(n-1/2)\,\pi}{L}\, x$$

$$xt = \sum_{n=1}^{\infty} s_n(t) \sin \frac{(n-1/2)\,\pi}{L}\, x \qquad \Rightarrow \qquad s_n(t) = \frac{\int_0^L xt \sin \frac{(n-1/2)\,\pi}{L}\, x\, dx}{\int_0^L \sin^2 \frac{(n-1/2)\,\pi}{L}\, x\, dx}$$

$$v(x,0) = \sin x = \sum_{n=1}^{\infty} v_n(0) \sin \frac{(n-1/2)\,\pi}{L}\, x \qquad \Rightarrow \qquad v_n(0) = \frac{\int_0^L \sin x \sin \frac{(n-1/2)\,\pi}{L}\, x\, dx}{\int_0^L \sin^2 \frac{(n-1/2)\,\pi}{L}\, x\, dx}$$

$$v_t(x,0) = -x = \sum_{n=1}^{\infty} \dot{v}_n(0) \sin \frac{(n-1/2)\,\pi}{L}\, x \qquad \dot{v}_n(0) = \frac{-\int_0^L x \sin \frac{(n-1/2)\,\pi}{L}\, x\, dx}{\int_0^L \sin^2 \frac{(n-1/2)\,\pi}{L}\, x\, dx}$$

$$\ddot{v}_n + \left(\frac{(n-1/2)\,\pi}{L}\right)^2 c^2 v_n = s_n(t)$$

$$\Rightarrow v_n = c_1 \cos c\sqrt{\lambda_n}\, t + c_2 \sin c\sqrt{\lambda_n}\, t + \int_0^t s_n(\tau) \frac{\sin c\sqrt{\lambda_n}\,(t - \tau)}{c\sqrt{\lambda_n}}\, d\tau$$

$v_n(0) = c_1$

$\dot{v}_n(0) = c_2\, c\sqrt{\lambda_n}$

continue as in 3b.

d. $u_{tt} - c^2 u_{xx} = xt$

$u(x, 0) = \sin x$

$u_t(x, 0) = 0$

$$\left.\begin{aligned} u_x(0, t) &= 0 \\ u_x(L, t) &= 1 \end{aligned}\right\} \quad \Rightarrow \quad w(x, t) = \frac{x^2}{2L}; \qquad w_t = 0$$

$$w_{tt} = 0 \qquad w_{xx} = \tfrac{1}{L}$$

Thus $w(x,0) = \dfrac{x^2}{2L}, \qquad w_t(x,0) = 0$

$$v_{tt} - c^2 v_{xx} = \underbrace{xt + \frac{c^2}{L}}_{s(x,t)}$$

$$v_x(0, t) = v_x(L, t) = 0 \qquad \Rightarrow \qquad \lambda_n = \left(\tfrac{n\pi}{L}\right)^2 \qquad \phi_n = \cos \tfrac{n\pi}{L} x \qquad n = 0, 1, 2, \cdots$$

$v(x, 0) = \sin x - \frac{x^2}{2L}$

$v_t(x, 0) = 0$

$$v(x, t) = \frac{1}{2} v_0(t) + \sum_{n=1}^{\infty} v_n(t) \cos \frac{n\pi}{L} x$$

$$s(x,t) = \frac{1}{2} s_0(t) + \sum_{n=1}^{\infty} s_n(t) \cos \frac{n\pi}{L} x$$

$$\Rightarrow \quad s_n(t) = \frac{1}{L} \int_0^L \left(xt + \frac{c^2}{L}\right) \cos \frac{n\pi}{L} x \, dx \qquad n = 0, 1, 2, \cdots$$

$$v(x,0) = \sin x - \frac{x^2}{2L} = \frac{1}{2} v_0(0) + \sum_{n=1}^{\infty} v_n(0) \cos \frac{n\pi}{L} x$$

$$\Rightarrow \quad v_n(0) = \frac{1}{L} \int_0^L \left(\sin x - \frac{x^2}{2L}\right) \cos \frac{n\pi}{L} x \, dx \qquad n = 0, 1, 2, \cdots$$

$$v_t(x,0) = 0 = \frac{1}{2} \dot{v}_0(0) + \sum_{n=1}^{\infty} \dot{v}_n(0) \cos \frac{n\pi}{L} x \qquad \dot{v}_n(0) = 0 \qquad n = 0, 1, 2, \cdots$$

$$\ddot{v}_n + \left(\tfrac{n\pi}{L}\right)^2 c^2 v_n = s_n(t)$$

$$\ddot{v}_0 = s_0(t)$$

The solution of the ODE for $n = 0$ is obtained by integration twice and using the initial conditions

$$v_0(t) \; = \; \int_0^t \left( \int_0^\xi s_0(\tau) d\tau \right) d\xi \; + \; v_0(0)$$

$$v_n \; = \; C_n \, \cos \, c \, \frac{n\pi}{L} \, t \; + \; D_n \, \sin \, c \, \frac{n\pi}{L} \, t \; + \; \int_0^t \, s_n(\tau) \, \frac{\sin \, c \, \frac{n\pi}{L} \, (t \; - \; \tau)}{c \, \frac{n\pi}{L}} \, d\tau$$

$$v_n(0) \; = \; C_n$$

$$\dot{v}_n(0) \; = \; 0 \; = \; D_n \, c \, \tfrac{n\pi}{L} \qquad \Rightarrow D_n \; = \; 0$$

$$\boxed{v_n(t) \; = \; v_n(0) \cos \frac{cn\pi}{L} t \; + \; \int_0^t \, s_n(\tau) \, \frac{\sin \, \frac{cn\pi}{L} \, (t \; - \; \tau)}{\frac{cn\pi}{L}} \, d\tau}$$

Now that we have all the coefficients in the expansion of $v$, recall that $u \; = \; v \; + \; w$.

5. $u_{tt} - u_{xx} = 1$

$u(x,\,0) = f(x)$

$u_t(x,\,0) = g(x)$

$$\left.\begin{array}{l} u(0,\,t) = 1 \\ \\ u_x(L,\,t) = B(t) \end{array}\right\} \quad \Rightarrow \quad w(x,\,t) = xB(t) + 1; \qquad w_t = x\dot{B}(t)$$

$$w_{tt} = x\ddot{B}(t) \quad w_{xx} = 0$$
$$w(x,0) = xB(0) + 1 \quad w_t(x,0) = x\dot{B}(0)$$

$v_{tt} - v_{xx} = 1 - x\ddot{B}(t) + 0 \equiv S(x,t)$

$v(x,0) = f(x) - xB(0) - 1 \equiv F(x)$

$v_t(x,0) = g(x) - x\dot{B}(0) \equiv G(x)$

$v(0,\,t) = v_t(L,\,t) = 0$

$$\Rightarrow \qquad \lambda_n = \left(\frac{(n-1/2)\,\pi}{L}\right)^2 \qquad \phi_n = \sin\frac{(n-1/2)\,\pi}{L}\,x \qquad n = 1,\,2,\,\cdots$$

$$v(x,\,t) = \sum_{n=1}^{\infty} v_n(t)\,\sin\frac{(n-1/2)\,\pi}{L}\,x$$

$$S(x,t) = \sum_{n=1}^{\infty} s_n(t)\,\sin\frac{(n-1/2)\,\pi}{L}\,x$$

$$\Rightarrow \quad s_n(t) = \frac{\int_0^L S(x,t)\,\sin\frac{(n-1/2)\,\pi}{L}\,x\,dx}{\int_0^L \sin^2\frac{(n-1/2)\,\pi}{L}\,x\,dx}$$

$$F(x) = \sum_{n=1}^{\infty} v_n(0)\,\sin\frac{(n-1/2)\,\pi}{L}\,x \qquad \Rightarrow \quad v_n(0) = \frac{\int_0^L F(x)\,\sin\frac{(n-1/2)\,\pi}{L}\,x\,dx}{\int_0^L \sin^2\frac{(n-1/2)\,\pi}{L}\,x\,dx}$$

$$G(x) = \sum_{n=1}^{\infty} \dot{v}_n(0)\,\sin\frac{(n-1/2)\,\pi}{L}\,x \qquad \Rightarrow \dot{v}_n(0) = \frac{\int_0^L G(x)\sin\frac{(n-1/2)\,\pi}{L}\,x\,dx}{\int_0^L \sin^2\frac{(n-1/2)\,\pi}{L}\,x\,dx}$$

$$\ddot{v}_n + \left(\frac{(n-1/2)\,\pi}{L}\right)^2 v_n = s_n(t)$$

$$\Rightarrow v_n = c_1\cos\sqrt{\lambda_n}\,t + c_2\sin\sqrt{\lambda_n}\,t + \int_0^t s_n(\tau)\,\frac{\sin\sqrt{\lambda_n}\,(t-\tau)}{\sqrt{\lambda_n}}\,d\tau$$

$v_n(0) = c_1$

$\dot{v}_n(0) = c_2\sqrt{\lambda_n}$

continue as in 3b.

## 5.4 Poisson's Equation

### 5.4.1 Homogeneous Boundary Conditions

### 5.4.2 Inhomogeneous Boundary Conditions

**Problems**

1. Solve

$$\nabla^2 u = S(x,y), \qquad 0 < x < L, \quad 0 < y < H,$$

a.

$$u(0,y) = u(L,y) = 0$$

$$u(x,0) = u(x,H) = 0$$

Use a Fourier sine series in y.

b.

$$u(0,y) = 0 \qquad u(L,y) = 1$$

$$u(x,0) = u(x,H) = 0$$

Hint: Do NOT reduce to homogeneous boundary conditions.

c.

$$u_x(0,y) = u_x(L,y) = 0$$

$$u_y(x,0) = u_y(x,H) = 0$$

In what situations are there solutions?

2. Solve the following Poisson's equation

$$\nabla^2 u = e^{2y} \sin x, \qquad 0 < x < \pi, \quad 0 < y < L,$$

$$u(0,y) = u(\pi,y) = 0,$$

$$u(x,0) = 0,$$

$$u(x,L) = f(x).$$

1. a. $\nabla^2 u = s(x, y)$

$$u(0, y) = u(L, y) = 0$$

$$u(x, 0) = u(x, H) = 0 \quad \Rightarrow \quad \sin \tfrac{n\pi}{H} y$$

Use a Fourier sine series in $y$ (we can also use a Fourier sine series in $x$ or a double Fourier sine series, because of the boundary conditions)

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x) \sin \frac{n\pi}{H} y$$

$$S(x, y) = \sum_{n=1}^{\infty} s_n(x) \sin \frac{n\pi}{H} y$$

$$s_n(x) = \frac{\int_0^H s(x, y) \sin \frac{n\pi}{H} y \, dy}{\int_0^H \sin^2 \frac{n\pi}{H} y \, dy}$$

$$\sum_{n=1}^{\infty} \left\{ \underbrace{-\left(\frac{n\pi}{H}\right)^2 u_n \sin \frac{n\pi}{H} y}_{u_{yy}} + \underbrace{\ddot{u}_n \sin \frac{n\pi}{H} y}_{u_{xx}} \right\} = \sum_{n=1}^{\infty} s_n(x) \sin \frac{n\pi}{H} y$$

$$\underline{\ddot{u}_n(x) - \left(\tfrac{n\pi}{H}\right)^2 u_n(x) = s_n(x)}$$

Boundary conditions are coming from $u(0, y) = u(L, y) = 0$

$$\sum_{n=1}^{\infty} u_n(0) \sin \frac{n\pi}{H} y = 0 \quad \Rightarrow \quad \underline{u_n(0) = 0}$$

$$\sum_{n=1}^{\infty} u_n(L) \sin \frac{n\pi}{H} y = 0 \quad \Rightarrow \quad \underline{u_n(L) = 0}$$

$$(*)\ u_n(x) = \frac{\sinh \frac{n\pi}{H}(L - x)}{-\frac{n\pi}{H} \sinh \frac{n\pi}{H} L} \int_0^x s_n(\xi) \sinh \frac{n\pi}{H} \xi \, d\xi$$
$$+ \frac{\sinh \frac{n\pi}{H} x}{-\frac{n\pi}{H} \sinh \frac{n\pi}{H} L} \int_x^L s_n(\xi) \sinh \frac{n\pi}{H} (L - \xi) \, d\xi$$

Let's check by using (*)

$u_n(0) =$ $1^{st}$ term the integral is zero since limits are same

$2^{nd}$ term the numerator is zero $= \sinh \frac{n\pi}{H} \cdot 0$

$u_n(L) =$ $1^{st}$ term the numerator $\sinh \frac{n\pi}{H} (L - L) = 0$

$2^{nd}$ term the integral is zero since limits of integration are the same.

$$\dot{u}_n = \frac{-\frac{n\pi}{H}\cosh\frac{n\pi}{H}(L-x)}{-\frac{n\pi}{H}\sinh\frac{n\pi}{H}L}\int_0^x s_n(\xi)\sinh\frac{n\pi}{H}\xi\,d\xi + \frac{\sinh\frac{n\pi}{H}(L-x)}{-\frac{n\pi}{H}\sinh\frac{n\pi}{H}L}\quad\underbrace{s_n(x)\sinh\frac{n\pi}{H}x}_{\text{integrand at upper limit}}$$

$$+\frac{\frac{n\pi}{H}\cosh\frac{n\pi}{H}x}{-\frac{n\pi}{H}\sinh\frac{n\pi}{H}L}\int_x^L s_n(\xi)\sinh\frac{n\pi}{H}(L-\xi)\,d\xi + \frac{\sinh\frac{n\pi}{H}x}{-\frac{n\pi}{H}\sinh\frac{n\pi}{H}L}\underbrace{\left[-s_n(x)\sinh\frac{n\pi}{H}(L-x)\right]}_{\text{integrand at lower limit}}$$

Let's add the second and fourth terms up

$$-\frac{1}{\frac{n\pi}{H}\sinh\frac{n\pi}{H}L}s_n(x)\left\{\underbrace{\sinh\frac{n\pi}{H}(L-x)\sinh\frac{n\pi}{H}x-\sinh\frac{n\pi}{H}x\sinh\frac{n\pi}{H}(L-x)}_{=0}\right\}$$

$$\ddot{u}_n = \frac{\left(-\frac{n\pi}{H}\right)^2\sinh\frac{n\pi}{H}(L-x)}{-\frac{n\pi}{H}\sinh\frac{n\pi}{H}L}\int_0^x s_n(\xi)\sinh\frac{n\pi}{H}\xi\,d\xi + \frac{-\frac{n\pi}{H}\cosh\frac{n\pi}{H}(L-x)}{-\frac{n\pi}{H}\sinh\frac{n\pi}{H}L}s_n(x)\sinh\frac{n\pi}{H}x$$

$$+\frac{\left(\frac{n\pi}{H}\right)^2\sinh\frac{n\pi}{H}x}{-\frac{n\pi}{H}\sinh\frac{n\pi}{H}L}\int_x^L s_n(\xi)\sinh\frac{n\pi}{H}(L-\xi)\,d\xi + \frac{\frac{n\pi}{H}\cosh\frac{n\pi}{H}x}{-\frac{n\pi}{H}\sinh\frac{n\pi}{H}L}\left[-s_n(x)\sinh\frac{n\pi}{H}(L-x)\right]$$

Let's add the second and fourth terms up

$$\frac{s_n(x)}{-\sinh\frac{n\pi}{H}L}\qquad\left\{\underbrace{\sinh\frac{n\pi}{H}x\cosh\frac{n\pi}{H}(L-x)+\cosh\frac{n\pi}{H}x\sinh\frac{n\pi}{H}(L-x)}_{=\sinh\frac{n\pi}{H}(x-(L-x))=\sin h\frac{n\pi}{H}L}\right\} = s_n(x)$$

The integral terms in $\ddot{u}_n$ are exactly $\left(\frac{n\pi}{H}\right)^2 u_n$ and thus the ODE is satisfied.

b. $\nabla^2 u = S(x, y)$

$$u(0, y) = 0 \qquad u(L, y) = 1$$

$$u(x, 0) = u(x, H) = 0 \quad \Rightarrow \quad \sin \tfrac{n\pi}{H} y$$

Use a Fourier sine series in $y$

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x) \sin \frac{n\pi}{H} y$$

$$S(x, y) = \sum_{n=1}^{\infty} s_n(x) \sin \frac{n\pi}{H} y$$

$$s_n(x) = \frac{\int_0^H S(x, y) \sin \frac{n\pi}{H} y \, dy}{\int_0^H \sin^2 \frac{n\pi}{H} y \, dy}$$

$$\sum_{n=1}^{\infty} \left\{ \underbrace{-\left(\frac{n\pi}{H}\right)^2 u_n \sin \frac{n\pi}{H} y}_{u_{yy}} + \underbrace{\ddot{u}_n \sin \frac{n\pi}{H} y}_{u_{xx}} \right\} = \sum_{n=1}^{\infty} s_n(x) \sin \frac{n\pi}{H} y$$

$$\underline{\ddot{u}_n(x) - \left(\tfrac{n\pi}{H}\right)^2 u_n(x) = s_n(x)}$$

Boundary conditions are coming from $u(0, y) = 0 \qquad u(L, y) = 1$

$$\sum_{n=1}^{\infty} u_n(0) \sin \frac{n\pi}{H} y = 0 \quad \Rightarrow \quad \underline{u_n(0) = 0}$$

$$\sum_{n=1}^{\infty} u_n(L) \sin \frac{n\pi}{H} y = 1 \quad \Rightarrow \quad u_n(L) = \frac{1}{H} \int_0^H 1 \cdot \sin \frac{n\pi}{H} y \, dy$$

$\Rightarrow \quad u_n(L) = \dfrac{4}{n\pi}$ for $n$ odd and 0 for $n$ even. (see (5.8.1)

For $n$ even the solution is as in 1a (since $u_n(L) = 0$)

For $n$ odd, how would the solution change?

Let $\omega_n = \dfrac{4}{n\pi L} x$, then $\ddot{\omega}_n = 0$

Let $v_n = u_n - \omega_n$ then $v_n(0) = 0$ and $v_n(L) = u_n(L) - \omega_n(L) = 0$

and

$$\ddot{v}_n - \left(\tfrac{n\pi}{H}\right)^2 v_n = \underbrace{\left(\frac{n\pi}{H}\right)^2 \frac{4x}{n\pi L} + s_n}_{\text{This is the } s_n \text{ to be used in (*) in 1a}}$$

c. $\nabla^2 u = S(x, y)$

$$u_x(0, y) = 0 \qquad u_x(L, y) = 0 \quad \Rightarrow \quad \cos \tfrac{n\pi}{L} x$$

$$u_y(x, 0) = u_y(x, H) = 0 \quad \Rightarrow \quad \cos \tfrac{m\pi}{H} y$$

Use a double Fourier cosine series

$$u(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_{nm} \cos \frac{n\pi}{L} x \cos \frac{m\pi}{H} y$$

$$S(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} s_{nm} \cos \frac{n\pi}{L} x \cos \frac{m\pi}{H} y$$

$$s_{nm} = \frac{\int_0^H \int_0^L S(x, y) \cos \frac{m\pi}{H} y \cos \frac{n\pi}{L} x \, dx \, dy}{\int_0^H \int_0^L \cos^2 \frac{m\pi}{H} y \cos^2 \frac{n\pi}{L} x \, dx \, dy}$$

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-u_{nm}) \left[ \left( \frac{n\pi}{L} \right)^2 + \left( \frac{m\pi}{H} \right)^2 \right] \cos \frac{n\pi}{L} x \cos \frac{m\pi}{H} y = S(x, y)$$

Thus

$$-u_{nm} \left[ \left( \tfrac{n\pi}{L} \right)^2 + \left( \tfrac{m\pi}{H} \right)^2 \right] = s_{nm}$$

Substituing for $s_{nm}$, we get the unknowns $u_{nm}$

$$u_{nm} = \frac{\int_0^H \int_0^L S(x, y) \cos \frac{m\pi}{H} y \cos \frac{n\pi}{L} x \, dx \, dy}{\left[ \left( \frac{n\pi}{L} \right)^2 + \left( \frac{m\pi}{H} \right)^2 \right] \int_0^H \int_0^L \cos^2 \frac{m\pi}{H} y \cos^2 \frac{n\pi}{L} x \, dx \, dy}$$

What if $\lambda_{nm} = 0$? (i.e. $n = m = 0$)
Then we cannot divide by $\lambda_{nm}$ but in this case we have zero on the left
$\Rightarrow \int_0^H \int_0^L S(x, y) \, dx \, dy = 0$

This is typical of Neumann boundary conditions.

2. $\nabla^2 u = e^{2y} \sin x$

$$u(0,\, y) = 0 \qquad u(\pi,\, y) = 0 \quad \Rightarrow \quad \sin nx$$

$$u(x,\, 0) = 0 \qquad u(x,\, L) = f(x)$$

Use a Fourier sine series in $x$

$$u(x,\, y) = \sum_{n=1}^{\infty} u_n(y) \sin nx$$

$S(x,\, y)$ is already in a Fourier sine series in $x$ with the coefficients $s_1(y) = e^{2y}$ and all the other coefficients are zero.

$$\sum_{n=1}^{\infty} \left\{ \underbrace{-n^2 u_n \sin nx}_{u_{xx}} + \underbrace{\ddot{u}_n \sin nx}_{u_{yy}} \right\} = \sum_{n=1}^{\infty} s_n(x) \sin nx$$

$$\ddot{u}_n(y) - n^2 u_n(y) = 0 \qquad \text{for } n \neq 1$$

$$\ddot{u}_1(y) - u_1(y) = e^{2y}$$

Boundary conditions are coming from $u(x,\, 0) = 0 \qquad u(x,\, L) = f(x)$

$$\sum_{n=1}^{\infty} u_n(0) \sin nx = 0 \quad \Rightarrow \quad \underline{u_n(0) = 0}$$

$$\sum_{n=1}^{\infty} u_n(L) \sin nx = f(x) \quad \Rightarrow \quad u_n(L) = \frac{1}{L} \int_0^L f(x) \sin nx \, dx$$

The solution of the ODEs is

$$u_1(y) = \underbrace{\frac{1}{3} e^{2y}}_{\text{particular solution}} + \alpha_1 \sinh y + \beta_1 \cosh y$$

and

$$u_n(y) = \alpha_n \sinh ny + \beta_n \cosh ny \qquad n \neq 1$$

Since $u_1(0) = 0$ we have $\frac{1}{3} + \beta_1 = 0 \qquad \Rightarrow \beta_1 = -\frac{1}{3}$

Since $u_n(0) = 0$ we have $\beta_n = 0 \qquad n \neq 1$

Using $u_1(L)$ we have $\frac{1}{3} e^{2L} + \alpha_1 \sinh L - \frac{1}{3} \cosh L = u_1(L)$. This gives a value for $\alpha_1$

$$\alpha_1 = \frac{u_1(L) + \frac{1}{3} \cosh L - \frac{1}{3} e^{2L}}{\sinh L}$$

Using $u_n(L)$ we get a vlaue for $\alpha_n$

$$\alpha_n = \frac{u_n(L)}{\sinh nL} \qquad n \neq 1$$

Now we can write the solution

$$u(x,y) = \left(\alpha_1 \sinh y - \frac{1}{3}\cosh y + \frac{1}{3}e^{2L}\right)\sin x + \sum_{n=2}^{\infty}\left(\frac{u_n(L)}{\sinh nL}\sinh ny\right)\sin nx$$

with $\alpha_1$ as above.

### 5.4.3 One Dimensional Boundary Value Problems

**Problems**

1. Find the eigenvalues and corresponding eigenfunctions in each of the following boundary value problems.

(a) $y'' - \lambda^2 y = 0 \quad 0 < x < a \quad y'(0) = y'(a) = 0$

(b) $y'' - \lambda^2 y = 0 \quad 0 < x < a \quad y(0) = 0 \quad y(a) = 1$

(c) $y'' + \lambda^2 y = 0 \quad 0 < x < a \quad y(0) = y'(a) = 0$

(d) $y'' + \lambda^2 y = 0 \quad 0 < x < a \quad y(0) = 1 \quad y'(a) = 0$

2. Find the eigenfunctions of the following boundary value problem.

$$y'' + \lambda^2 y = 0 \qquad 0 < x < 2\pi \qquad y(0) = y(2\pi) \quad y'(0) = y'(2\pi)$$

3. Obtain the eigenvalues and eigenfunctions of the problem.

$$y'' + y' + (\lambda + 1)y = 0 \qquad 0 < x < \pi \qquad y(0) = y(\pi) = 0$$

4. Obtain the orthonormal set of eigenfunctions for the problem.

(a) $y'' + \lambda y = 0 \quad 0 < x < \pi \quad y'(0) = 0 \quad y(\pi) = 0$

(b) $y'' + (1 + \lambda)y = 0 \quad 0 < x < \pi \quad y(0) = 0 \quad y(\pi) = 0$

(c) $y'' + \lambda y = 0 \quad -\pi < x < \pi \quad y'(-\pi) = 0 \quad y'(\pi) = 0$

1a.

$$y'' - \lambda^2 y \, = \, 0 \qquad 0 < x < a$$
$$y'(0) \, = \, y'(a) \, = \, 0$$

case 1: $\lambda = 0$

$$y'' = 0$$

implies $y = Ax + B$.

Using the boundary conditions, we get $y'(x) = A = 0$. Thus the solution in this case is

$$y(x) = B$$

case 2: $\lambda \neq 0$

The solution is

$$y = Ae^{\lambda x} + Be^{-\lambda x}$$

or

$$y = C \cosh \lambda x + D \sinh \lambda x$$

To use the boundary conditions we need

$$y'(x) = C\lambda \sinh \lambda x + D\lambda \cosh \lambda x$$

The condition $y'(0) = 0$ implies $\lambda D = 0$ and since in this case $\lambda \neq 0$, we have $D = 0$. Now apply the other boundary condition

$$\lambda C \sinh \lambda a = 0$$

This implies $C = 0$ which yield the trivial solution

**or** $\sinh \lambda a = 0$ which means $\lambda a = n\pi i$. So $\lambda = \dfrac{n\pi i}{a}$ and the eingenvalues $\lambda^2$ are

$$\lambda^2 = -\left(\frac{n\pi}{a}\right)^2$$

with eigenfunctions

$$\sinh n\pi i = \sin n\pi, \qquad n \neq 0$$

1b.

$$y'' - \lambda^2 y = 0 \qquad 0 < x < a$$
$$y(0) = 0, \qquad y(a) = 1$$

case 1: $\lambda = 0$

$$y'' = 0$$

implies $y = Ax + B$.

Using the boundary conditions, we get $y(0) = B = 0$, and $y(a) = Aa + B = 1$, or $A = \frac{1}{a}$. Thus the solution in this case is

$$y(x) = \frac{1}{a}x$$

case 2: $\lambda \neq 0$

The solution is

$$y = Ae^{\lambda x} + Be^{-\lambda x}$$

or

$$y = C\cosh\lambda x + D\sinh\lambda x$$

The first boundary condition gives $C = 0$. Now apply the other boundary condition

$$D\sinh\lambda a = 1$$

This implies $D = \frac{1}{\sinh\lambda a}$. The problem is when $\lambda a = n\pi i$ (causing the denominator to vanish). So for **any** $\lambda \neq \frac{n\pi i}{a}$ the eigenfunctions

$$y(x) = \frac{\sinh\lambda x}{\sinh\lambda a}$$

1c.

$$y'' + \lambda^2 y = 0 \qquad 0 < x < a$$
$$y(0) = 0, \qquad y'(a) = 0$$

The eigenvalues and eigenfunctions are

$$\lambda = (n + \frac{1}{2})\,\frac{\pi}{a}$$

$$y = \sin(n + \frac{1}{2})\,\frac{\pi x}{a}$$
$$n = 0, 1, \ldots.$$

1d.

$$y'' + \lambda^2 y = 0 \qquad 0 < x < a$$
$$y(0) = 1, \qquad y'(a) = 0$$

case 1: $\lambda = 0$

$$y'' = 0$$

implies $y = Ax + B$.

Using the boundary conditions, we get $y(0) = B = 1$, and $y'(a) = A = 0$. Thus the solution in this case is

$$y(x) = B$$

case 2: $\lambda \neq 0$

The solution is

$$y = C \cos \lambda x + D \sin \lambda x$$

The first boundary condition gives $C = 1$. Now apply the other boundary condition

$$-\lambda \sin \lambda a + \lambda D \cos \lambda a = 0$$

Suppose $\sin \lambda a \neq 0$, then this implies $D = \tan \lambda a$ (since $\lambda \neq 0$).

The eigenfunctions are

$$y(x) = \cos \lambda x + \tan \lambda a \sin \lambda x, \text{ where } \sin \lambda a \neq 0, \, \lambda \neq (n + \frac{1}{2})\frac{\pi}{a}$$

The last condition ensures $\tan \lambda a \neq 0$.

When $\cos \lambda a = 0$ or $\lambda = (n + \frac{1}{2})\frac{\pi}{a}$ the eigenfunctions

$$y(x) = \cos(n + \frac{1}{2})\frac{\pi}{a}x \text{ clearly } \sin \lambda a \neq 0 \text{ in this case}$$

2.

$$y'' + \lambda^2 y = 0 \qquad 0 < x < 2\pi$$
$$y(0) = y(2\pi)$$
$$y'(0) = y'(2\pi)$$

The eigenvalues

$$\lambda_n = n,\ n = 0, 1, 2, \ldots$$

The eigenfunctions

$$y_n = \begin{cases} \cos nx \\ \sin nx \end{cases}$$

3.

$$y'' + y' + (\lambda + 1)y = 0 \qquad 0 < x < \pi$$
$$y(0) = y(\pi) = 0$$

Try $y = e^{\mu x}$, then

$$\mu^2 + \mu + \lambda + 1 = 0$$

The characteristic values $\mu$ are then

$$\mu = \frac{-1 \pm \sqrt{1 - 4(\lambda + 1)}}{2}.$$

There are 3 possible cases.

case 1: $1 - 4(\lambda + 1) > 0$, then we have two real $\mu$.
$\mu_1 = -\frac{1}{2} + r$, $\mu_2 = -\frac{1}{2} - r$,
where $r = \frac{\sqrt{1 - 4(\lambda + 1)}}{2}$.
The solution is $y(x) = e^{-x/2}\left(Ae^{rx} + Be^{-rx}\right)$.
Now use the boundary conditions to get a system of two equations for $A, B$.

$$A + B = 0$$
$$Ae^{r\pi} + Be^{-r\pi} = 0$$

The determinat must be zero to get a nontrivial solution, i.e.

$$\begin{vmatrix} 1 & 1 \\ e^{r\pi} & e^{-r\pi} \end{vmatrix} = 0$$

or

$$\sinh r\pi = 0$$

This implies $r\pi = n\pi i$, or $r = ni$. In terms of $\lambda$, we have

$$\frac{\sqrt{1 - 4(\lambda + 1)}}{2} = (ni)$$

$$1 - 4(1+\lambda) = -4n^2$$
$$1 + 4n^2 = 4(1+\lambda)$$
$$\lambda_n = \frac{1+4n^2}{4} - 1 = n^2 - \frac{3}{4}$$

The eigenfunctions are then

$$y = 2B \sinh nix = 2B \sin nx$$

case 2: $1 - 4(\lambda + 1) = 0$, then we have two real identical $\mu$.

$$\mu_1 = \mu_2 = -\frac{1}{2}$$
$$y(x) = e^{-x/2}(Ax + B)$$

The first boundary condition gives $B = 0$. The second boundary condition gives $A\pi e^{-\pi/2} = 0$ or $A = 0$. Thus the soultion is trivial in this case.

case 3: $1 - 4(\lambda + 1) < 0$, then we have two complex conjugate $\mu$.
$\mu_1 = -\frac{1}{2} + si$, $\mu_2 = -\frac{1}{2} - si$,
where $s = \frac{\sqrt{4(\lambda+1) - 1}}{2}$.
The solution is $y(x) = e^{-x/2}\left(A \cos sx + B \sin sx\right)$.
Now use the boundary conditions to get a system of two equations for $A, B$.

$$A = 0$$
$$e^{-pi/2} B \sin s\pi = 0$$

Clearly $B \neq 0$ to get a nontrivial solution, thus $\sin s\pi = 0$ or $s\pi = n\pi$, i.e. $s = n$. In terms of $\lambda$, we have

$$\frac{\sqrt{4(\lambda+1) - 1}}{2} = n$$
$$4(\lambda + 1) = 4n^2 + 1$$
$$\lambda_n = \frac{4n^2 - 3}{4}$$
$$\lambda_n = n^2 - \frac{3}{4}$$

The eigenfunctions are then

$$y = B \sin nx, \; n = 1, 2, \ldots$$

4a.

$$y'' + \lambda y = 0 \qquad 0 < x < \pi$$
$$y'(0) = 0 \qquad y(\pi) = 0$$

$$\lambda_n = \left(n + \frac{1}{2}\right)^2$$
$$\cos(n + \frac{1}{2})x$$

To get orthonormal set, we need to divide by the normalization factor,

$$||y_n|| = \sqrt{\int_0^\pi \cos^2(n + \frac{1}{2})x dx} = \sqrt{\pi/2}$$

The normalized eigenfunctions are

$$\frac{\cos(n + \frac{1}{2})x}{\sqrt{\pi/2}}$$

4b.

$$y'' + (1 + \lambda)y = 0 \qquad 0 < x < \pi$$
$$y(0) = 0 \qquad y(\pi) = 0$$

Let $\mu = 1 + \lambda$, then

$$y_n = \sin nx$$
$$\mu_n = n^2$$

or

$$\lambda_n = n^2 - 1$$

The normalization factor is the same as before.

4c.

$$y'' + \lambda y = 0 \qquad -\pi < x < \pi$$
$$y'(-\pi) = 0 \qquad y'(\pi) = 0$$

case 1: $\lambda = 0$, the solution is $y = Ax + B$
The boundary conditions give $A = 0$
The solution is then $y = B$ for $\lambda = 0$.

case 2: $\lambda < 0$, the solution is trivial

case 3: $\lambda > 0$, the solution is $y = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$
Differentiate and use the boundary conditions, we get

$$\sqrt{\lambda}(-A\sin\sqrt{\lambda}\pi + B\cos\sqrt{\lambda}\pi) = 0$$

$$\sqrt{\lambda}(A\sin\sqrt{\lambda}\pi + B\cos\sqrt{\lambda}\pi) = 0$$

To solve the homogeneous system (we can drop the factor $\sqrt{\lambda}$ since it is not zero), we must have the determinat equals zero

$$\begin{vmatrix} -\sin\sqrt{\lambda}\pi & \cos\sqrt{\lambda}\pi \\ \sin\sqrt{\lambda}\pi & \cos\sqrt{\lambda}\pi \end{vmatrix} = 0$$

$$-2\sin\sqrt{\lambda}\pi\cos\sqrt{\lambda}\pi = 0$$

If $\cos\sqrt{\lambda}\pi = 0$, then

$$\lambda_n = (n - \frac{1}{2})^2,\ n = 1, 2, \ldots$$

and the system is

$$A\sin\sqrt{\lambda}\pi = 0$$

or

$$A = 0$$

and $B$ is any value since it doesn't show in the system. Thus the eigenfunction is

$$B\sin\sqrt{\lambda}x = B\sin(n - \frac{1}{2})x$$

If $\sin\sqrt{\lambda}\pi = 0$, then

$$\lambda_n = n^2,\ n = 1, 2, \ldots$$

and the system is

$$B\cos\sqrt{\lambda}\pi = 0$$

or

$$B = 0$$

and $A$ is any value since it doesn't show in the system. Thus the eigenfunction is

$$A\cos\sqrt{\lambda}x = A\cos nx$$

In summary

$$\begin{array}{ll} \lambda = 0 & y = B \\ \lambda_n = n^2 y = A\cos nx & \\ \lambda_n = (n - \frac{1}{2})^2 & y = B\sin(n - \frac{1}{2})x \end{array}$$

Now normalize each.

# 6 Classification and Characteristics

## 6.1 Physical Classification

## 6.2 Classification of Linear Second Order PDEs

### Problems

1. Classify each of the following as hyperbolic, parabolic or elliptic at every point $(x, y)$ of the domain

a. $x\, u_{xx} + u_{yy} = x^2$
b. $x^2\, u_{xx} - 2xy\, u_{xy} + y^2 u_{yy} = e^x$
c. $e^x u_{xx} + e^y u_{yy} = u$
d. $u_{xx} + u_{xy} - x u_{yy} = 0$ in the left half plane $(x \leq 0)$
e. $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} + xy u_x + y^2 u_y = 0$
f. $u_{xx} + x u_{yy} = 0$ (Tricomi equation)

2. Classify each of the following constant coefficient equations

a. $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$
b. $u_{xx} + u_{xy} + u_{yy} + u_x = 0$
c. $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$
d. $u_{xx} + 2u_{xy} + 3u_{yy} + 4u_x + 5u_y + u = e^x$
e. $2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0$
f. $u_{xx} + 5u_{xy} + 4u_{yy} + 7u_y = \sin x$

3. Use any symbolic manipulator (e.g. MACSYMA or MATHEMATICA) to prove (6.1.19). This means that a transformation does NOT change the type of the PDE.

1a. $A = x$ $B = 0$ $C = 1$ $\triangle = -4x$

hyperbolic for $x < 0$

parabolic $x = 0$

elliptic $x > 0$

b. $A = x^2$ $B = 2xy$ $C = y^2$ $\triangle = 0$ parabolic

c. $A = e^x$ $B = 0$ $C = e^y$ $\triangle = -4e^x e^y$ elliptic

d. $A = 1$ $B = 1$ $C = -x$ $\triangle = 1 + 4x$

hyperbolic $0 \geq x > -\frac{1}{4}$

parabolic $x = -\frac{1}{4}$

elliptic $x < -\frac{1}{4}$

e. $A = x^2$ $B = 2xy$ $C = y^2$ $\triangle = 0$ parabolic

f. $A = 1$ $B = 0$ $C = x$ $\triangle = -4x$

hyperbolic $x < 0$

parabolic $x = 0$

elliptic $x > 0$

2.

| | A | B | C | Discriminant | |
|---|---|---|---|---|---|
| a. | 4 | 5 | 1 | 25 - 16 > 0 | hyperbolic |
| b. | 1 | 1 | 1 | 1 - 4 < 0 | elliptic |
| c. | 3 | 10 | 3 | 100 - 36 > 0 | hyperbolic |
| d. | 1 | 2 | 3 | 4 - 12 < 0 | elliptic |
| e. | 2 | -4 | 2 | 16 - 16 = 0 | parabolic |
| f. | 1 | 5 | 4 | 25 - 16 > 0 | hyperbolic |

3. We substitute for $A^*, B^*, C^*$ given by (6.1.12)-(6.1.14) in $\Delta^*$.

$$\begin{aligned}
\Delta^* &= (B^*)^2 - 4A^*C^* \\
&= \left[2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y\right]^2 - \\
&\quad 4\left[A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2\right]\left[A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2\right] \\
&= 4A^2\xi_x^2\eta_x^2 + 4A\xi_x\eta_x B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 8A\xi_x\eta_x C\xi_y\eta_y \\
&+ B^2\left(\xi_x\eta_y + \xi_y\eta_x\right)^2 + 4B\left(\xi_x\eta_y + \xi_y\eta_x\right)C\xi_y\eta_y \\
&+ 4C^2\xi_y^2\eta_y^2 - 4A^2\xi_x^2\eta_x^2 - 4A\xi_x^2 B\eta_x\eta_y - 4A\xi_x^2 C\eta_y^2 \\
&- 4B\xi_x\xi_y A\eta_x^2 - 4B^2\xi_x\xi_y\eta_x\eta_y - 4B\xi_x\xi_y C\eta_y^2 \\
&- 4C\xi_y^2 A\eta_x^2 - 4C\xi_y^2 B\eta_x\eta_y - 4C^2\xi_y^2\eta_y^2.
\end{aligned}$$

Collect terms to find

$$\begin{aligned}
\Delta^* &= 4AB\xi_x^2\eta_x\eta_y + 4AB\xi_x\xi_y\eta_x^2 + 8AC\xi_x\xi_y\eta_x\eta_y \\
&+ B^2(\xi_x^2\eta_y^2 + 2\xi_x\xi_y\eta_x\eta_y + \xi_y^2\eta_x^2) \\
&+ 4BC\xi_x\xi_y\eta_y^2 + 4BC\eta_x\eta_y\xi_y^2 - 4AB\xi_x^2\eta_x\eta_y \\
&- 4AC\xi_x^2\eta_y^2 - 4AB\xi_x\xi_y\eta_x^2 - 4B^2\xi_x\xi_y\eta_x\eta_y \\
&- 4BC\xi_x\xi_y\eta_y^2 - 4AC\xi_y^2\eta_x^2 - 4BC\xi_y^2\eta_x\eta_y
\end{aligned}$$

$$\begin{aligned}
\Delta^* &= -4AC\left(\xi_x^2\eta_y^2 - 2\xi_x\xi_y\eta_x\eta_y + \xi_y^2\eta_x^2\right) \\
&+ B^2\left(\xi_x^2\eta_y^2 - 2\xi_x\xi_y\eta_x\eta_y + \xi_y^2\eta_x^2\right) \\
&= J^2\Delta,
\end{aligned}$$

since $J = (\xi_x\eta_y - \xi_y\eta_x)$.

## 6.3 Canonical Forms

### Problems

1. Find the characteristic equation, characteristic curves and obtain a canonical form for each

a. $x\,u_{xx} + u_{yy} = x^2$
b. $u_{xx} + u_{xy} - xu_{yy} = 0 \qquad (x \le 0,\ \text{all } y)$
c. $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} + xyu_x + y^2u_y = 0$
d. $u_{xx} + xu_{yy} = 0$
e. $u_{xx} + y^2u_{yy} = y$
f. $\sin^2 xu_{xx} + \sin 2xu_{xy} + \cos^2 xu_{yy} = x$

2. Use Maple to plot the families of characteristic curves for each of the above.

3. Classify the following PDEs:

(a) $\dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial u}{\partial x} = -e^{-kt}$

(b) $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial u}{\partial y} = 4$

4. Find the characteristics of each of the following PDEs:

(a) $\dfrac{\partial^2 u}{\partial x^2} + 3\dfrac{\partial^2 u}{\partial x \partial y} + 2\dfrac{\partial^2 u}{\partial y^2} = 0$

(b) $\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$

5. Obtain the canonical form for the following elliptic PDEs:

(a) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$

(b) $\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x \partial y} + 5\dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial u}{\partial y} = 0$

6. Transform the following parabolic PDEs to canonical form:

(a) $\dfrac{\partial^2 u}{\partial x^2} - 6\dfrac{\partial^2 u}{\partial x \partial y} + 9\dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial u}{\partial x} - e^{xy} = 1$

(b) $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} + 7\dfrac{\partial u}{\partial x} - 8\dfrac{\partial u}{\partial y} = 0$

1a. $x u_{xx} + u_{yy} = x^2$

$A = x$ $\quad B = 0$ $\quad C = 1$ $\quad \triangle = B^2 - 4AC = -4x$

If $x > 0$ then $\triangle < 0$ elliptic

$= 0$ $\quad = 0$ parabolic

$< 0$ $\quad > 0$ hyperbolic

characteristic equation

$$\frac{dy}{dx} = \frac{\pm\sqrt{-4x}}{2x} = \frac{\pm\sqrt{-x}}{x}$$

Suppose $x < 0$ (hyperbolic)

Let $z = -x$ (then $z > 0$)

then $dz = -dx$

and

$\frac{dy}{dz} = -\frac{dy}{dx} = -\frac{\pm\sqrt{z}}{-z} = \pm\frac{1}{\sqrt{z}}$

$dy = \pm\frac{dz}{z^{1/2}}$

$y = \pm 2\sqrt{z} + c$

$y \mp 2\sqrt{z} = c$

characteristic curves: $y \mp 2\sqrt{z} = c$
2 families as expected.

Transformation: $\xi = y - 2\sqrt{z}$

$\eta = y + 2\sqrt{z}$

$u_{xx} = u_{\xi\xi}\,\xi_x^2 + 2u_{\xi\eta}\,\xi_x\,\eta_x + u_{\eta\eta}\,\eta_x^2 + u_\xi\,\xi_{xx} + u_\eta\,\eta_{xx}$

$\xi_x = \xi_z\, z_x = -\xi_z$

$\eta_x = \eta_z\, z_x = -\eta_z$

$\xi_z = -2\left(\frac{1}{2}z^{-1/2}\right) = -\frac{1}{\sqrt{z}} \Rightarrow \xi_x = \frac{1}{\sqrt{z}}$

$$\eta_z = 2\left(\frac{1}{2}z^{-1/2}\right) = \frac{1}{\sqrt{z}} \Rightarrow \eta_x = -\frac{1}{\sqrt{z}}$$

$$\xi_y = 1$$

$$\eta_y = 1$$

$$\xi_{xx} = (\xi_x)_x = \left(\frac{1}{\sqrt{z}}\right)_x = \left(\frac{1}{\sqrt{z}}\right)_z z_x = -\left(-\frac{1}{2}\,z^{-3/2}\right) = \frac{1}{2z^{3/2}}$$

$$\eta_{xx} = (\eta_x)_x = \left(-\frac{1}{\sqrt{z}}\right)_x = \left(-\frac{1}{\sqrt{z}}\right)_z z_x = -\left(\frac{1}{2}\,z^{-3/2}\right) = \frac{-1}{2z^3/2}$$

$$\xi_{xy} = \xi_{yy} = \eta_{xy} = \eta_{yy} = 0$$

$$u_{xx} = \frac{1}{z}\,u_{\xi\,\xi} - \frac{2}{z}\,u_{\xi\,\eta} + \frac{1}{z}\,u_{\eta\,\eta} + \frac{1}{2z^{3/2}}\,u_\xi - \frac{1}{2z^{3/2}}\,u_\eta$$

$$u_{yy} = u_{\xi\,\xi}\,\underbrace{\xi_y^2}_{=1} + 2u_{\xi\,\eta}\,\xi_y\,\eta_y + u_{\eta\,\eta}\,\underbrace{\eta_y^2}_{=1} + u_\xi\,\underbrace{\xi_{yy}}_{=0} + u_\eta\,\underbrace{\eta_{yy}}_{=0}$$

$$= u_{\xi\,\xi} + 2u_{\xi\,\eta} + u_{\eta\,\eta}$$

Substitute in the equation

$$\underbrace{x}_{-z}\left\{\frac{1}{z}\,u_{\xi\,\xi} - \frac{2}{z}\,u_{\xi\,\eta} + \frac{1}{z}\,u_{\eta\,\eta} + \frac{1}{2z^{3/2}}\,u_\xi - \frac{1}{2z^{3/2}}\,u_\eta\right\} + u_{\xi\,\xi} + 2u_{\xi\,\eta} + u_{\eta\,\eta} = \underbrace{x^2}_{(-z)^2}$$

$$-u_{\xi\,\xi} + 2u_{\xi\,\eta} - u_{\eta\,\eta} - \frac{1}{2z^{1/2}}\,u_\xi + \frac{1}{2z^{1/2}}\,u_\eta + u_{\xi\,\xi} + 2u_{\xi\,\eta} - u_{\eta\,\eta} = z^2$$

$$4u_{\xi\,\eta} - \frac{1}{2\sqrt{z}}\,u_\xi + \frac{1}{2\sqrt{z}}\,u_\eta = z^2$$

The last step is to get rid of z

$$\xi - \eta = -4\sqrt{z} \quad \text{(using the transformation)}$$

$$\sqrt{z} = \frac{\eta - \xi}{4} \Rightarrow 2\sqrt{z} = \frac{\eta - \xi}{2}\ ;\ z = \left(\frac{\eta - \xi}{4}\right)^2$$

$$\boxed{4u_{\xi\,\eta} - \frac{2}{\eta - \xi}\,u_\xi + \frac{2}{\eta - \xi}\,u_\eta = \left(\frac{\eta - \xi}{4}\right)^4}$$

For the elliptic case $\underline{x > 0}$

$$\frac{dy}{dx} = \frac{\pm i}{\sqrt{x}}$$

$$dy = \pm i \frac{dx}{\sqrt{x}}$$

$$y = \pm i \, 2\sqrt{x} \; dx + c$$

$$\xi = y - 2i \sqrt{x}$$

$$\eta = y - 2i \sqrt{x}$$

$$\alpha = \frac{1}{2}(\xi + \eta) = y$$

$$\beta = \frac{1}{2i}(\xi - \eta) = -2\sqrt{x}$$

$$u_{xx} = u_{\alpha\alpha} \, \alpha_x^2 + 2u_{\alpha\beta} \, \alpha_x \, \beta_x + u_{\beta\beta} \, \beta_x^2 + u_\alpha \, \alpha_{xx} + u_\beta \, \beta_{xx}$$

$$u_{yy} = u_{\alpha\alpha} \, \alpha_y^2 + 2u_{\alpha\beta} \, \alpha_y \, \beta_y + u_{\beta\beta} \, \beta^2 + u_\alpha \, \alpha_{yy} + u_\beta \, \beta_{yy}$$

$$\alpha_x = 0 \; ; \; \alpha_y = 1 \, ; \, \alpha_{xx} = \alpha_{yy} = 0$$

$$\beta_x = -2 \cdot \frac{1}{2} x^{-1/2} = -x^{-1/2} \, ; \, \beta_y = 0 \, ; \, \beta_{xx} = \frac{1}{2} x^{-3/2} \, ; \, \beta_{yy} = 0$$

$$u_{xx} = u_{\beta\beta}(-x^{-1/2})^2 + u_\beta \left(\frac{1}{2}x^{-3/2}\right)$$

$$u_{yy} = u_{\alpha\alpha}$$

$$x \left[ u_{\beta\beta} \cdot x^{-1} + \frac{1}{2} u_\beta \, x^{-3/2} \right] + u_{\alpha\alpha} = x^2$$

$$\boxed{u_{\beta\beta} + u_{\alpha\alpha} + \frac{1}{2} x^{-1/2} \, u_\beta = x^2}$$

Again, substitute for x:

$$-2\sqrt{x} = \beta$$

$$\Rightarrow \sqrt{x} = -\frac{1}{2}\beta$$

$$\Rightarrow x = \frac{1}{4}\beta^2$$

$$u_{\alpha\alpha} + u_{\beta\beta} + \frac{1}{2}\frac{1}{-\frac{1}{2}\beta}\,u_\beta = \left(\frac{1}{4}\beta^2\right)^2$$

$$\boxed{u_{\alpha\alpha} + u_{\beta\beta} = \frac{1}{\beta}u_\beta + \frac{1}{16}\beta^4}$$

For the parabolic case $\underline{x = 0}$ the equation becomes:

$$0 \cdot u_{xx} + u_{yy} = 0$$

or $\boxed{u_{yy} = 0}$

which is already in a canonical form

This parabolic case can be solved. Integrate with respect to $y$ holding $x$ fixed (the constant of integration may depend on $x$)

$$u_y = f(x)$$

Integrate again:

$$\boxed{u(x, y) = y\,f(x) + g(x)}$$

1b. $u_{xx} + u_{xy} - x\,u_{yy} = 0$

$A = 1 \qquad B = 1 \qquad C = -x$

$\triangle = 1 + 4x \qquad > 0 \qquad \text{if } x > -\frac{1}{4}$ <u>hyperbolic</u>

$= 0 \qquad = -\frac{1}{4}$ <u>parabolic</u>

$< 0 \qquad < -\frac{1}{4}$ <u>elliptic</u>

$$\frac{dy}{dx} = \frac{1 \pm \sqrt{1 + 4x}}{2}$$

Consider the hyperbolic case:

$$2dy = (1 \pm \sqrt{1 + 4x})\,dx$$

Integrate to get characteristics

$$2y = x \pm \frac{2}{3} \cdot \frac{1}{4}(1 + 4x)^{3/2} + c$$

$$2y - x \mp \frac{1}{6}(1 + 4x)^{3/2} = c$$

$$\xi = 2y - x - \frac{1}{6}(1 + 4x)^{3/2}$$

$$\eta = 2y - x + \frac{1}{6}(1 + 4x)^{3/2}$$

$$\xi_x = -1 - \frac{1}{6} \cdot \frac{3}{2} \cdot 4\,(1 + 4x)^{1/2} = -1 - \sqrt{1 + 4x}$$

$$\xi_{xx} = -\frac{1}{2}(1 + 4x)^{-1/2} \cdot 4 = -2\,(1 + 4x)^{-1/2}$$

$$\xi_y = 2 \qquad \xi_{yy} = 0 \qquad \xi xy = 0$$

$$\eta_x = -1 + \frac{1}{6} \cdot \frac{3}{2} \cdot 4\,(1 + 4x)^{1/2} = -1 + \sqrt{1 + 4x}$$

$$\eta_{xx} = +2\,(1 + 4x)^{-1/2}$$

$$\eta_y = 2 \qquad \eta_{xy} = 0 \qquad \eta_{yy} = 0$$

Now we can compute the new coefficients or compute each of the derivative in the equation. We chose the latter.

$u_{xx} \;=\; u_{\xi\xi}(-1 \,-\, \sqrt{1 \,+\, 4x})^2 \,+\, 2u_{\xi\eta}(-1 \,-\, \sqrt{1 \,+\, 4x})(-1 \,+\, \sqrt{1 \,+\, 4x})$

$+\,u_{\eta\eta}(-1 \,+\, \sqrt{1 \,+\, 4x})^2 \,+\, u_\xi\,[-2\,(1 \,+\, 4x)^{-1/2}] \,+\, u_\eta\,[2\,(1 \,+\, 4x)^{-1/2}]$

$=\, u_{\xi\xi}[1 \,+\, 2\,\sqrt{1 \,+\, 4x} \,+\, 1 \,+\, 4x] \,+\, 2u_{\xi\eta}(1 \,-\, (1 \,+\, 4x))$

$+\,u_{\eta\eta}[1 \,-\, 2\,\sqrt{1 \,+\, 4x} \,+\, 1 \,+\, 4x] \,-\, 2\,(1 \,+\, 4x)^{-1/2}\,u_\xi \,+\, 2(1 \,+\, 4x)^{-1/2}\,u_\eta$

$u_{xy} \;=\; 2u_{\xi\xi}\,(-1 \,-\, \sqrt{1 \,+\, 4x}) \,+\, u_{\xi\eta}\,[2\,(-1 \,-\, \sqrt{1 \,+\, 4x}) \,+\, 2\,(-1 \,+\, \sqrt{1 \,+\, 4x})]$

$+\,u_{\eta\eta}\,2(-1 \,+\, \sqrt{1 \,+\, 4x})$

$u_{yy} \;=\; 4u_{\xi\xi} \,+\, 2u_{\xi\eta}\,\cdot\,4 \,+\, u_{\eta\eta}\,\cdot\,4$

$\Rightarrow\; u_{xx} \,+\, u_{xy} \,-\, x\,u_{yy} \;=$

$u_{\xi\xi}\,[2 \,+\, 4x \,+\, 2\,\sqrt{1 \,+\, 4x}] \,+\, 2u_{\xi\eta}\,(-\,4x) \,+\, u_{\eta\eta}\,(2 \,+\, 4x \,-\, 2\,\sqrt{1 \,+\, 4x})$

$-\,2\,(1 \,+\, 4x)^{-1/2}\,u_\xi \,+\, 2\,(1 \,+\, 4x)^{-1/2}\,u_\eta$

$+\,2u_{\xi\xi}\,(-1 \,-\, \sqrt{1 \,+\, 4x}) \,+\, u_{\xi\eta}\,(-4) \,+\, 2u_{\eta\eta}\,(-1 \,+\, \sqrt{1 \,+\, 4x})\,-$

$4x\,(u_{\xi\xi} \,+\, 2u_{\xi\eta} \,+\, u_{\eta\eta}) \;=$

$(2 + 4x + 2\sqrt{1+4x} - 2 - 2\sqrt{1+4x} - 4x)\,u_{\xi\xi} \,+\, (-8x - 4 - 8x)u_{\xi\eta}$

$+\,(2 + 4x - 2\sqrt{1+4x} - 2 - 2\sqrt{1+4x} - 4x)\,u_{\eta\eta} \,-\, 2(1 \,+\, 4x)^{-1/2}\;(u_\xi \,-\, u_\eta) \;=\; 0$

$-4\,(1+4x)u_{\xi\eta} \,-\, 2(1 \,+\, 4x)^{-1/2}\;(u_\xi \,-\, u_\eta) \;=\; 0$

$u_{\xi\eta} \,+\, \dfrac{1}{2}(1 \,+\, 4x)^{-3/2}\;(u_\xi \,-\, u_\eta) \;=\; 0$

Now find $(1 \,+\, 4x)^{-3/2}$ in terms of $\xi$, $\eta$ and substitute

$\xi \,-\, \eta \;=\; -\dfrac{1}{3}\,(1 \,+\, 4x)^{3/2}$

$3(\eta \,-\, \xi) \;=\; (1 \,+\, 4x)^{3/2}$

$(1 \,+\, 4x)^{-3/2} \;=\; [3(\eta \,-\, \xi)]^{-1}$

$u_{\xi\eta} \;=\; -\dfrac{1}{2[3(\eta \,-\, \xi)]}\,(u_\xi \,-\, u_\eta)$

$$u_{\xi\eta} = \frac{1}{6(\eta - \xi)}(u_\eta - u_\xi)$$

The parabolic case is easier, the only characteristic is

$$y = \frac{1}{2}x + K$$

and so the transformation is

$$\xi = y - \frac{1}{2}x$$

$$\eta = x$$

The last equation is an arbitrary function and one should check the Jacobian. The details are left to the reader. One can easily show that

$$A^* = B^* = 0$$

Also

$$C^* = 1$$

and the rest of the coefficients are zero. Therefore the equation is

$$u_{\eta\eta} = 0$$

In the elliptic case, one can use the transformation $z = -(1+4x)$ so that the characteristic equation becomes

$$\frac{dy}{dx} = \frac{1 \pm \sqrt{z}}{2}$$

or if we eliminate the $x$ dependence

$$\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz} = -\frac{1}{4}\frac{1 \pm \sqrt{z}}{2}$$

Now integrate, and take the real and imaginary part to be the functions $\xi$ and $\eta$. The rest is left for the reader.

1c. $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} + xy\, u_x + y^2 u_y = 0$

$A = x^2 \qquad\qquad B = 2xy \qquad\qquad C = y^2$

$\triangle = 4x^2 y^2 - 4x^2 y^2 = 0$ <u>parabolic</u>

$$\frac{dy}{dx} = \frac{2xy}{2x^2} = \frac{y}{x}$$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\xi = \ell n\, y - \ell n\, x \quad \Rightarrow \quad \xi = \ell n \left(\frac{y}{x}\right) \Rightarrow e^{\xi} = \frac{y}{x}$$

$\eta = x$ arbitrarily chosen since this is parabolic

$$\xi_x = \frac{-1}{x} \qquad \xi_y = \frac{1}{y} \qquad \xi_{xx} = \frac{1}{x^2} \qquad \xi_{xy} = 0 \qquad \xi_{yy} = -\frac{1}{y^2}$$

$$\eta_x = 1 \quad \eta_y = \eta_{xx} = \eta_{xy} = \eta_{yy} = 0$$

$$u_{xx} = \frac{1}{x^2} u_{\xi\xi} + 2u_{\xi\eta}(-\frac{1}{x}) + u_{\eta\eta} + \frac{1}{x^2} u_\xi$$

$$u_{xy} = -\frac{1}{xy} u_{\xi\xi} + u_{\xi\eta}\frac{1}{y}$$

$$u_{yy} = \frac{1}{y^2} u_{\xi\xi} - \frac{1}{y^2} u_\xi$$

$$u_{\xi\xi} - 2xu_{\xi\eta} + x^2 u_{\eta\eta} + u_\xi - 2u_{\xi\xi} + 2xu_{\xi\eta} + u_{\xi\xi} - u_\xi +$$

$$xy(-\frac{1}{x} u_\xi + u_\eta) + y^2(\frac{1}{y} u_\xi) = 0$$

$$x^2 u_{\eta\eta} + xyu_\eta = 0$$

$\boxed{u_{\eta\eta} = -e^{\xi} u_\eta}$ $\qquad y = e^{\xi} x$ therefore $y/x = e^{\xi}$

This equation can be solved.

1d. $u_{xx} + x\,u_{yy} = 0$

$A = 1 \qquad B = 0 \qquad C = x$

$\triangle = -4x \qquad > 0 \qquad$ if $x < 0$ <u>hyperbolic</u>

$= 0 \qquad x = 0$ <u>parabolic</u>

$< 0 \qquad x > 0$ <u>elliptic</u>

<u>Parabolic</u> $x = 0 \quad \Rightarrow \quad u_{xx} = 0$ already in canonical form

<u>Hyperbolic</u> $x < 0$ Let $\zeta = -x$

$$\triangle = 4\zeta > 0$$

$\dfrac{dy}{dx} = \pm \dfrac{2\sqrt{\zeta}}{2} = \pm\sqrt{\zeta}$ Note: $dx = -d\zeta$

$dy = \pm\sqrt{\zeta}\,(-d\zeta)$

$y \pm \dfrac{2}{3}\zeta^{3/2} = c$

$\xi = y + \dfrac{2}{3}\zeta^{3/2}$

$\eta = y - \dfrac{2}{3}\zeta^{3/2}$

Continue as in example in class (See 1a)

1e. $u_{xx} + y^2 u_{yy} = y$

$A = 1 \qquad\qquad B = 0 \qquad\qquad C = y^2$

$\triangle = -4y^2 < 0$ $\underline{\text{elliptic}}$ if $y \neq 0$

For $y = 0$ the equation is $\underline{\text{parabolic}}$ and it is in canonical form $u_{xx} = 0$

$$\frac{dy}{dx} = \frac{\pm\sqrt{-4y^2}}{2} = \pm iy$$

$$\frac{dy}{y} = \pm i dx$$

$$\ln y = \pm ix + c$$

$$\xi = \ln y + ix$$

$$\eta = \ln y - ix$$

$$\alpha = \ln y \qquad \alpha_x = 0 \qquad \alpha_y = \frac{1}{y}$$

$$\beta = x \qquad \beta_x = 1 \qquad \beta_y = 0$$

$$u_x = u_\beta \beta_x + u_\alpha \alpha_x = u_\beta$$

$$u_y = u_\alpha \frac{1}{y} + u_\beta \beta_y = \frac{1}{y} u_\alpha$$

$$u_{xx} = (u_\beta)_x = u_{\beta\beta}$$

$$u_{yy} = \left(\frac{1}{y}\right)_y u_\alpha + \frac{1}{y}(u_\alpha)_y = -\frac{1}{y^2} u_\alpha + \frac{1}{y^2} u_{\alpha\alpha}$$

$$\Rightarrow \quad u_{\beta\beta} + y^2\left(-\frac{1}{y^2} u_\alpha + \frac{1}{y^2} u_{\alpha\alpha}\right) = y$$

$$u_{\alpha\alpha} + u_{\beta\beta} - u_\alpha = y$$

But $y = e^\alpha$

$$\boxed{\Rightarrow \quad u_{\alpha\alpha} + u_{\beta\beta} - u_\alpha = e^\alpha}$$

1f. $\sin^2 x\, u_{xx} + \sin 2x\, u_{xy} + \cos^2 x\, u_{yy} = x$

$A = \sin^2 x \qquad B = \sin 2x = 2\sin x \cos x \qquad C = \cos^2 x$

$\triangle = 0$ <u>parabolic</u>

$$\frac{dy}{dx} = \frac{2\sin x \cos x}{2\sin^2 x} = \cot x$$

$y = \ln \sin x + c$

$\xi = y - \ln \sin x \qquad \xi_x = -\cot x \qquad \xi_y = 1$

$\eta = y \qquad \eta_x = 0 \qquad \eta_y = 1$

$u_x = -\cot x\, u_\xi + u_\eta \eta_x = -\cot x\, u_\xi$

$u_y = u_\xi + u_\eta$

$$u_{xx} = (-\cot x\, u_\xi)_x = \frac{1}{\sin^2 x} u_\xi + \cot^2 x\, u_{\xi\xi}$$

$u_{xy} = -\cot x\, (u_\xi)_y = -\cot x\, (u_{\xi\xi} + u_{\xi\eta})$

$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$$L.\ H.\ S. = u_\xi + \sin^2 x \frac{\cos^2 x}{\sin^2 x} u_{\xi\xi} + 2\sin x \cos x\, (-\cot x)(u_{\xi\xi} + u_{\xi\eta})$$

$$+ \cos^2 x\, (u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta})$$

$L.\ H.\ S. = \cos^2 x\, u_{\eta\eta} + u_\xi$

Therefore the equation becomes:

$\cos^2 x\, u_{\eta\eta} + u_\xi = x$

$\ln \sin x = y - \xi = \eta - \xi$

$\sin x = e^{\eta - \xi} \quad \Rightarrow \quad \cos^2 x = 1 - \sin^2 x = 1 - e^{2(\eta - \xi)}$

$x = \arcsin e^{\eta - \xi}$

$$\boxed{[1 - e^{2(\eta - \xi)}]\, u_{\eta\eta} + u_\xi = \arcsin e^{\eta - \xi}}$$

2a. $y \pm 2\sqrt{z} = c \qquad z > 0$

eq: $y + 2 * sqrt(z) = c$; $\longleftarrow$ maple command to give the equation

char:=solve (eq, $y$); $\longleftarrow$ maple command to solve for $y$

chars:=seq (char, $c$= -5..5); $\longleftarrow$ maple command to create several characteristic curves for a variety of $c$'s.

plot ( {chars} , $z = 0..10, y = -5..5$); $\longleftarrow$ maple command to plot all those curves

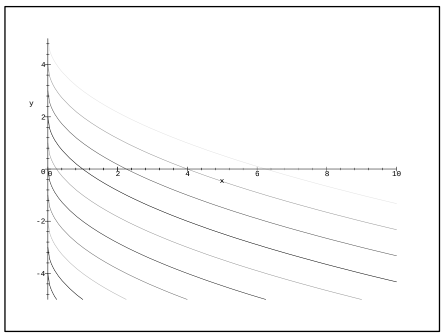

Figure 35: Maple plot of characteristics for 6.2 2a

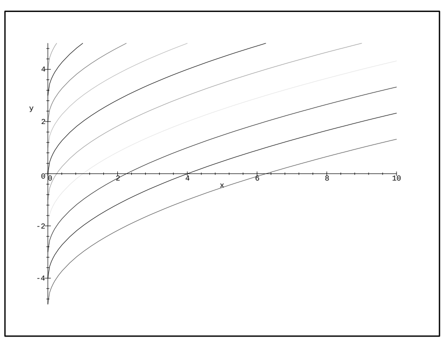

Figure 36: Maple plot of characteristics for 6.2 2a

2b. $y = \frac{1}{2}x \pm \frac{1}{12}(1 + 4x)^{3/2} + c$

$$1 + 4x \geq 0$$

$$4x \geq -1$$

$$x \geq -.25$$

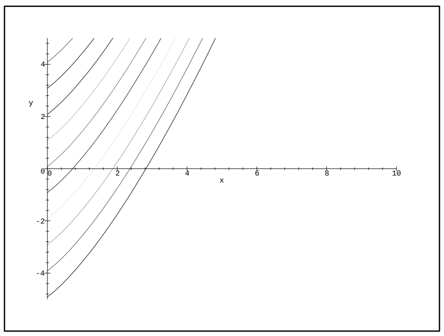


Figure 37: Maple plot of characteristics for 6.2 2b

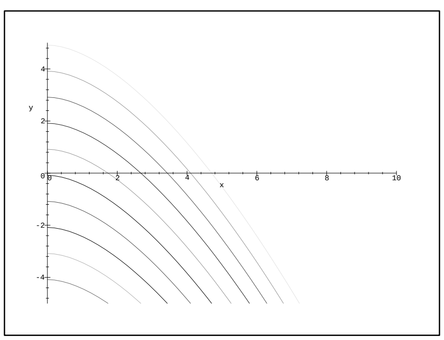


Figure 38: Maple plot of characteristics for 6.2 2b

2c. $\ln \dfrac{y}{x} = c$ parabolic

$$\ln y = xe^c = kx$$

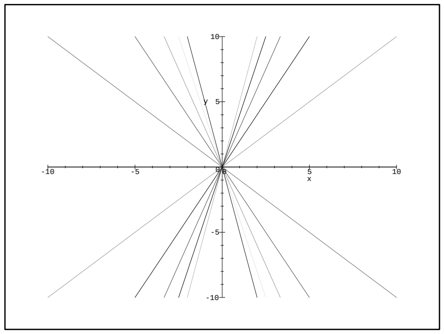


Figure 39: Maple plot of characteristics for 6.2 2c

2d. $y \pm \dfrac{2}{3} z^{3/2} = c$

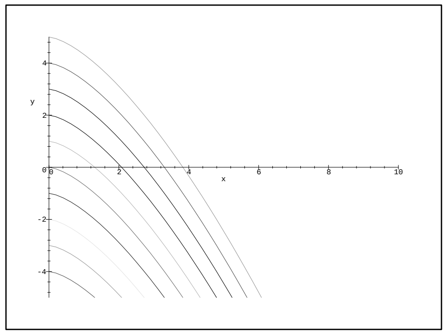


Figure 40: Maple plot of characteristics for 6.2 2d

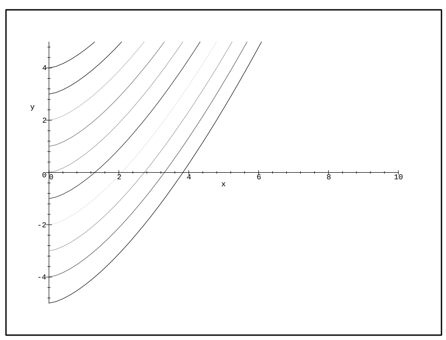


Figure 41: Maple plot of characteristics for 6.2 2d

2e. elliptic. no real characteristic

2f. $y = \ln \sin x + c$

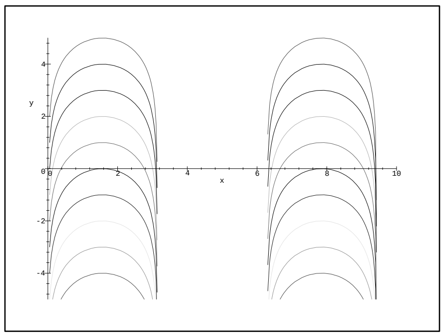


Figure 42: Maple plot of characteristics for 6.2 2f

3. a. $\dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial u}{\partial x} = -e^{-kt}$

$$A = 1,\ B = 0,\ C = 1.$$

The discriminant $\Delta = 0^2 - 4 \cdot 1 \cdot 1 = -4 < 0$

Therefore the problem is elliptic.

3. b. $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial u}{\partial y} = 4$

$$A = 1,\ B = -1,\ C = 0.$$

The discriminant $\Delta = (-1)^2 - 4 \cdot 1 \cdot 0 = 1 > 0$

Therefore the problem is hyperbolic.

4. a. $\dfrac{\partial^2 u}{\partial x^2} + 3\dfrac{\partial^2 u}{\partial x \partial y} + 2\dfrac{\partial^2 u}{\partial y^2} = 0$

$$A = 1,\ B = 3,\ C = 2.$$

The discriminant $\Delta = 3^2 - 4 \cdot 1 \cdot 2 = 9 - 8 = 1 > 0$

Therefore the problem is hyperbolic.

The characteristic equation is

$$\frac{dy}{dx} = \frac{B \pm \sqrt{\Delta}}{2A} = \frac{3 \pm 1}{2}$$

The first equation is

$$\frac{dy}{dx} = 2$$

and the second is

$$\frac{dy}{dx} = 1$$

Integrating, we get

$$y = 2x + C$$

and

$$y = x + D$$

Both characteristic families are straight lines.

4. b. $\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$$A = 1,\ B = -2,\ C = 1.$$

The discriminant $\Delta = (-2)^2 - 4 \cdot 1 \cdot 1 = 4 - 4 = 0$

Therefore the problem is parabolic.

The characteristic equation is

$$\frac{dy}{dx} = \frac{B \pm \sqrt{\Delta}}{2A} = \frac{-2}{2} = -1$$

Integrating, we get

$$y = -x + C$$

The characteristic family is straight lines.

5. a. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} \; = \; 0$

$$A = 1, \; B = 1, \; C = 1.$$

The discriminant $\Delta \; = \; 1^2 - 4 \cdot 1 \cdot 1 \; = \; 1 - 4 \; = \; -3 < 0$
Therefore the problem is elliptic.
The characteristic equation is

$$\frac{dy}{dx} \; = \; \frac{B \pm \sqrt{\Delta}}{2A} \; = \; \frac{1 \pm i\sqrt{3}}{2}$$

The solutions are

$$y = \frac{1}{2}x \pm i\frac{\sqrt{3}}{2}x + C$$

The transformation is

$$\xi = y - \frac{1}{2}x - i\frac{\sqrt{3}}{2}x$$

$$\eta = y - \frac{1}{2}x + i\frac{\sqrt{3}}{2}x$$

In elliptic problems we use another transformation (to stay with real functions)

$$\alpha = y - \frac{1}{2}x$$

$$\beta = \frac{\sqrt{3}}{2}x$$

Since these are linear functions, we only need the first partials

$$\alpha_x = -\frac{1}{2}, \; \alpha_y = 1$$

$$\beta_x = \frac{\sqrt{3}}{2}, \; \beta_y = 0$$

Use the formulae for $A^*$, $B^*$ etc with $\alpha$ for $\xi$ and $\beta$ for $\eta$, we have

$$A^* = 1 \cdot (-\frac{1}{2})^2 + 1 \cdot (-\frac{1}{2}) \cdot 1 + 1 \cdot 1^2 = \frac{3}{4}$$

$$B^* = 0 \;\; \text{as should be for elliptic}$$

$$C^* = A^* = \frac{3}{4}$$

The rest are zero (since they were zero and the transformation is linear)

Thus the canonical form is

$$\frac{3}{4}\left(u_{\alpha\alpha}+u_{\beta\beta}\right)=0$$

or

$$u_{\alpha\alpha}+u_{\beta\beta}=0$$

5. b. $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + 5\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} \;=\; 0$

$$A = 1,\; B = -2,\; C = 5,\; D = 0,\; E = 1,\; F = G = 0.$$

The discriminant $\Delta \;=\; (-2)^2 - 4 \cdot 1 \cdot 5 \;=\; 4 - 20 \;=\; -16 < 0$
Therefore the problem is elliptic.
The characteristic equation is

$$\frac{dy}{dx} \;=\; \frac{B \pm \sqrt{\Delta}}{2A} \;=\; \frac{-2 \pm 4i}{2} \;=\; -1 + 2i$$

The solutions are

$$y = -x \pm 2ix + C$$

The transformation is

$$\xi = y + x + 2ix$$

$$\eta = y + x - 2ix$$

In elliptic problems we use another transformation (to stay with real functions)

$$\alpha = y + x$$

$$\beta = 2x$$

Since these are linear functions, we only need the first partials

$$\alpha_x = 1,\; \alpha_y = 1$$

$$\beta_x = 2,\; \beta_y = 0$$

Use the formulae for $A^*$, $B^*$ etc with $\alpha$ for $\xi$ and $\beta$ for $\eta$, we have

$$A^* = 1 \cdot 1^2 - 2 \cdot 1 \cdot 1 + 5 \cdot 1^2 = 1 - 2 + 5 = 4$$

$$B^* = 0 \;\text{ as should be for elliptic}$$

$$C^* = A^* = 4$$

$$D^* = 1 \cdot 1 = 1$$

$$E^* = 1 \cdot 0 = 0$$

$$F^* = G^* = 0$$

Thus the canonical form is

$$4\left(u_{\alpha\alpha}+u_{\beta\beta}\right)+u_{\alpha}=0$$

or

$$u_{\alpha\alpha}+u_{\beta\beta}=-\frac{1}{4}u_{\alpha}$$

6. a. $\frac{\partial^2 u}{\partial x^2} - 6\frac{\partial^2 u}{\partial x \partial y} + 9\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} - e^{xy} \ = \ 1$

$$A = 1,\ B = -6,\ C = 9,\ D = 1,\ E = F = 0,\ G = 1 + e^{xy}.$$

The discriminant $\Delta \ = \ (-6)^2 - 4 \cdot 1 \cdot 9 \ = \ 36 - 36 \ = \ 0$

Therefore the problem is parabolic.

The characteristic equation is

$$\frac{dy}{dx} \ = \ \frac{B \pm \sqrt{\Delta}}{2A} \ = \ \frac{-6}{2} \ = \ -3$$

The solution is

$$y = -3x + C$$

The transformation is

$$\xi = y + 3x$$

$$\eta = y \ \text{ arbitrary for parabolic}$$

Since these are linear functions, we only need the first partials

$$\xi_x = 3,\ \xi_y = 1$$

$$\eta_x = 0,\ \eta_y = 1$$

Note that the Jacobian of the transformation is NOT zero.

Use the formulae for $A^*$, $B^*$ etc, we have

$$A^* = 1 \cdot 9 - 6 \cdot 3 \cdot 1 + 9 \cdot 1 = 0$$

$$B^* = 0 \ \text{ as should be for parabolic}$$

$$C^* = 0 - 0 + 9 = 9$$

$$D^* = 1 \cdot 3 = 3$$

$$E^* = 0$$

$$F^* = 0$$

$$G^* = 1 + e^{xy}$$

Need to substitute for $x, y$ into $G^*$.

Note that from the transformation $y = \eta$ and $3x = \xi - \eta$, so

$$G^* = 1 + e^{\eta(\xi - \eta)/3}$$

Thus the canonical form is

$$u_{\eta\eta} + \frac{1}{3}u_\xi = \frac{1}{9}\left(1 + e^{\eta(\xi-\eta)/3}\right)$$

6. b. $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} + 7\dfrac{\partial u}{\partial x} - 8\dfrac{\partial u}{\partial y} \;=\; 0$

$$A = 1,\; B = 2,\; C = 1,\; D = 7,\; E = -8,\; F = G = 0.$$

The discriminant $\Delta \;=\; 2^2 - 4 \cdot 1 \cdot 1 \;=\; 4 - 4 \;=\; 0$

Therefore the problem is parabolic.

The characteristic equation is

$$\frac{dy}{dx} \;=\; \frac{B \pm \sqrt{\Delta}}{2A} \;=\; \frac{2}{2} \;=\; 1$$

The solution is

$$y = x + C$$

The transformation is

$$\xi = y - x$$

$$\eta = y \;\;\text{arbitrary for parabolic}$$

Since these are linear functions, we only need the first partials

$$\xi_x = -1,\; \xi_y = 1$$

$$\eta_x = 0,\; \eta_y = 1$$

Note that the Jacobian of the transformation is NOT zero.

Use the formulae for $A^*$, $B^*$ etc, we have

$$A^* = 0$$

$$B^* = 0 \;\;\text{as should be for parabolic}$$

$$C^* = 0 + 0 + 1 = 1$$

$$D^* = 7 \cdot (-1) - 8 \cdot 1 = -15$$

$$E^* = -8 \cdot 1 = -8$$

$$F^* = 0$$

$$G^* = 0$$

Thus the canonical form is

$$u_{\eta\eta} - 15u_\xi - 8u_\eta = 0$$

## 6.4 Equations with Constant Coefficients

### Problems

1. Find the characteristic equation, characteristic curves and obtain a canonical form for

   a. $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$
   b. $u_{xx} + u_{xy} + u_{yy} + u_x = 0$
   c. $3u_{xx} + 10u_{xy} + 3u_{yy} = x + 1$
   d. $u_{xx} + 2u_{xy} + 3u_{yy} + 4u_x + 5u_y + u = e^x$
   e. $2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0$
   f. $u_{xx} + 5u_{xy} + 4u_{yy} + 7u_y = \sin x$

2. Use Maple to plot the families of characteristic curves for each of the above.

1a. $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$

$A = 4$

$B = 5$

$C = 1$

$\triangle = 5^2 - 4 \cdot 4 \cdot 1 = 25 - 16 = 9 > 0$ <u>hyperbolic</u>

$$\frac{dy}{dx} = \frac{5 \pm \sqrt{9}}{2 \cdot 4} = \frac{5 \pm 3}{8} \begin{matrix} \nearrow 1/4 \\ \searrow 1 \end{matrix}$$

$$dy = dx \qquad dy = \frac{1}{4}\,dx$$

$$y = x + c \qquad y = \frac{1}{4}\,x + c$$

$$\xi = y - x \qquad \eta = y - \frac{1}{4}x$$

$$\xi_x = -1 \qquad \xi_y = 1 \qquad \eta_x = -\frac{1}{4} \qquad \eta_y = 1$$

$$\xi_{xx} = 0 \qquad \xi_{yy} = 0 \qquad \xi_{xy} = 0 \qquad \eta_{xx} = 0 \qquad \eta_{yy} = 0 \qquad \eta_{xy} = 0$$

$$u_{xx} = u_{\xi\xi}\,(-1)^2 + 2u_{\xi\eta}\,(-1)\left(-\frac{1}{4}\right) + u_{\eta\eta}\left(-\frac{1}{4}\right)^2 + u_\xi \cdot 0 + u_\eta \cdot 0$$

$$u_{yy} = u_{\xi\xi} \cdot 1^2 + 2u_{\xi\eta} \cdot 1 \cdot 1 \quad + u_{\eta\eta} \cdot 1^2 + u_\xi \cdot 0 + u_\eta \cdot 0$$

$$u_{xy} = u_{\xi\xi}\,(-1) \cdot 1 + u_{\xi\eta}\left(-1 \cdot 1 + 1 \cdot \left(-\frac{1}{4}\right)\right) + u_{\eta\eta}\left(-\frac{1}{4}\right) \cdot 1 + u_\xi \cdot 0 \cdot + u_\eta \cdot 0$$

$$u_x = u_\xi\,(-1) + u_\eta\left(-\frac{1}{4}\right)$$

$$u_y = u_\xi \cdot 1 + u_\eta \cdot 1$$

$$4\,u_{\xi\xi} + 2u_{\xi\eta} + \frac{1}{4}u_{\eta\eta} - 5\,u_{\xi\xi} - \frac{25}{4}\,u_{\xi\eta} - \frac{5}{4}\,u_{\eta\eta} + u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$- u_\xi - \frac{1}{4}\,u_\eta + u_\xi + u_\eta \qquad = 2$$

All $u_{\xi\xi}$, $u_{\eta\eta}$ and $u_\xi$ terms cancel out

$$-\frac{9}{4}\,u_{\xi\eta} + \frac{3}{4}\,u_\eta = 2$$

$$\boxed{u_{\xi\eta} = \tfrac{1}{3}u_\eta - \tfrac{8}{9}}$$

This equation can be solved as follows:

Let $\nu = u_\eta$ then $u_{\xi\eta} = \nu_\xi$

$\nu_\xi = \frac{1}{3}\nu - \frac{8}{9}$

This is Linear $1^{st}$ order ODE

$$\nu' - \frac{1}{3}\nu = -\frac{8}{9}$$

Integrating factor is $e^{-\frac{1}{3}\xi}$

$(\nu e^{-\frac{1}{3}\xi})' = -\dfrac{8}{9}e^{-\frac{1}{3}\xi}$

$\nu e^{-\frac{1}{3}\xi} = -\dfrac{8}{9}\displaystyle\int e^{-\frac{1}{3}\xi}d_\xi = \dfrac{8}{3}e^{-\frac{1}{3}\xi} + C(\eta)$

$\boxed{\nu = \frac{8}{3} + C(\eta)e^{\frac{1}{3}\xi}}$

To find $u$ we integrate with respect to $\eta$

$$u_\eta = \frac{8}{3} + C(\eta)e^{\frac{1}{3}\xi}$$

$$u = \frac{8}{3}\eta + e^{\frac{1}{3}\xi}\underbrace{c_1(\eta)}_{\text{integral of } C(\eta)} + K(\xi)$$

To check the solution, we differentiate it and substitute in the canonical form:

$$u_\xi = 0 + \frac{1}{3}e^{\frac{1}{3}\xi}c_1(\eta) + K'(\xi)$$

$$u_{\xi\eta} = \frac{1}{3}e^{\frac{1}{3}\xi}c_1'(\eta)$$

$$u_\eta = \frac{8}{3} + e^{\frac{1}{3}\xi}c_1'(\eta)$$

$$\Rightarrow \quad \frac{1}{3}\,u_\eta \,=\, \frac{8}{9} \,+\, \frac{1}{3}\,e^{\frac{1}{3}\,\xi}\,c_1'\,(\eta)$$

Substitute in the PDE in canonical form

$$\frac{1}{3}\,e^{\frac{1}{3}\,\xi}\,c_1'\,(\xi) \,=\, \frac{8}{9} \,+\, \frac{1}{3}\,e^{\frac{1}{3}\,\xi}\,c_1'\,(\eta) \,-\, \frac{8}{9}$$

Identity

| |
|---|
| In terms of original variables $\quad u(x,\,y) \,=\, \frac{8}{3}\,(y \,-\, \frac{1}{4}x) \,+\, e^{\frac{1}{3}\,(y-x)}\,c_1\,(y \,-\, \frac{1}{4}x) \,+\, K\,(y-x)$ |

1b. $u_{xx} + u_{xy} + u_{yy} + u_x = 0$

$A = 1 \qquad B = 1 \qquad C = 1 \qquad \triangle = 1 - 4 = -3 < 0$ elliptic

$$\frac{dy}{dx} = \frac{1 \pm \sqrt{-3}}{2}$$

$$2dy = (1 \pm \sqrt{3}i)\, dx$$

$$\xi = 2y - (1 + \sqrt{3}i)x \qquad \eta = 2y - (1 - \sqrt{3}i)x$$

$$\alpha = \frac{1}{2}(\xi + \eta) = 2y - x$$

$$\beta = \frac{1}{2i}(\xi - \eta) = -\sqrt{3}x$$

$$\alpha_x = -1 \qquad \alpha_y = 2 \qquad \alpha_{xx} = 0 \qquad \alpha_{xy} = 0 \qquad \alpha_{yy} = 0$$

$$\beta_x = -\sqrt{3} \qquad \beta_y = 0 \qquad \beta_{xx} = 0 \qquad \beta_{xy} = 0 \qquad \beta_{yy} = 0$$

$$u_{\alpha\alpha} + 2u_{\alpha\beta}(-1)(-\sqrt{3}) + u_{\beta\beta} \cdot 3 + \underbrace{u_{\alpha\alpha}(-2) + u_{\alpha\beta}(-2\sqrt{3})}_{u_{xy}} + 4u_{\alpha\alpha} - u_\alpha - \sqrt{3}u_\beta = 0$$

$$3u_{\alpha\alpha} + 3u_{\beta\beta} - u_\alpha - \sqrt{3}\, u_\beta = 0$$

$$\boxed{u_{\alpha\alpha} + u_{\beta\beta} = \frac{1}{3}u_\alpha + \frac{\sqrt{3}}{3}\, u_\beta}$$

1c. $3u_{xx} + 10u_{xu} + 3u_{yy} = x + 1$

$A = C = 3 \qquad B = 10 \qquad \triangle = 100 - 36 = 64 > 0$ <u>hyperbolic</u>

$$\frac{dy}{dx} = \frac{10 \pm 8}{6} \begin{array}{l} \nearrow 3 \\ \searrow 1/3 \end{array}$$

$$\xi = y - 3x \qquad \eta = y - \frac{1}{3}x$$

$$\xi_x = -3 \qquad \xi_y = 1 \qquad \xi_{xx} = 0 \qquad \xi_{xy} = 0 \qquad \xi_{yy} = 0$$

$$\eta_x = -\frac{1}{3} \qquad \eta_y = 1 \qquad \eta_{xx} = 0 \qquad \eta_{xy} = 0 \qquad \eta_{yy} = 0$$

$$3\left(u_{\xi\xi}(-3)^2 + 2u_{\xi\eta}(-3)\left(-\frac{1}{3}\right) + u_{\eta\eta}\left(-\frac{1}{3}\right)^2\right)$$

$$+10\left(u_{\xi\xi}(-3) + u_{\xi\eta}\left(-3 - \frac{1}{3}\right) + u_{\eta\eta}\left(-\frac{1}{3}\right)\right)$$

$$+3\left(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\right) = x + 1$$

$$-\tfrac{64}{3}u_{\xi\eta} = x + 1$$

$$\left.\begin{array}{l} \xi = y - 3x \\ \underline{\eta = y - \frac{1}{3}x} \end{array}\right\} -$$

$$\xi - \eta = -\frac{8}{3}x$$

$$x = \frac{3}{8}(\eta - \xi)$$

$$-\frac{64}{3}u_{\xi\eta} = \frac{3}{8}(\eta - \xi) + 1$$

$$\boxed{u_{\xi\eta} = -\tfrac{9}{512}(\eta - \xi) - \tfrac{3}{64}}$$

To Find the general solution !

$$u_{\xi\eta} = -\frac{9}{512}(\eta - \xi) - \frac{3}{64}$$

$$u_{\xi} = -\frac{9}{512}(\frac{1}{2}\eta^2 - \eta\xi) - \frac{3}{64}\eta + f(\xi)$$

$$u = -\frac{9}{512}(\frac{1}{2}\eta^2\xi - \frac{1}{2}\xi^2\eta) - \frac{3}{64}\eta\xi + F(\xi) + G(\eta)$$

$$= \frac{9}{512}\cdot\frac{1}{2}\eta\xi(\xi - \eta) - \frac{3}{64}\eta\xi + F(\xi) + G(\eta)$$

$$u(x, y) = \frac{9}{1024}\left(y - \frac{1}{3}x\right)(y - 3x)\underbrace{\left(\frac{1}{3}x - 3x\right)}_{-\frac{8}{3}x} - \frac{3}{64}\left(y - \frac{1}{3}x\right)(y - 3x)$$

$$+ F(y - 3x) + G(y - \tfrac{1}{3}x)$$

$$= \frac{9}{1024}\cdot\frac{-8}{3}x\left(y - \frac{1}{3}x\right)(y - 3x) - \frac{3}{64}\left(y - \frac{1}{3}x\right)(y - 3x) + F(y - 3x)$$
$$+ G(y - \tfrac{1}{3}x)$$

$$\boxed{u(x, y) = \left(-\frac{3}{128}x - \frac{3}{64}\right)(y - \frac{1}{3}x)(y - 3x) + F(y - 3x) + G\left(y - \frac{1}{3}x\right)}$$

check !

$$u_x = -\frac{3}{128}(y - \frac{1}{3}x)(y - 3x) + (-\frac{3}{128}x - \frac{3}{64})(-\frac{1}{3})(y - 3x)$$

$$+ \left(-\frac{3}{128}x - \frac{3}{64}\right)\left(y - \frac{1}{3}x\right)(-3) - 3F'(y - 3x) - \frac{1}{3}G'\left(y - \frac{1}{3}x\right)$$

$$u_y = \left(-\frac{3}{128}x - \frac{3}{64}\right)(y - 3x) + \left(-\frac{3}{128}x - \frac{3}{64}\right)\left(y - \frac{1}{3}x\right) + F'(y - 3x) + G'\left(y - \frac{1}{3}x\right)$$

$$u_{xx} = -\frac{3}{128}\left(-\frac{1}{3}\right)(y - 3x) + \frac{9}{128}(y - \frac{1}{3}x) + (-\frac{3}{128}x - \frac{3}{64}) - \frac{1}{3}(-\frac{3}{128})(y - 3x)$$

$$-3\left(-\frac{3}{128}\right)\left(y - \frac{1}{3}x\right) - 3(-\frac{1}{3})(-\frac{3}{128}x - \frac{3}{64}) + 9F'' + \frac{1}{9}G''$$

$$u_{xx} = \frac{1}{64}(y - 3x) + \frac{9}{64}(y - \frac{1}{3}x) + 2(-\frac{3}{128}x - \frac{3}{64}) + 9F''(y - 3x) + \frac{1}{9}G''(y - \frac{1}{3}x)$$

$$u_{xy} \,=\, -\frac{3}{128}\,(y \,-\, 3x) \,-\, 3\,(-\frac{3}{128}x \,-\, \frac{3}{64}) \,-\, \frac{3}{128}\,(y - \frac{1}{3}x) \,-\, \frac{1}{3}\,(-\frac{3}{128}x \,-\, \frac{3}{64})$$

$$-\,3F''\,(y \,-\, 3x) \,-\, \frac{1}{3}\,G''\,(y - \frac{1}{3}x)$$

$$u_{yy} \,=\, -\frac{3}{128}x \,-\, \frac{3}{64} \,-\, \frac{3}{128}x - \frac{3}{64} \,+\, F''\,(y \,-\, 3x) \,+\, G''\left(y - \frac{1}{3}x\right)$$

$$3u_{xx} + 10u_{xy} + 3u_{yy} \,=\, \frac{3}{64}\,(y - 3x) + \frac{27}{64}\,(y - \frac{1}{3}x) + 6\left(-\frac{3}{128}x \,-\, \frac{3}{64}\right) + 27F'' + \frac{1}{3}\,G''$$

$$-\frac{30}{128}\,(y \,-\, 3x) \,-\, \frac{15}{64}\,(y - \frac{1}{3}x) \,-\, \frac{100}{3}\left(-\frac{3}{128}x \,-\, \frac{3}{64}\right) \,-\, 30F'' \,-\, \frac{10}{3}\,G''$$

$$+\,6\left(-\frac{3}{128}x \,-\, \frac{3}{64}\right) \,+\, 3F'' \,+\, 3G''$$

$$=\, -\,\frac{12}{64}\,(y \,-\, 3x) + \frac{12}{64}\,(y - \frac{1}{3}x) \,-\, \frac{64}{3}\,(-\frac{3}{128}x \,-\, \frac{3}{64})$$

$$=\, \frac{9}{16}x \,-\, \frac{1}{16}x \,+\, \frac{1}{2}x \,+\, 1 \qquad =\, x \,+\, 1$$

checks

1d. $u_{xx} + 2u_{xy} + 3u_{yy} + 4u_x + 5u_y + u = e^x$

$A = 1 \qquad B = 2 \qquad C = 3 \qquad \triangle = 4 - 12 = -8 < 0 \qquad \underline{\text{elliptic}}$

$$\frac{dy}{dx} = \frac{2 \pm \sqrt{-8}}{2} = 1 \pm i\sqrt{2}$$

$$y = (1 \pm i\sqrt{2})x + C$$

$$\xi = y - (1 + i\sqrt{2})x$$

$$\eta = y - (1 - i\sqrt{2})x$$

$$\alpha = y - x$$

$$\beta = -\sqrt{2}x \qquad \Rightarrow x = -\frac{\beta}{\sqrt{2}}$$

$$\alpha_x = -1 \qquad \alpha_y = 1 \qquad \alpha_{xx} = \alpha_{xy} = \alpha_{yy} = 0$$

$$\beta_x = -\sqrt{2} \qquad \beta_y = 0 \qquad \beta_{xx} = \beta_{xy} = \beta_{yy} = 0$$

$$u_{\alpha\alpha}(-1)^2 + 2u_{\alpha\beta} \cdot \sqrt{2} + u_{\beta\beta} \cdot 2$$

$$+2\left(-u_{\alpha\alpha} + u_{\alpha\beta}(-\sqrt{2})\right) + 3\,u_{\alpha\alpha} + 4\left(-u_\alpha - \sqrt{2}\,u_\beta\right) + 5u_\alpha + u = e^x$$

$$2u_{\alpha\alpha} + 2u_{\beta\beta} + u_\alpha - 4\sqrt{2}\,u_\beta + u = e^x$$

$$\boxed{u_{\alpha\alpha} + u_{\beta\beta} = -\tfrac{1}{2}u_\alpha + 2\sqrt{2}\,u_\beta - \tfrac{1}{2}u + \tfrac{1}{2}e^{-\beta/\sqrt{2}}}$$

1e. $2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0$

$A = C = 2 \qquad B = -4 \qquad \triangle = 16 - 16 = 0 \qquad$ parabolic

$$\frac{dy}{dx} = \frac{-4 \pm 0}{4} = -1$$

$dy = -dx$

$$\begin{cases} \xi = y + x & \xi_x = 1 \quad \xi_y = 1 \quad \xi_{xx} = \xi_{xy} = \xi_{yy} = 0 \\ \eta = x & \eta_x = 1 \quad \eta_y = 0 \quad \eta_{xx} = \eta_{xy} = \eta_{yy} = 0 \end{cases}$$

$2\left(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\right) - 4\left(u_{\xi\xi} + u_{\xi\eta}\right) + 2\,u_{\xi\xi} + 3u = 0$

$2u_{\eta\eta} + 3u = 0$

$$\boxed{u_{\eta\eta} = -\tfrac{3}{2}u}$$

1f. $u_{xx} + 5u_{xy} + 4u_{yy} + 7u_y = \sin x$

$A = 1 \qquad B = 5 \qquad C = 4 \qquad \triangle = 25 - 16 = 9 > 0$ <u>hyperbolic</u>

$$\frac{dy}{dx} = \frac{5 \pm 3}{2} \begin{matrix} \nearrow 4 \\ \searrow 1 \end{matrix}$$

$$\begin{cases} \xi = y - 4x & \xi_x = -4 \quad \xi_y = 1 \quad \xi_{xx} = \xi_{xy} = \xi_{yy} = 0 \\ \eta = y - x & \eta_x = -1 \quad \eta_y = 1 \quad \eta_{xx} = \eta_{xy} = \eta_{yy} = 0 \end{cases}$$

$16u_{\xi\xi} + 2u_{\xi\eta} \cdot 4 + u_{\eta\eta} + 5\left(-4u_{\xi\xi} + u_{\xi\eta}(-4-1) + u_{\eta\eta}(-1)\right)$

$+4\left(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\right) + 7\left(u_\xi + u_\eta\right) = \sin x$

$-9\,u_{\xi\eta} + 7(u_\xi + u_\eta) = \sin x$

$$u_{\xi\eta} = \frac{7}{9}\left(u_\xi + u_\eta\right) - \frac{1}{9}\sin x$$

$\xi - \eta = -3x$

$$x = \frac{\eta - \xi}{3}$$

$$\boxed{u_{\xi\eta} = \tfrac{7}{9}\left(u_\xi + u_\eta\right) - \tfrac{1}{9}\sin\tfrac{\eta-\xi}{3}}$$

2a. $y = x + c$

$y = \frac{1}{4}x + c$

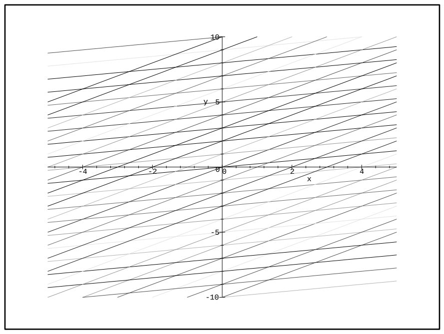


Figure 43: Maple plot of characteristics for 6.3 2a

2b. elliptic . no real characteristics

2c. $y = 3x + c$

$y = \frac{1}{3}x + c$

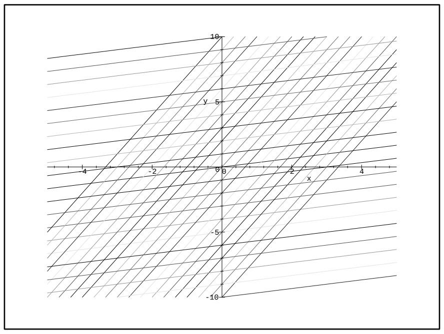

Figure 44: Maple plot of characteristics for 6.3 2c

2d. elliptic . no real characteristics

2e. $y = x + c$ see 2a

2f. $y = 4x + c$

$y = x + c \quad \rightarrow \quad$ (see 2a)

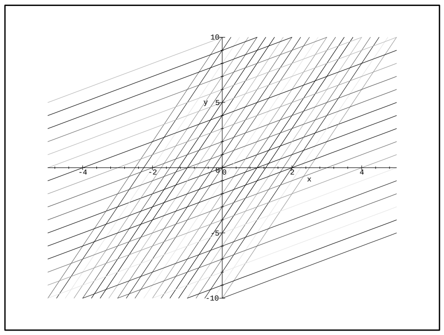

Figure 45: Maple plot of characteristics for 6.3 2f

## 6.5 Linear Systems

### Problems

1. Classify the behavior of the following system of PDEs in $(t, x)$ and $(t, y)$ space:

$$\begin{aligned}
\frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= 0 \\
\frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0
\end{aligned}$$

1. Let the vector $U$ be defined as

$$U = \begin{pmatrix} u \\ v \end{pmatrix}$$

Then the equation can be written as

$$U_t = AU_x + BU_y$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The eigenvalues of $A$ are

$$\begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix}$$

or

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

So the system behaves like elliptic in the $x, t$ space

The eigenvalues of $B$ are

$$\begin{vmatrix} \lambda + 1 & 0 \\ 0 & \lambda - 1 \end{vmatrix}$$

or

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

So the system behaves like hyperbolic in the $y, t$ space

## 6.6 General Solution

### Problems

1. Determine the general solution of

   a. $u_{xx} - \frac{1}{c^2} u_{yy} = 0 \qquad c = \text{constant}$
   b. $u_{xx} - 3u_{xy} + 2u_{yy} = 0$
   c. $u_{xx} + u_{xy} = 0$
   d. $u_{xx} + 10u_{xy} + 9u_{yy} = y$

2. Transform the following equations to

$$U_{\xi\eta} = cU$$

by introducing the new variables

$$U = ue^{-(\alpha\xi + \beta\eta)}$$

where $\alpha$, $\beta$ to be determined

   a. $u_{xx} - u_{yy} + 3u_x - 2u_y + u = 0$
   b. $3u_{xx} + 7u_{xy} + 2u_{yy} + u_y + u = 0$

(Hint: First obtain a canonical form)

3. Show that

$$u_{xx} = au_t + bu_x - \frac{b^2}{4}u + d$$

is parabolic for $a$, $b$, $d$ constants. Show that the substitution

$$u(x,t) = v(x,t)e^{\frac{b}{2}x}$$

transforms the equation to

$$v_{xx} = av_t + de^{-\frac{b}{2}x}$$

1a. $u_{xx} - \frac{1}{c^2} u_{yy} = 0$

$A = 1 \qquad B = 0 \qquad C = -\frac{1}{c^2} \qquad \triangle = \frac{4}{c^2} > 0 \qquad \underline{\text{hyperbolic}}$

$\frac{dy}{dx} = \frac{\pm \frac{2}{c}}{2} = \pm \frac{1}{c}$

$y = \pm \frac{1}{c} x + K$

$\xi = y + \frac{1}{c} x$

$\eta = y - \frac{1}{c} x$

Canonical form:

$u_{\xi \eta} = 0$

The solution is:

$u = f(\xi) + g(\eta)$

Substitute for $\xi$ and $\eta$ to get the solution in the original domain:

$u(x, y) = f(y + \frac{1}{c} x) + g(y - \frac{1}{c} x)$

1b. $u_{xx} - 3u_{xy} + 2u_{yy} = 0$

$A = 1 \qquad B = -3 \qquad C = 2 \qquad \triangle = 9 - 8 = 1 \qquad$ <u>hyperbolic</u>

$$\frac{dy}{dx} = \frac{-3 \pm 1}{2} \begin{matrix} \nearrow -2 \\ \searrow -1 \end{matrix}$$

$y = -2x + K_1$

$y = -x + K_2$

$\xi = y + 2x \qquad \xi_x = 2 \qquad \xi_y = 1$

$\eta = y + x \qquad \eta_x = 1 \qquad \eta_y = 1$

$u_x = 2u_\xi + u_\eta$

$u_y = u_\xi + u_\eta$

$u_{xx} = 2\left(2u_{\xi\xi} + u_{\xi\eta}\right) + 2u_{\xi\eta} + u_{\eta\eta}$

$\Rightarrow \quad u_{xx} = 4u_{\xi\xi} + 4u_{\xi\eta} + u_{\eta\eta}$

$u_{xy} = 2\left(u_{\xi\xi} + u_{\xi\eta}\right) + u_{\xi\eta} + u_{\eta\eta} = 2u_{\xi\xi} + 3u_{\xi\eta} + u_{\eta\eta}$

$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$$\begin{aligned} u_{xx} - 3u_{xy} + 2u_{yy} &= 4u_{\xi\xi} + 4u_{\xi\eta} + u_{\eta\eta} - 3\left(2u_{\xi\xi} + 3u_{\xi\eta} + u_{\eta\eta}\right) + 2\left(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\right) \\ &= -u_{\xi\eta} \end{aligned}$$

$\Rightarrow u_{\xi\eta} = 0$

The solution in the original domain is then:

$u(x, y) = f\left(y + 2x\right) + g\left(y + x\right)$

1c. $u_{xx} + u_{xy} = 0$

$A = 1 \qquad B = 1 \qquad C = 0 \qquad \triangle = 1 \qquad \underline{\text{hyperbolic}}$

$$\frac{dy}{dx} = \frac{+1 \pm 1}{2} \begin{matrix} \nearrow +1 \\ \searrow 0 \end{matrix}$$

$y = +x + K_1$

$y = K_2$

$$\begin{cases} \xi = y - x & \xi_x = -1 & \xi_y = 1 \\ \eta = y & \eta_x = 0 & \eta_y = 1 \end{cases}$$

$u_x = -u_\xi + u_\eta \underbrace{\eta_x}_{=0} = -u_\xi$

$u_y = u_\xi + u_\eta$

$$u_{xx} = u_{\xi\xi}$$

$$u_{xy} = -u_{\xi\xi} - u_{\xi\eta}$$

$u_{xx} + u_{xy} = -u_{\xi\eta} = 0$

The solution in the original domain is then:
$u = f(y - x) + g(y)$

1d. $u_{xx} + 10u_{xy} + 9u_{yy} = y$

$A = 1 \qquad B = 10 \qquad C = 9 \qquad \triangle = 100 - 36 = 64 \qquad$ <u>hyperbolic</u>

$$\frac{dy}{dx} = \frac{10 \pm 8}{2} \begin{matrix} \nearrow 9 \\ \searrow 1 \end{matrix}$$

$\xi = y - 9x \quad \xi_x = -9 \quad \xi_y = 1$

$\eta = y - x \quad \eta_x = -1 \quad \eta_y = 1$

$u_x = -9u_\xi - u_\eta$

$u_y = u_\xi + u_\eta$

$$\begin{aligned} u_{xx} &= -9\left(-9u_{\xi\xi} - u_{\xi\eta}\right) - \left(-9u_{\xi\eta} - u_{\eta\eta}\right) \\ &= 81u_{\xi\xi} + 18u_{\xi\eta} + u_{\eta\eta} \end{aligned}$$

$u_{xy} = -9\left(u_{\xi\xi} + u_{\xi\eta}\right) - \left(u_{\xi\eta} + u_{yy}\right)$

$= -9u_{\xi\xi} - 10u_{\xi\eta} - u_{\eta\eta}$

$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$$u_{xx} + 10u_{xy} + 9u_{yy} = \underbrace{(81 - 90 + 9)}_{=0} u_{\xi\xi} + (18 - 100 + 18)u_{\xi\eta} + \underbrace{(1 - 10 + 9)}_{=0} u_{\eta\eta} = y$$

$$-64u_{\xi\eta} = y$$

Substitute for $y$ by using the transformation

$$\left.\begin{aligned} \xi &= y - 9x \\ 9\eta &= 9y - 9x \end{aligned}\right\} -$$

$$\overline{\xi - 9\eta = -8y}$$

$$y = \frac{9\eta - \xi}{8}$$

$$u_{\xi\eta} = \frac{\frac{9\eta - \xi}{8}}{-64} = \frac{\xi}{512} - \frac{9\eta}{512}$$

$$u_{\xi\eta} = \frac{\xi}{512} - \frac{9\eta}{512}$$

To solve this PDE let $\xi$ be fixed and integrate with respect to $\eta$

$$\Rightarrow\ u_\xi = \frac{\xi}{512}\,\eta - \frac{9}{512}\,\frac{1}{2}\eta^2 + f\,(\xi)$$

$$u = \frac{1}{2}\,\frac{\xi^2\eta}{512} - \frac{9}{2}\,\frac{1}{512}\,\xi\,\eta^2 + F(\xi) + g\,(\eta)$$

The solution in $xy$ domain is:

$$u(x,\,y) = \frac{(y-9x)^2(y-x)}{1024} - \frac{9}{1024}\,(y-9x)(y-x)^2 + F(y-9x) + g(y-x)$$

2a. $u_{xx} - u_{yy} + 3u_x - 2u_y + u = 0$

$$U = u\,e^{-(\alpha\xi + \beta\eta)}$$

$A = 1 \qquad B = 0 \qquad C = -1 \qquad \triangle = 4 \qquad$ <u>hyperbolic</u>

$$\frac{dy}{dx} = \frac{\pm 2}{2} = \pm 1$$

$$\xi = y - x$$

$$\eta = y + x$$

$$u_x = -u_\xi + u_\eta$$

$$u_y = u_\xi + u_\eta$$

$$u_{xx} = -(-u_{\xi\xi} + u_{\xi\eta}) + (-u_{\xi\eta} + u_{\eta\eta}) = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$$

$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$-4u_{\xi\eta} - 3u_\xi + 3u_\eta - 2u_\xi - 2u_\eta + u = 0$$

$$-4u_{\xi\eta} - 5u_\xi + u_\eta + u = 0$$

$$U = u\,e^{-(\alpha\xi + \beta\eta)} \Rightarrow u = U\,e^{(\alpha\xi + \beta\eta)}$$

$$u_\xi = U_\xi\,e^{(\alpha\xi + \beta\eta)} + \alpha\,U\,e^{(\alpha\xi + \beta\eta)}$$

$$u_\eta = U_\eta\,e^{(\alpha\xi + \beta\eta)} + \beta\,U\,e^{(\alpha\xi + \beta\eta)}$$

$$u_{\xi\eta} = U_{\xi\eta}\,e^{(\alpha\xi + \beta\eta)} + \beta\,U_\xi\,e^{(\alpha\xi + \beta\eta)} + \alpha\,U_\eta\,e^{(\alpha\xi + \beta\eta)} + \alpha\,\beta\,U\,e^{(\alpha\xi + \beta\eta)}$$

$$-4U_{\xi\eta} - 4\beta\,U_\xi - 4\alpha\,U_\eta - 4\alpha\,\beta\,U - 5U_\xi - 5\alpha\,U + U_\eta + \beta\,U + U = 0$$

$$-4U_{\xi\eta} + (-4\beta - 5)U_\xi + (-4\alpha + 1)U_\eta + (-4\alpha\,\beta - 5\alpha + \beta + 1)\,U = 0$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$\beta = -5/4 \qquad \alpha = 1/4 \qquad -4(1/4)(-5/4) - 5(1/4) + (-5/4) + 1 = -1/4$$

$$-4U_{\xi\eta} - \frac{1}{4}U = 0$$

$$\boxed{U_{\xi\eta} = -\frac{1}{16}\,U} \qquad \text{required form}$$

2b. $3u_{xx} + 7u_{xy} + 2u_{yy} + u_y + u = 0$

$A = 3 \qquad B = 7 \qquad C = 2 \qquad \triangle = 49 - 24 = 25$ <u>hyperbolic</u>

$$\frac{dy}{dx} = \frac{7 \pm 5}{6} \begin{matrix} \nearrow 2 \\ \searrow \frac{1}{3} \end{matrix}$$

$\xi = y - 2x \qquad \xi_x = -2 \qquad \xi_y = 1$

$\eta = y - \frac{1}{3}x \qquad \eta_x = -\frac{1}{3} \qquad \eta_y = 1$

$$u_x = -2u_\xi - \frac{1}{3}u_\eta$$

$$u_y = u_\xi + u_\eta$$

$$u_{xx} = -2\left(-2u_{\xi\xi} - \frac{1}{3}u_{\xi\eta}\right) - \frac{1}{3}\left(-2u_{\xi\eta} - \frac{1}{3}u_{\eta\eta}\right)$$

$$u_{xx} = 4u_{\xi\xi} + \frac{4}{3}u_{\xi\eta} + \frac{1}{9}u_{\eta\eta}$$

$$u_{xy} = -2\left(u_{\xi\xi} + u_{\xi\eta}\right) - \frac{1}{3}\left(u_{\xi\eta} + u_{\eta\eta}\right)$$

$$u_{xy} = -2u_{\xi\xi} - \frac{7}{3}u_{\xi\eta} - \frac{1}{3}u_{\eta\eta}$$

$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$4u_{\xi\eta} - \frac{49}{3}u_{\xi\eta} + 4u_{\xi\eta} + u_\xi + u_\eta + u = 0$$

$$\boxed{-\tfrac{25}{3}u_{\xi\eta} + u_\xi + u_\eta + u = 0}$$

Use last page:

$$\frac{-25}{3}\left(U_{\xi\eta} + \beta U_\xi + \alpha U_\eta + \alpha\beta U\right) + U_\xi + \alpha U + U_\eta + \beta U + U = 0$$

$$\frac{-25}{3}U_{\xi\eta} + \left(\frac{-25}{3}\beta + 1\right)U_\xi + \left(\frac{-25}{3}\alpha + 1\right)U_\eta + \left(\frac{-25}{3}\alpha\beta + \alpha + \beta + 1\right)U = 0$$

$\Downarrow \qquad \Downarrow \qquad \Downarrow$

$\beta = 3/25 \qquad \alpha = 3/25 \qquad -\frac{3}{25} + \frac{3}{25} + \frac{3}{25} + 1 = \frac{28}{25}$

$$\frac{-25}{3}u_{\xi\eta} + \frac{28}{25}U = 0 \quad \Rightarrow \quad \boxed{U_{\xi\eta} = \tfrac{3}{25}\,\tfrac{28}{25}U}$$

3. $u_{xx} = au_t + bu_x - \frac{b^2}{4}u + d$

$A = 1 \qquad B = C = 0 \qquad \Rightarrow \qquad \triangle = 0$ <u>parabolic</u>

$\frac{dx}{dt} = 0$ already in canonical form since $u_{xx}$ is the only $2^{nd}$ order term

$u = \upsilon e^{\frac{b}{2}x}$

$u_x = \upsilon_x e^{\frac{b}{2}x} + \frac{b}{2}\upsilon e^{\frac{b}{2}x}$

$u_{xx} = \upsilon_{xx} e^{\frac{b}{2}x} + b\,\upsilon_x e^{\frac{b}{2}x} + \frac{b^2}{4}\upsilon e^{\frac{b}{2}x}$

$u_t = \upsilon_t e^{\frac{b}{2}x}$

$\Rightarrow \quad \upsilon_{xx} + b\upsilon_x + \frac{b^2}{4}\upsilon = a\upsilon_t + b\left(\upsilon_x + \frac{b}{2}\upsilon\right) - \frac{b^2}{4}\upsilon + de^{-\frac{b}{2}x}$

Since $\upsilon_x$ and $\upsilon$ terms cancel out we have:

$\upsilon_{xx} = a\upsilon_t + de^{-\frac{b}{2}x}$

# 7 Method of Characteristics

## 7.1 Advection Equation (first order wave equation)

### Problems

1. Solve

$$\frac{\partial w}{\partial t} - 3\frac{\partial w}{\partial x} = 0$$

subject to

$$w(x,0) = \sin x$$

2. Solve using the method of characteristics

   a. $\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = e^{2x}$ subject to $u(x,0) = f(x)$

   b. $\dfrac{\partial u}{\partial t} + x\dfrac{\partial u}{\partial x} = 1$ subject to $u(x,0) = f(x)$

   c. $\dfrac{\partial u}{\partial t} + 3t\dfrac{\partial u}{\partial x} = u$ subject to $u(x,0) = f(x)$

   d. $\dfrac{\partial u}{\partial t} - 2\dfrac{\partial u}{\partial x} = e^{2x}$ subject to $u(x,0) = \cos x$

   e. $\dfrac{\partial u}{\partial t} - t^2\dfrac{\partial u}{\partial x} = -u$ subject to $u(x,0) = 3e^{x}$

3. Show that the characteristics of

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0$$

$$u(x,0) = f(x)$$

are straight lines.

4. Consider the problem

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = 0$$

$$u(x,0) = f(x) = \begin{cases} 1 & x < 0 \\ 1 + \frac{x}{L} & 0 < x < L \\ 2 & L < x \end{cases}$$

   a. Determine equations for the characteristics
   b. Determine the solution $u(x,t)$
   c. Sketch the characteristic curves.

d. Sketch the solution $u(x,t)$ for fixed $t$.

5. Solve the initial value problem for the damped unidirectional wave equation

$$v_t + cv_x + \lambda v = 0 \qquad v(x,0) = F(x)$$

where $\lambda > 0$ and $F(x)$ is given.

6. (a) Solve the initial value problem for the inhomogeneous equation

$$v_t + cv_x = f(x,t) \qquad v(x,0) = F(x)$$

where $f(x,t)$ and $F(x)$ are specified functions.
(b) Solve this problem when $f(x,t) = xt$ and $F(x) = \sin x$.

7. Solve the "signaling" problem

$$v_t + cv_x = 0 \qquad v(0,t) = G(t) \quad -\infty < t < \infty$$

in the region $x > 0$.

8. Solve the initial value problem

$$v_t + e^x v_x = 0 \qquad v(x,0) = x$$

9. Show that the initial value problem

$$u_t + u_x = x \qquad u(x,x) = 1$$

has no solution. Give a reason for the problem.

1. The PDE can be rewrriten as a system of two ODEs

$$\frac{dx}{dt} = -3$$

$$\frac{dw}{dt} = 0$$

The solution of the first gives the characteristic curve

$$x + 3t = x_0$$

and the second gives

$$w(x(t), t) = w(x(0), 0) = \sin x_0 = \sin(x + 3t)$$

$$\boxed{w(x, t) = \sin(x + 3t)}$$

2. a. The ODEs in this case are

$$\frac{dx}{dt} = c$$

$$\frac{du}{dt} = e^{2x}$$

Solve the characteristic equation

$$x = ct + x_0$$

Now solve the second ODE. To do that we have to plug in for $x$

$$\frac{du}{dt} = e^{2(x_0 + ct)} = e^{2x_0} e^{2ct}$$

$$u(x, t) = e^{2x_0} \frac{1}{2c} e^{2ct} + K$$

The constant of integration can be found from the initial condition

$$f(x_0) = u(x_0, 0) = \frac{1}{2c} e^{2x_0} + K$$

Therefore

$$K = f(x_0) - \frac{1}{2c} e^{2x_0}$$

Plug this $K$ in the solution

$$u(x, t) = \frac{1}{2c} e^{2x_0 + 2ct} + f(x_0) - \frac{1}{2c} e^{2x_0}$$

Now substitute for $x_0$ from the characteristic curve $\boxed{u(x, t) = \frac{1}{2c} e^{2x} + f(x - ct) - \frac{1}{2c} e^{2(x - ct)}}$

2. b. The ODEs in this case are

$$\frac{dx}{dt} = x$$
$$\frac{du}{dt} = 1$$

Solve the characteristic equation

$$\ln x = t + \ln x_0 \qquad \text{or} \qquad x = x_0\, e^t$$

The solution of the second ODE is

$$u = t + K \qquad \text{and the constant is} \qquad f(x_0)$$

$$u(x,t) = t + f(x_0)$$

Substitute $x_0$ from the characteristic curve $\boxed{u(x,t) = t + f\left(x\, e^{-t}\right)}$

2. c. The ODEs in this case are

$$\frac{dx}{dt} = 3t$$
$$\frac{du}{dt} = u$$

Solve the characteristic equation

$$x = \frac{3}{2}\, t^2 + x_0$$

The second ODE can be written as

$$\frac{du}{u} = dt$$

Thus the solution of the second ODE is

$$\ln u = t + \ln K \qquad \text{and the constant is} \qquad f(x_0)$$

$$u(x,t) = f(x_0)\, e^t$$

Substitute $x_0$ from the characteristic curve $\boxed{u(x,t) = f\left(x - \frac{3}{2}\, t^2\right)\, e^t}$

2. d. The ODEs in this case are

$$\frac{dx}{dt} = -2$$
$$\frac{du}{dt} = e^{2x}$$

Solve the characteristic equation

$$x = -2t + x_0$$

Now solve the second ODE. To do that we have to plug in for $x$

$$\frac{du}{dt} = e^{2(x_0 - 2t)} = e^{2x_0}\, e^{-4t}$$

$$u(x,t) = e^{2x_0}\left(-\frac{1}{4}\, e^{-4t}\right) + K$$

The constant of integration can be found from the initial condition

$$\cos(x_0) = u(x_0, 0) = -\frac{1}{4}\, e^{2x_0} + K$$

Therefore

$$K = \cos(x_0) + \frac{1}{4}\, e^{2x_0}$$

Plug this $K$ in the solution and substitute for $x_0$ from the characteristic curve

$$u(x,t) = -\frac{1}{4}\, e^{2(x+2t)}\, e^{-4t} + \cos(x + 2t) + \frac{1}{4}\, e^{2(x+2t)}$$

$$\boxed{u(x,t) = \frac{1}{4}\, e^{2x}\left(e^{4t} - 1\right) + \cos(x + 2t)}$$

To check the answer, we differentiate

$$u_x = \frac{1}{2}\, e^{2x}\left(e^{4t} - 1\right) - \sin(x + 2t)$$

$$u_t = \frac{1}{4}\, e^{2x}\left(4\, e^{4t}\right) - 2\sin(x + 2t)$$

Substitute in the PDE

$$u_t - 2u_x = e^{2x}\, e^{4t} - 2\sin(x + 2t) - 2\left\{\frac{1}{2}\, e^{2x}\left(e^{4t} - 1\right) - \sin(x + 2t)\right\}$$
$$= e^{2x}\, e^{4t} - 2\sin(x + 2t) - e^{2x}\, e^{4t} + e^{2x} + 2\sin(x + 2t)$$
$$= e^{2x} \qquad \text{which is the right hand side of the PDE}$$

2. e. The ODEs in this case are

$$\frac{dx}{dt} = -t^2$$

$$\frac{du}{dt} = -u$$

Solve the characteristic equation

$$x = -\frac{t^3}{3} + x_0$$

Now solve the second ODE. To do that we rewrite it as

$$\frac{du}{u} = -dt$$

Therefore the solution as in 2c

$$\ln u = -t + \ln K \qquad \text{and the constant is} \qquad 3\,e^{x_0}$$

Plug this $K$ in the solution and substitute for $x_0$ from the characteristic curve

$$\ln u(x,t) = \ln \left[3\,e^{x + \frac{1}{3}t^3}\right] - t$$

$$\boxed{u(x,t) = 3\,e^{x + \frac{1}{3}t^3}\,e^{-t}}$$

To check the answer, we differentiate

$$u_t = 3\,e^x \left(t^2 - 1\right) e^{\frac{1}{3}t^3 - t}$$

$$u_x = 3\,e^x\,e^{\frac{1}{3}t^3 - t}$$

Substitute in the PDE

$$u_t - t^2 u_x = 3\,e^x\,e^{\frac{1}{3}t^3 - t} - t^2 \left\{3\,e^x \left(t^2 - 1\right) e^{\frac{1}{3}t^3 - t}\right\}$$

$$= 3\,e^x\,e^{\frac{1}{3}t^3 - t} \left[\left(t^2 - 1\right) - t^2\right] = -3\,e^{x + \frac{1}{3}t^3 - t} = -u$$

3. The ODEs in this case are

$$\frac{dx}{dt} = 2u$$
$$\frac{du}{dt} = 0$$

Since the first ODE contains $x$, $t$ and $u$, we solve the second ODE first

$$u(x,t) = u(x(0),0) = f(x(0))$$

Plug this $u$ in the first ODE, we get

$$\frac{dx}{dt} = 2f(x(0))$$

The solution is

$$x = x_0 + 2tf(x_0)$$

These are characteristics lines all with slope

$$\frac{1}{2f(x_0)}$$

Note that the characteristic through $x_1(0)$ will have a different slope than the one through $x_2(0)$. That is the straight line are NOT parallel.

4. The ODEs in this case are

$$\frac{dx}{dt} = 2u$$
$$\frac{du}{dt} = 0$$

with

$$u(x,0) = f(x) = \begin{cases} 1 & x < 0 \\ 1 + \frac{x}{L} & 0 < x < L \\ 2 & L < x \end{cases}$$

a. Since the first ODE contains $x$, $t$ and $u$, we solve the second ODE first

$$u(x,t) = u(x(0),0) = f(x(0))$$

Plug this $u$ in the first ODE, we get

$$\frac{dx}{dt} = 2f(x(0))$$

The solution is

$$x = x_0 + 2tf(x_0)$$

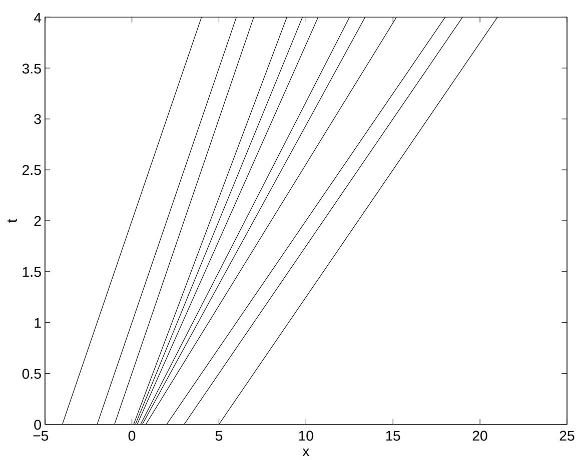


Figure 46: Characteristics for problem 4

b. For $x_0 < 0$ then $f(x_0) = 1$ and the solution is

$$u(x,t) = 1 \qquad \text{on } x = x_0 + 2t$$

or

$$u(x,t) = 1 \qquad \text{on } x < 2t$$

For $x_0 > L$ then $f(x_0) = 2$ and the solution is

$$u(x,t) = 2 \qquad \text{on } x > 4t + L$$

For $0 < x_0 < L$ then $f(x_0) = 1 + x_0/L$ and the solution is

$$u(x,t) = 1 + \frac{x_0}{L} \qquad \text{on } x = 2t\left(1 + \frac{x_0}{L}\right) + x_0$$

That is

$$x_0 = \frac{x - 2t}{2t + L} L$$

Thus the solution on this interval is

$$u(x,t) = 1 + \frac{x - 2t}{2t + L} = \frac{2t + L + x - 2t}{2t + L} = \frac{x + L}{2t + L}$$

Notice that $u$ is continuous.

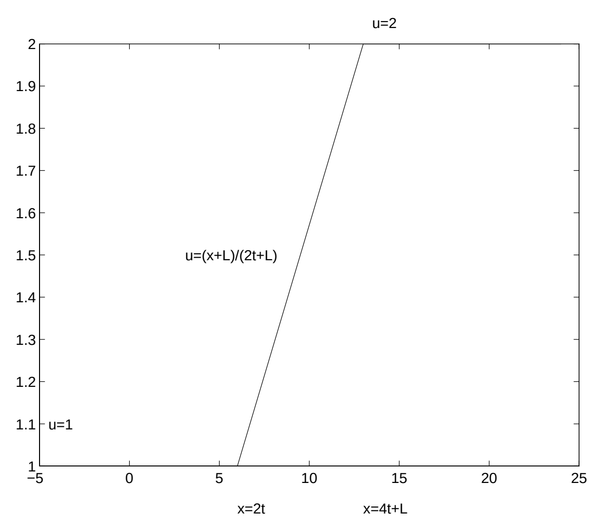


Figure 47: Solution for problem 4

5. $v_t + cv_x + \lambda v = 0 \qquad v(x,0) = F(x)$
where $\lambda > 0$ and $F(x)$ is given.
The ODEs are

$$\frac{dx}{dt} = c, \qquad \frac{dv}{dt} = -\lambda v$$

The initial condition for each:

$$x(0) = \xi, \qquad v(\xi, 0) = F(\xi)$$

Solve the characteristic equation to get

$$x = ct + \xi.$$

Now solve the other ODE to get

$$v(x(t), t) = Ke^{-\lambda t}$$

Use the initial condition to get

$$v(x(0), 0) = K = F(x(0))$$

and so

$$v(x(t), t) = F(\xi)e^{-\lambda t}$$

Now get $\xi$ from the characteristics and substitute here to get

$$\xi = x - ct$$

and

$$v(x,t) = F(x - ct)e^{-\lambda t}$$

6. a. Solve the initial value problem for the inhomogeneous equation

$$v_t + cv_x = f(x,t) \qquad v(x,0) = F(x)$$

where $f(x,t)$ and $F(x)$ are specified functions.
The ODEs are

$$\frac{dx}{dt} = c, \qquad \frac{dv}{dt} = f(x(t),t)$$

The initial condition for each:

$$x(0) = \xi, \qquad v(\xi,0) = F(\xi)$$

Solve the characteristic equation to get

$$x = ct + \xi.$$

Now solve the other ODE to get

$$v(x(t),t) = \int_0^t f(x(\tau),\tau)d\tau + F(\xi)$$

Substitute the value of $x(\tau)$ to get

$$v(x(t),t) = \int_0^t f(c\tau + \xi, \tau)d\tau + F(\xi)$$

Now get $\xi$ from the characteristics and substitute here to get

$$\xi = x - ct$$

$$v(x,t) = \int_0^t f(x - ct + c\tau, \tau)d\tau + F(x - ct)$$

To check the answer, we can differentiate the solution.

$$v_t = (-c) \cdot F'(x - ct) + f(x,t) + \int_0^t \frac{\partial f(x - ct + c\tau, \tau)}{\partial(x - ct + c\tau)} \underbrace{\frac{\partial(x - ct + c\tau)}{\partial t}}_{-c} d\tau$$

$$v_x = F'(x - ct) + \int_0^t \frac{\partial f(x - ct + c\tau, \tau)}{\partial(x - ct + c\tau)} \underbrace{\frac{\partial(x - ct + c\tau)}{\partial x}}_{1} d\tau$$

Substitute these two derivatives into the left side of the equation and find that the only term left is $f(x,t)$ which is on the right. We can also check the initial condition by substituting $t = 0$ in the solution. In this case the integral is zero and we get $F(x)$.

6. b. Solve this problem when $f(x,t) = xt$ and $F(x) = \sin x$.
In this case the integral becomes

$$\begin{aligned}\int_0^t (x - ct + c\tau)\tau d\tau &= x\frac{t^2}{2} - ct\frac{t^2}{2} + c\frac{t^3}{3} \\ &= \frac{1}{2}xt^2 - \frac{1}{6}ct^3\end{aligned}$$

Thus the solution is

$$v(x,t) = \left(x - \frac{1}{3}ct\right)\frac{t^2}{2} + \sin(x - ct)$$

7. Solve the "signaling" problem

$$v_t + cv_x = 0 \qquad v(0,t) = G(t) \quad -\infty < t < \infty$$

in the region $x > 0$.

The easiest way is to reverse the role of $x$ and $t$. So the problem is now

$$v_x + cv_t = 0 \qquad v(x,0) = G(x) \quad -\infty < x < \infty$$

or

$$v_t + \frac{1}{c}v_x = 0 \qquad v(x,0) = G(x) \quad -\infty < x < \infty$$

The ODEs are

$$\frac{dx}{dt} = \frac{1}{c}, \qquad \frac{dv}{dt} = 0$$

and the initial condition for each

$$x(0) = \xi, \qquad v(\xi, 0) = G(\xi)$$

Solve the characteristic equation

$$x(t) = \frac{1}{c}t + \xi$$

The solution of the other ODE

$$v(x(t), t) = G(\xi)$$

Solve for $\xi$ and substitute in $v$ to get

$$v(x,t) = G\left(x - \frac{1}{c}t\right)$$

Now change the variables back

$$v(x,t) = G\left(t - \frac{1}{c}x\right).$$

8. Solve the initial value problem

$$v_t + e^x v_x = 0 \qquad v(x,0) = x$$

The ODEs are

$$\frac{dx}{dt} = e^x, \qquad \frac{dv}{dt} = 0$$

and the initial condition for each

$$x(0) = \xi, \qquad v(\xi, 0) = \xi$$

Solve the characteristic equation

$$e^{-x} dx = dt$$

$$-e^{-x} = t + C$$

Now use the initial condition

$$-e^{-\xi} = 0 + C$$

or

$$e^{-x} = -t + e^{-\xi}$$

The solution of the other ODE

$$v(x(t), t) = K = v(x(0), 0) = \xi$$

Solve the characteristic equation for $\xi$

$$\xi = -\ln(t + e^{-x})$$

and substitute in $v$ to get

$$v(x,t) = -\ln(t + e^{-x})$$

9. Show that the initial value problem

$$u_t + u_x = x \qquad u(x,x) = 1$$

has no solution. Give a reason for the problem.

Note that the initial line in this case is NOT the $x$ axis ($t = 0$) but the line $t = x$.

The ODEs are

$$\frac{dx}{dt} = 1, \qquad \frac{du}{dt} = x$$

and the initial condition for each

$$x(t = x) = \xi, \qquad u(x(t = x), t = x) = 1$$

Solve the characteristic equation

$$x = t + K$$

and use the initial condition

$$\xi = \xi + K$$

so $K = 0$ and the characteristic is

$$x = t$$

This is the only line, there is no family as in other problems.

Substitute in the $u$ equation. But wait, if $x = t$ is a characteristic curve how do we get to the initial line? There is NO way. Therefore there is NO solution.

If we try to solve the $u$ equation

$$\frac{du}{dt} = t, \qquad (\text{since } x = t)$$

we get

$$u = \frac{1}{2}t^2 + C$$

Now use the initial condition

$$1 = u(x(t = x), t = x) = \frac{1}{2}x^2 + C$$

We can't find $C$ constant to satisfy this. Therefore, there is NO solution.

## 7.2 Quasilinear Equations

### 7.2.1 The Case $S = 0$, $c = c(u)$

**Problems**

1. Solve the following

   a. $\dfrac{\partial u}{\partial t} = 0$ subject to $u(x,0) = g(x)$

   b. $\dfrac{\partial u}{\partial t} = -3xu$ subject to $u(x,0) = g(x)$

2. Solve

$$\frac{\partial u}{\partial t} = u$$

subject to

$$u(x,t) = 1 + \cos x \qquad \text{along} \qquad x + 2t = 0$$

3. Let

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \qquad c = \text{constant}$$

   a. Solve the equation subject to $u(x,0) = \sin x$

   b. If $c > 0$, determine $u(x,t)$ for $x > 0$ and $t > 0$ where

$$\begin{aligned} u(x,0) &= f(x) \qquad \text{for } x > 0 \\ u(0,t) &= g(t) \qquad \text{for } t > 0 \end{aligned}$$

4. Solve the following linear equations subject to $u(x,0) = f(x)$

   a. $\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = e^{-3x}$ b. $\dfrac{\partial u}{\partial t} + t\dfrac{\partial u}{\partial x} = 5$

   c. $\dfrac{\partial u}{\partial t} - t^2\dfrac{\partial u}{\partial x} = -u$

   d. $\dfrac{\partial u}{\partial t} + x\dfrac{\partial u}{\partial x} = t$

   e. $\dfrac{\partial u}{\partial t} + x\dfrac{\partial u}{\partial x} = x$

5. Determine the parametric representation of the solution satisfying $u(x,0) = f(x)$,

   a. $\dfrac{\partial u}{\partial t} - u^2\dfrac{\partial u}{\partial x} = 3u$

b. $\dfrac{\partial u}{\partial t} + t^2 u \dfrac{\partial u}{\partial x} = -u$

6. Solve

$$\frac{\partial u}{\partial t} + t^2 u \frac{\partial u}{\partial x} = 5$$

subject to

$$u(x, 0) = x.$$

7. Using implicit differentiation, verify that $u(x, t) = f(x - tu)$ is a solution of

$$u_t + u u_x = 0$$

8. Consider the damped quasilinear wave equation

$$u_t + u u_x + c u = 0$$

where $c$ is a positive constant.

(a) Using the method of characteristics, construct a solution of the initial value problem with $u(x, 0) = f(x)$, in implicit form. Discuss the wave motion and the effect of the damping.

(b) Determine the breaking time of the solution by finding the envelope of the characteristic curves and by using implicit differentiation. With $\tau$ as the parameter on the initial line, show that unless $f'(\tau) < -c$, no breaking occurs.

9. Consider the one-dimensional form of Euler's equations for isentropic flow and assume that the pressure $p$ is a constant. The equations reduce to

$$\rho_t + \rho u_x + u \rho_x = 0 \qquad u_t + u u_x = 0$$

Let $u(x, 0) = f(x)$ and $\rho(x, 0) = g(x)$. By first solving the equation for $u$ and then the equation for $\rho$, obtain the implicit solution

$$u = f(x - ut) \qquad \rho = \frac{g(x - ut)}{1 + t f'(x - ut)}$$

1.

a. Integrate the PDE assuming $x$ fixed, we get

$$u(x,t) = K(x)$$

Since $dx/dt = 0$ we have $x = x_0$ and thus

$$u(x,t) = u(x_0,0) = K(x_0) = g(x_0) = g(x)$$
$$u(x,t) = g(x)$$

b. For a fixed $x$, we can integrate the PDE with respect to $t$

$$\int \frac{du}{u} = -3xt + K(x)$$

$$\ln u - \ln c(x) = -3xt$$
$$u(x,t) = ce^{-3xt}$$

Using the initial condition

$$u(x,t) = f(x)\,e^{-3xt}$$

2. The set of ODEs are

$$\frac{dx}{dt} = 0 \qquad \text{and} \quad \frac{du}{dt} = u$$

The characteristics are $x =$ constant and the ODE for $u$ can be written

$$\frac{du}{u} = dt$$

Thus

$$u(x,t) = k(x)\, e^t$$

On $x = -2t$ or $x + 2t = 0$ we have

$$1 + \cos x = k(x)\, e^t|_{x=-2t} = k(x)\, e^{-\frac{x}{2}}$$

Thus the constant of integration is

$$k(x) = e^{\frac{x}{2}}\ (1 + \cos x)$$

Plug this in the solution $u$ we get

$$u(x,t) = (1 + \cos x)\ e^{\frac{x}{2}+t}$$

Another way of getting the solution is by a rotation so that the line $x + 2t = 0$ becomes horizontal. Call that axis $\xi$, the line perpendicular to it is given by $t - 2x = 0$, which we call $\eta$.

So here is the transformation

$$\xi = x + 2t$$
$$\eta = t - 2x.$$

The PDE becomes:

$$u_\xi + \frac{1}{2}u_\eta = \frac{1}{2}u$$

and the intial condition is:

$$u(\eta, \xi = 0) = 1 + \cos\frac{\xi - 2\eta}{5}|_{\xi=0} = 1 + \cos\frac{2}{5}\eta$$

Rewrite this as a system of two first order ODEs,

$$\frac{d\eta}{d\xi} = \frac{1}{2}$$

$$\eta(0) = \alpha$$

$$\frac{du(\eta(\xi), \xi)}{d\xi} = \frac{1}{2}u$$

$$u(\eta(0),0) \;=\; 1+\cos\frac{2}{5}\alpha.$$

The solution of the first ODE, gives the characteristics in the transformed domain:

$$\eta \;=\; \frac{1}{2}\xi+\alpha$$

The solution of the second ODE:

$$u(\eta(\xi),\xi) \;=\; Ke^{\frac{1}{2}\xi}$$

Using the initial condition

$$1+\cos\frac{2}{5}\alpha \;=\; K$$

Thus

$$u(\eta(\xi),\xi) \;=\; (1+\cos\frac{2}{5}\alpha)e^{\frac{1}{2}\xi}$$

But $\alpha = \eta - \frac{1}{2}\xi$ thus

$$u(\eta(\xi),\xi) \;=\; (1+\cos\frac{2}{5}(\eta-\frac{1}{2}\xi))e^{\frac{1}{2}\xi}$$

Now substitute back:

$$\frac{1}{2}\xi \;=\; \frac{1}{2}x+t$$

$$\eta-\frac{1}{2}\xi \;=\; (t-2x)-(\frac{1}{2}x+t) \;=\; \frac{5}{2}x$$

Thus

$$u(x,t) \;=\; (1+\cos x)e^{\frac{1}{2}x+t}.$$

3. a. The set of ODEs to solve is

$$\frac{dx}{dt} = c \qquad\qquad \frac{du}{dt} = 0$$

The characteristics are:

$$x = x_0 + ct$$

The solution of the second ODE is

$$u(x,t) = \text{constant} = u(x_0, 0) = \sin x_0$$

Substitute for $x_0$, we get

$$u(x,t) = \sin(x - ct)$$

b. For $x > ct$ the solution is $u(x,t) = f(x - ct)$

But $f(x)$ is defined only for positive values of the independent variable $x$, therefore $f(x - ct)$ cannot be used for $x < ct$.

In this case ($x < ct$) we must use the condition

$$u(0,t) = g(t)$$

The characteristics for which $x_0 < 0$ is given by $x = x_0 + ct$ and it passes through the point $(0, t_0)$ (see figure). Thus $x = c(t - t_0)$ and $u(0, t_0) = g(t_0) = g\left(t - \frac{x}{c}\right)$

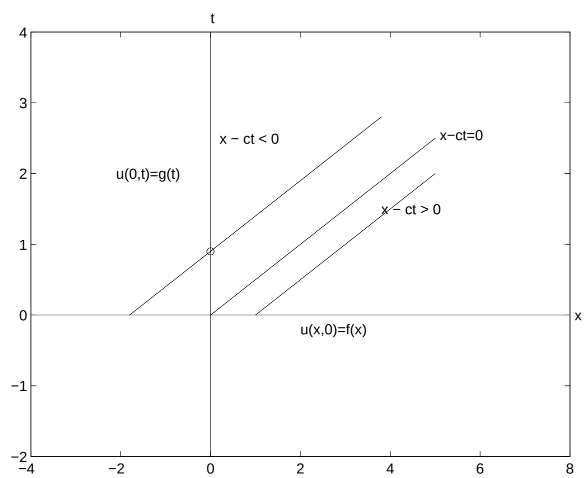


Figure 48: Domain and characteristics for problem 3b

The solution is therefore given by

$$u(x,t) = \begin{cases} f(x - ct) & \text{for } x - ct > 0 \\ g\left(t - \frac{x}{c}\right) & \text{for } x - ct < 0 \end{cases}$$

4. a. The set of ODEs is

$$\frac{dx}{dt} = c \qquad\qquad \frac{du}{dt} = e^{-3x}$$

The solution of the first is

$$x = x_0 + ct$$

Substituting $x$ in the second ODE

$$\frac{du}{dt} = e^{-3(x_0+ct)}$$

Now integrate

$$u(x,t) = K + e^{-3x_0}\frac{1}{-3c}e^{-3ct}$$

At $t = 0$ we get

$$f(x_0) = u(x_0,0) = K + e^{-3x_0}\frac{1}{-3c}$$

Therefore the constant of integration $K$ is

$$K = f(x_0) + e^{-3x_0}\frac{1}{3c}$$

Substitute this $K$ in the solution

$$u(x,t) = f(x_0) + e^{-3x_0}\frac{1}{3c} - e^{-3x_0}\frac{1}{3c}e^{-3ct}$$

Recall that $x_0 = x - ct$ thus

$$u(x,t) = f(x-ct) + \frac{1}{3c}e^{-3(x-ct)} - \frac{1}{3c}e^{-3x}$$

b. The set of ODEs is

$$\frac{dx}{dt} = t \qquad\qquad \frac{du}{dt} = 5$$

The solution of the first is

$$x = x_0 + \frac{1}{2}t^2$$

Now integrate the second ODE

$$u(x,t) = 5t + K$$

At $t = 0$ the solution is

$$u(x_0,0) = f(x_0) = K \qquad \text{plug } t = 0 \text{ in the solution } u$$

Thus when substituting for $x_0$ in the solution

$$u(x,t) = 5t + f\left(x - \frac{1}{2}t^2\right)$$

c. The set of ODEs is

$$\frac{dx}{dt} = -t^2 \qquad\qquad \frac{du}{dt} = u$$

The solution of the first is

$$x = x_0 - \frac{1}{3}t^3$$

Now integrate the second ODE

$$\ln u(x,t) = -t + \ln K$$

or

$$u(x,t) = K\, e^{-t}$$

At $t = 0$ the solution is

$$u(x_0, 0) = f(x_0) = K \qquad \text{plug } t = 0 \text{ in the solution } u$$

Thus when substituting for $x_0$ in the solution

$$u(x,t) = e^{-t} f\left(x + \frac{1}{3}t^3\right)$$

d. The set of ODEs is

$$\frac{dx}{dt} = x \qquad\qquad \frac{du}{dt} = t$$

The solution of the first is

$$\ln x = \ln x_0 + t$$

or

$$x = x_0\, e^t$$

Now integrate the second ODE

$$u(x,t) = \frac{1}{2}t^2 + K$$

At $t = 0$ the solution is

$$u(x_0, 0) = f(x_0) = K \qquad \text{plug } t = 0 \text{ in the solution } u$$

Thus when substituting for $x_0$ in the solution

$$u(x,t) = \frac{1}{2}t^2 + f\left(x\, e^{-t}\right)$$

e. The set of ODEs is

$$\frac{dx}{dt} = x \qquad\qquad \frac{du}{dt} = x$$

The solution of the first is

$$\ln x = \ln x_0 + t$$

or

$$x = x_0 \, e^t$$

Now substitute $x$ in the second ODE

$$\frac{du}{dt} = x_0 \, e^t$$

and integrate it

$$u(x,t) = e^t \, x_0 + K$$

At $t = 0$ the solution is

$$u(x_0, 0) = f(x_0) = K + x_0 \qquad \text{plug } t = 0 \text{ in the solution } u$$

Thus when substituting $K$ in $u$

$$u(x,t) = x_0 \, e^t + f(x_0) - x_0$$

Now substitute for $x_0$ in the solution

$$u(x,t) = x + f\left(x \, e^{-t}\right) - x \, e^{-t}$$

5. a. The set of ODEs is

$$\frac{dx}{dt} = -u^2 \qquad\qquad \frac{du}{dt} = 3u$$

The solution of the first ODE requires the yet unknown $u$ thus we tackle the second ODE

$$\frac{du}{u} = 3\,dt$$

Now integrate this

$$\ln u(x,t) = 3t + K \qquad\qquad \text{or } u(x,t) = C\,e^{3t}$$

At $t = 0$ the solution is

$$u(x_0, 0) = f(x_0) = C$$

Thus

$$u(x,t) = f(x_0)\,e^{3t}$$

Now substitute this solution in the characteristic equation (first ODE)

$$\frac{dx}{dt} = -\left(f(x_0)\,e^{3t}\right)^2 = -\left(f(x_0)\right)^2\,e^{6t}$$

Integrating

$$x = -\left(f(x_0)\right)^2 \int e^{6t}dt = -\frac{1}{6}\left(f(x_0)\right)^2 e^{6t} + K$$

For $t = 0$ we get

$$x_0 = -\frac{1}{6}\left(f(x_0)\right)^2 + K$$

Thus

$$K = x_0 + \frac{1}{6}\left(f(x_0)\right)^2$$

and the characteristics are

$$x = -\frac{1}{6}\left(f(x_0)\right)^2\,e^{6t} + x_0 + \frac{1}{6}\left(f(x_0)\right)^2$$

"Solve" this for $x_0$ and subsitute for $u$. The quote is because one can only solve this for special cases of the function $f(x_0)$.

The implicit solution is given by

$$\boxed{\begin{aligned} u(x,t) &= f(x_0)\,e^{3t} \\ x &= -\tfrac{1}{6}\left(f(x_0)\right)^2\,e^{6t} + x_0 + \tfrac{1}{6}\left(f(x_0)\right)^2 \end{aligned}}$$

b. The set of ODEs is

$$\frac{dx}{dt} = t^2 \, u \qquad\qquad \frac{du}{dt} = -u$$

The solution of the first ODE requires the yet unknown $u$ thus we tackle the second ODE

$$\frac{du}{u} = -\,dt$$

Now integrate this

$$\ln u(x,t) = -t + K \qquad\qquad \text{or } u(x,t) = C\,e^{-t}$$

At $t = 0$ the solution is

$$u(x_0, 0) = f(x_0) = C$$

Thus

$$u(x,t) = f(x_0)\,e^{-t}$$

Now substitute this solution in the characteristic equation (first ODE)

$$\frac{dx}{dt} = t^2\,f(x_0)\,e^{-t}$$

or

$$\int dx = f(x_0) \int t^2\,e^{-t}\,dt$$

Integrate and continue as in part a of this problem

$$x = f(x_0)\left[-t^2\,e^{-t} - 2t\,e^{-t} - 2\,e^{-t} + C\right]$$

For $t = 0$ we get

$$x_0 = f(x_0)\,[-2 + C]$$

Thus

$$C\,f(x_0) = x_0 + 2\,f(x_0)$$

and the characteristics are

$$x = f(x_0)\left[-t^2 - 2t - 2\right]\,e^{-t} + x_0 + 2\,f(x_0)$$

"Solve" this for $x_0$ and subsitute for $u$. The quote is because one can only solve this for special cases of the function $f(x_0)$.

The implicit solution is given by

$$\boxed{\begin{aligned} u(x,t) &= f(x_0)\,e^{-t} \\ x &= -\,f(x_0)\,[t^2 + 2t + 2]\,e^{-t} + x_0 + 2\,f(x_0) \end{aligned}}$$

6. The set of ODEs is

$$\frac{dx}{dt} = t^2 u \qquad\qquad \frac{du}{dt} = 5$$

The solution of the first ODE requires the yet unknown $u$ thus we tackle the second ODE

$$du = 5\,dt$$

Now integrate this

$$u(x,t) = 5t + K$$

At $t = 0$ the solution is

$$u(x_0, 0) = f(x_0) = x_0 = K$$

Thus

$$u(x,t) = x_0 + 5t$$

Now substitute this solution in the characteristic equation (first ODE)

$$\frac{dx}{dt} = 5t^3 + x_0\,t^2$$

Integrate

$$x = \frac{5}{4}\,t^4 + \frac{1}{3}\,t^3\,x_0 + C$$

For $t = 0$ we get

$$x_0 = 0 + 0 + C$$

Thus

$$C = x_0$$

and the characteristics are

$$x = \frac{5}{4}\,t^4 + \left(\frac{1}{3}\,t^3 + 1\right)\,x_0$$

Solve this for $x_0$

$$x_0 = \frac{x - \frac{5}{4}\,t^4}{1 + \frac{1}{3}\,t^3}$$

The solution is then given by $\boxed{u(x,t) = 5t + \frac{x - \frac{5}{4}\,t^4}{1 + \frac{1}{3}\,t^3}}$

7. To show that $u(x,t) = f(x - tu)$ is a solution of

$$u_t + uu_x = 0$$

we differentiate.

$$u_t = \frac{\partial f}{\partial(x - tu)} \underbrace{\frac{\partial(x - tu)}{\partial t}}_{=-u}$$

$$u_x = \frac{\partial f}{\partial(x - tu)} \underbrace{\frac{\partial(x - tu)}{\partial x}}_{=1}$$

Substitute in the equation:

$$u_t + uu_x = -u\frac{\partial f}{\partial(x - tu)} + u\frac{\partial f}{\partial(x - tu)} = 0$$

8. a. Use the method of characteristics to solve

$$u_t + uu_x + cu = 0$$

with $u(x,0) = f(x)$ where $c$ is a positive constant.

Let's rewrite the equation as

$$u_t + uu_x = -cu$$

Now the characteristic equation is

$$\frac{dx}{dt} = u$$

$$x(0) = \xi$$

In order to solve this, we have to first find $u$ from the second ODE, which is

$$\frac{du}{dt} = -cu$$

$$u(\xi, 0) = f(\xi)$$

This equation is separable

$$\frac{du}{u} = -cdt$$

$$\ln u = -ct + \ln f(\xi)$$

or $\boxed{u(x(t),t) = f(\xi)e^{-ct}}$

Taking this to the right hand side of the characteristic equation we have

$$\frac{dx}{dt} = f(\xi)e^{-ct}$$

$$x(0) = \xi$$

Integrate

$$x = f(\xi)\int e^{-ct}dt = -\frac{1}{c}f(\xi)e^{-ct} + K$$

Use the initial condition

$$\xi = x(0) = -\frac{1}{c}f(\xi) + K$$

so

$$K = \xi + \frac{1}{c}f(\xi)$$

Use this $K$ to get

$$x(t) = -\frac{1}{c}f(\xi)e^{-ct} + \frac{1}{c}f(\xi) + \xi$$

$$\boxed{x(t) = -\frac{1}{c}f(\xi)\left(e^{-ct} - 1\right) + \xi}$$

The two boxes give the implicit form of the solution.

Now from the solution $u$ we have

$$f(\xi) = ue^{ct}$$

so

$$x(t) = -\frac{1}{c}ue^{ct}\left(e^{-ct} - 1\right) + \xi$$

$$x(t) = -\frac{u}{c}\left(1 - e^{ct}\right) + \xi$$

$$\xi = x(t) + \frac{u}{c}\left(1 - e^{ct}\right)$$

so we get

$$u(x,t) = f\left(x + \frac{u}{c}\left(1 - e^{ct}\right)\right)e^{-ct}$$

The factor $e^{-ct}$ in the solution causes damping since $c$ is positive.

8. b. Breaking time can be found by looking at the time that $u_t$ or $u_x$ tend to $\infty$

$$u_t = -cu + f' \left\{ \frac{u_t}{c}(1 - e^{ct}) - ue^{ct} \right\} e^{-ct}$$

$$u_x = f' \left\{ 1 + \frac{u_x}{c}(1 - e^{ct}) \right\} e^{-ct}$$

Solve for $u_t$

$$u_t = \frac{-cu - uf'}{1 + \frac{f'}{c}(1 - e^{ct})}$$

and for $u_x$

$$u_x = \frac{f'e^{-ct}}{1 + \frac{f'}{c}(1 - e^{ct})}$$

The smallest $t$ for which the denominator is zero ($u_x$ and $u_t \to \infty$) is

$$1 + \frac{f'}{c}(1 - e^{ct}) = 0$$

Since $c, t$ are positive, the factor $(1 - e^{ct}) \geq 0$ and so

$$f'(\xi) < -c$$

Another way is by looking at the family of characteristics

$$x(t) = -\frac{1}{c} f(\xi) \left( e^{-ct} - 1 \right) + \xi$$

or

$$F(x, t, \xi) = x(t) + \frac{1}{c} f(\xi) \left( e^{-ct} - 1 \right) - \xi$$

To find the envelope means to find a solution for

$$F = 0, \quad \text{and} \quad \frac{\partial F}{\partial \xi} = 0$$

(i.e. the equation of characteristics is satisfied for all $\xi$. Now

$$\frac{\partial F}{\partial \xi} = \frac{f'(\xi)}{c} \left( e^{-ct} - 1 \right) - 1 = 0$$

Since $t > 0$, this last condition gives the same result as before $f'(\xi) < -c$

9.

$$\rho_t + \rho u_x + u\rho_x = 0 \qquad u_t + uu_x = 0$$

with $u(x,0) = f(x)$ and $\rho(x,0) = g(x)$.

The set of ODEs is

$$\frac{dx}{dt} = u$$

$$\frac{du}{dt} = 0$$

Solve the second ODE to get $u(x(t),t) = K = f(\xi)$.

Now susbstitute this in the first ODE and solve

$$x = \underbrace{f(\xi)}_{=u} t + \xi$$

So

$$\xi = x - tu$$

and $\boxed{u(x,t) = f(x - tu)}$

Now use this $u$ in the first PDE

$$\rho_t + \rho u_x + u\rho_x = 0$$

The ODEs for this are

$$\frac{dx}{dt} = u, \qquad u \text{ is known!}$$

and

$$\frac{d\rho}{dt} = -\rho u_x$$

Solve the first to get

$$x(t) = ut + \xi$$

To solve the second we write (recall $u_x$ can be found from the solution $u$)

$$\frac{d\rho}{\rho} = -u_x dt$$

Differentiate $u$ to get

$$u_x = (1 - tu_x)f'$$

or

$$u_x = \frac{f'}{1 + tf'}$$

$$\frac{d\rho}{\rho} = -\frac{f'}{1 + tf'}dt$$

Integrate and use the known $u$ and $u_x$

$$\ln \rho = -\ln \left(1 + tf'(\xi)\right) + K$$

Or

$$\rho = \frac{K}{1 + tf'(\xi)}$$

Use the initial condition

$$\rho(\xi, 0) = g(\xi)$$

and the value of $\xi$, we get

$$\rho = \frac{g(x - ut)}{1 + tf'(x - ut)}$$

### 7.2.2 Graphical Solution

### 7.2.3 Numerical Solution

### 7.2.4 Fan-like Characteristics

### 7.2.5 Shock Waves

**Problems**

1. Consider Burgers' equation

$$\frac{\partial \rho}{\partial t} + u_{max}\left[1 - \frac{2\rho}{\rho_{max}}\right]\frac{\partial \rho}{\partial x} = \nu\frac{\partial^2 \rho}{\partial x^2}$$

Suppose that a solution exists as a density wave moving without change of shape at a velocity $V$, $\rho(x,t) = f(x - Vt)$.

a. What ordinary differential equation is satisfied by $f$

b. Show that the velocity of wave propagation, $V$, is the same as the shock velocity separating $\rho = \rho_1$ from $\rho = \rho_2$ (occuring if $\nu = 0$).

2. Solve

$$\frac{\partial \rho}{\partial t} + \rho^2\frac{\partial \rho}{\partial x} = 0$$

subject to

$$\rho(x,0) = \begin{cases} 4 & x < 0 \\ 3 & x > 0 \end{cases}$$

3. Solve

$$\frac{\partial u}{\partial t} + 4u\frac{\partial u}{\partial x} = 0$$

subject to

$$u(x,0) = \begin{cases} 3 & x < 1 \\ 2 & x > 1 \end{cases}$$

4. Solve the above equation subject to

$$u(x,0) = \begin{cases} 2 & x < -1 \\ 3 & x > -1 \end{cases}$$

5. Solve the quasilinear equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

subject to

$$u(x,0) = \begin{cases} 2 & x < 2 \\ 3 & x > 2 \end{cases}$$

6. Solve the quasilinear equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

subject to

$$u(x,0) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & 1 \le x \end{cases}$$

7. Solve the inviscid Burgers' equation

$$u_t + uu_x = 0$$

$$u(x, 0) = \begin{cases} 2 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad 0 < x < 1 \\ 0 & \text{for} \quad x > 1 \end{cases}$$

Note that two shocks start at $t = 0$, and eventually intersect to create a third shock. Find the solution for all time (analytically), and graphically display your solution, labeling all appropriate bounding curves.

1. a. Since

$$\rho(x,t) = f(x - Vt)$$

we have (using the chain rule)

$$\rho_t = f'(x - Vt) \cdot (-V)$$
$$\rho_x = f'(x - Vt) \cdot 1$$
$$\rho_{xx} = f''(x - Vt)$$

Substituting these derivatives in the PDE we have

$$-V f'(x - Vt) + u_{max} \left(1 - \frac{2f(x - Vt)}{\rho_{max}}\right) f'(x - Vt) = \nu f''(x - Vt)$$

This is a second order ODE for $f$.

b. For the case $\nu = 0$ the ODE becomes

$$-V f'(x - Vt) + u_{max} \left(1 - \frac{2f(x - Vt)}{\rho_{max}}\right) f'(x - Vt) = 0$$

Integrate (recall that the integral of $2ff'$ is $f^2$)

$$-V f(x - Vt) + u_{max} \left(f(x - Vt) - \frac{f^2(x - Vt)}{\rho_{max}}\right) = C$$

To find the constant, we use the following
As $x \to \infty$, $\rho \to \rho_2$ and as $x \to -\infty$, $\rho \to \rho_1$, then

$$-V \rho_2 + u_{max} \left(\rho_2 - \frac{\rho_2^2}{\rho_{max}}\right) = C$$

$$-V \rho_1 + u_{max} \left(\rho_1 - \frac{\rho_1^2}{\rho_{max}}\right) = C$$

Subtract

$$V (\rho_1 - \rho_2) + u_{max} \left(\rho_2 - \frac{\rho_2^2}{\rho_{max}}\right) - u_{max} \left(\rho_1 - \frac{\rho_1^2}{\rho_{max}}\right) = 0$$

Solve for $V$

$$V = \frac{u_{max} \left(\rho_2 - \frac{\rho_2^2}{\rho_{max}}\right) - u_{max} \left(\rho_1 - \frac{\rho_1^2}{\rho_{max}}\right)}{\rho_2 - \rho_1} \tag{1}$$

This can be written as

$$V = u_{max} - \frac{u_{max}}{\rho_{max}} (\rho_1 + \rho_2)$$

Note that (1) is

$$V = \frac{[q]}{[\rho]}$$

since

$$q = u_{max} \left(\rho - \frac{\rho^2}{\rho_{max}}\right)$$

Thus $V$ given in (1) is exactly the shock speed.

2. The set of ODEs is

$$\frac{dx}{dt} = \rho^2 \qquad \frac{d\rho}{dt} = 0$$

The solution of the first ODE requires the yet unknown $\rho$ thus we tackle the second ODE

$$d\rho = 0$$

Now integrate this

$$\rho(x,t) = K$$

At $t = 0$ the solution is

$$\rho(x_0, 0) = K = \begin{cases} 4 & x < 0 \\ 3 & x > 0 \end{cases}$$

Thus

$$\rho(x,t) = \rho(x_0, 0)$$

Now substitute this solution in the characteristic equation (first ODE)

$$\frac{dx}{dt} = \rho^2(x_0, 0)$$

Integrate

$$x = \rho^2(x_0, 0)\, t + C$$

For $t = 0$ we get

$$x_0 = 0 + C$$

Thus

$$C = x_0$$

and the characteristics are

$$x = \rho^2(x_0, 0)\, t + x_0$$

For $x_0 < 0$ then $\rho(x_0, 0) = 4$ and the characteristic is then given by $x = x_0 + 16t$
Therefore for $x_0 = x - 16t < 0$ the solution is $\rho = 4$.

For $x_0 > 0$ then $\rho(x_0, 0) = 3$ and the characteristic is then given by $x = x_0 + 9t$
Therefore for $x_0 = x - 9t < 0$ the solution is $\rho = 3$.

Notice that there is a shock (since the value of $\rho$ is decreasing with increasing $x$). The shock characteristc is given by

$$\frac{dx_s}{dt} = \frac{\frac{1}{3}\cdot 4^3 - \frac{1}{3}\cdot 3^3}{4 - 3} = \frac{\frac{1}{3}(64 - 27)}{1} = \frac{37}{3}$$

The solution of this ODE is

$$x_s = \frac{37}{3}t + x_s(0)$$

$x_s(0)$ is where the shock starts, i.e. the discontinuity at time $t = 0$.

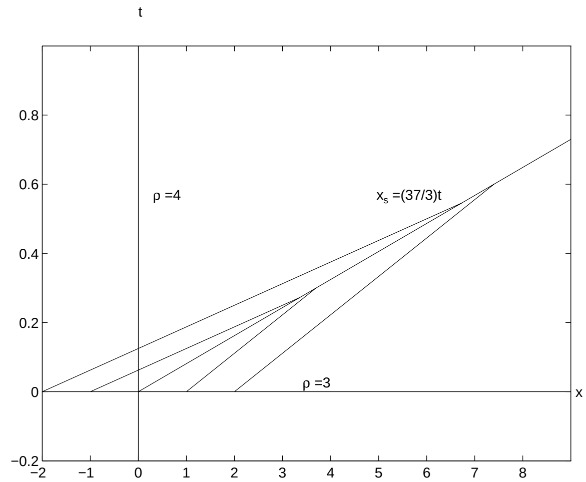


Figure 49: Characteristics for problem 2

Thus $x_s(0) = 0$ and the shock characteristic is

$$x_s = \frac{37}{3} t$$

See figure for the characteristic curves including the shock's. The solution in region I above the shock chracteristic is $\rho = 4$ and below (region II) is $\rho = 3$.

3.

$$u_t + 4uu_x = 0 \qquad\qquad \text{or } u_t + (2u^2)_x = 0$$

$$u(x, 0) = \begin{cases} 3 & x < 1 \\ 2 & x > 1 \end{cases}$$

Shock again

The shock characteristic is obtained by solving:

$$\frac{dx_s}{dt} = \frac{2 \cdot 3^2 - 2 \cdot 2^2}{3 - 2} = 10$$

$$x_s = 10t + \underbrace{x_s(0)}_{=1}$$

$$\underline{x_s = 10t + 1}$$

Now we solve the ODE for $u$:

$$\frac{du}{dt} = 0 \quad \Rightarrow \quad \underline{u(x, t) = u(x_0, 0)} \qquad \text{away from shock}$$

The ODE for $x$ is:

$$\frac{dx}{dt} = 4u = 4u(x_0, 0)$$

$$\underline{x = 4u(x_0, 0))t + x_0}$$

If $\quad x_0 < 1 \qquad x_0 = x - 12t \qquad \Rightarrow \qquad \underline{x < 1 + 12t}$

$\quad x_0 < 1 \qquad x_0 = x - 8t \qquad \Rightarrow \qquad \underline{x > 1 + 8t}$

4. Solve

$$u_t + 4uu_x = 0$$

$$u(x,0) = \begin{cases} 2 & x < -1 \\ 3 & x > -1 \end{cases}$$

$$\frac{du}{dt} = 0 \qquad u(x,t) = u(x_0, 0)$$

$$\frac{dx}{dt} = 4u = 4u(x_0, 0)$$

$$dx = 4u\,(x_0, 0)\,dt$$

$$\underline{x = 4u\,(x_0, 0)\,t + x_0}$$

For $x_0 < -1 \qquad x = 8t + x_0 \qquad \Rightarrow \qquad x - 8t < -1$

$x_0 > -1 \qquad x = 12t + x_0 \qquad \Rightarrow \qquad x - 12t > -1$

$$u(x,t) = \begin{cases} 2 & x < 8t - 1 \\ ? & 8t - 1 < x < 12t - 1 \\ 3 & x > 12t - 1 \end{cases}$$

$$x = 4u\,t + \underbrace{x_0}_{=-1 \text{ discontinuity}}$$

$$u = \frac{x+1}{4t}$$

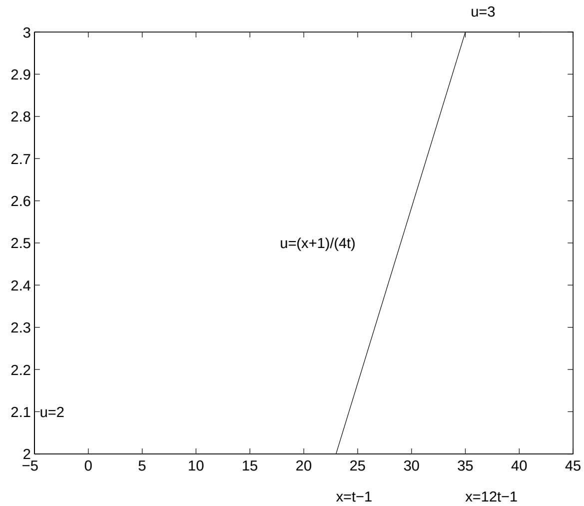


Figure 50: Solution for 4

5.

$$u_t + uu_x = 0$$

$$u(x, 0) = \begin{cases} 2 & x < 2 \\ 3 & x > 2 \end{cases}$$

fan

$$\frac{du}{dt} = 0 \quad \Rightarrow \quad u(x, t) = u(x_0, 0)$$

$$\frac{dx}{dt} = u = u(x_0, 0)$$

$$\Downarrow$$

$$x = t\,u\,(x_0, 0) + x_0$$

For $x_0 < 2$ $\quad x = 2t + x_0 \quad \Rightarrow \quad x - 2t < 2$

For $x_0 > 2$ $\quad x = 3t + x_0 \quad \Rightarrow \quad x - 3t > 2$

$x = t\,u\,(x_0, 0) + x_0$ at discontinuity $x_0 = 2$

we get $x = t\,u + 2$

$$u = \frac{x - 2}{t}$$

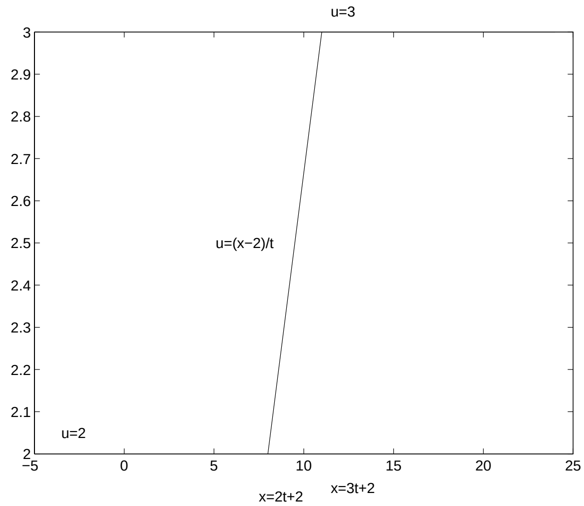


Figure 51: Solution for 5

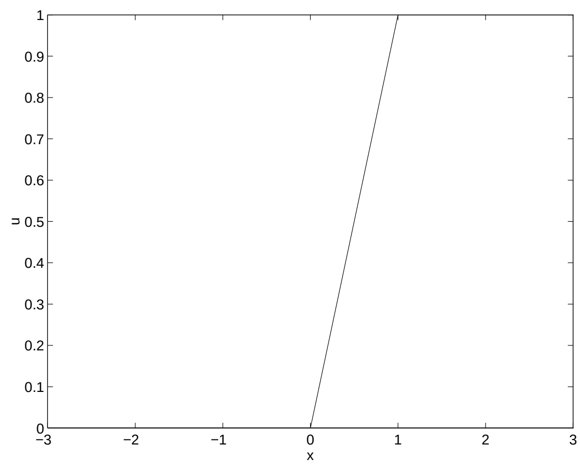


Figure 52: Sketch of initial solution

6.

$$u_t + uu_x = 0$$

$$u(x, 0) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

$$\frac{du}{dt} = 0 \quad \Rightarrow \quad u(x, t) = u(x_0, 0)$$

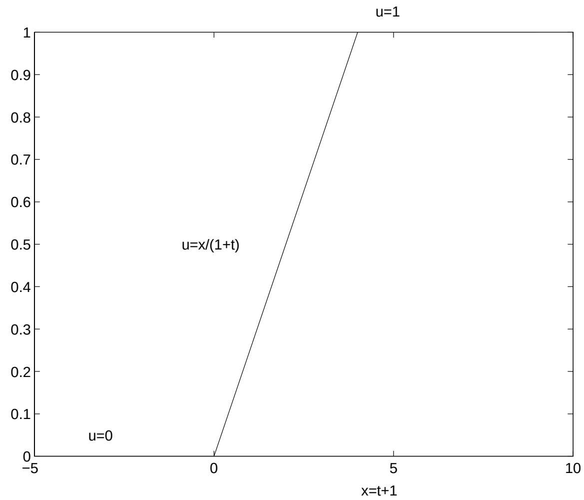


Figure 53: Solution for 6

$$\frac{dx}{dt} = u = u(x_0, 0) \Rightarrow \quad x = t\,u\,(x_0, 0) + x_0$$

| | | | | | |
|---|---|---|---|---|---|
| For | $x_0 < 0$ | $x = t \cdot 0 + x_0$ | $\Rightarrow$ | $x = x_0$ | $u = 0$ |
| | $0 \leq x_0 < 1$ | $x = t\,x_0 + x_0$ | $\Rightarrow$ | $x = x_0\,(1 + t)$ | $u = x_0 = \frac{x}{1+t}$ |
| | $1 \leq x_0$ | $x = t + x_0$ | $\Rightarrow$ | $x = t + x_0$ | $u = 1$ |

Basically the interval $[0, 1]$ is stretched in time to $[0, 1 + \mathrm{t}]$.

7.

$$u_t + uu_x = 0$$

$$u(x, 0) = \begin{cases} 2 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad 0 < x < 1 \\ 0 & \text{for} \quad x > 1 \end{cases}$$

First find the shock characteristic for those with speed $u = 2$ and $u = 1$

$$[q] = \frac{1}{2}u^2\Big|_1^2 = \frac{1}{2}(2^2 - 1^2) = \frac{3}{2}$$

$$[u] = 2 - 1 = 1$$

Thus

$$\frac{dx_s}{dt} = \frac{3}{2}$$

and the characteristic through $x = 0$ is then

$$x_s = \frac{3}{2}t$$

Similarly for the shock characteristic for those with speed $u = 1$ and $u = 0$

$$[q] = \frac{1}{2}u^2\Big|_0^1 = \frac{1}{2}(1^2 - 0^2) = \frac{1}{2}$$

$$[u] = 1 - 0 = 1$$

Thus

$$\frac{dx_s}{dt} = \frac{1}{2}$$

and the characteristic through $x = 1$ is then

$$x_s = \frac{1}{2}t + 1$$

Now these two shock charateristic going to intersect. The point of intersection is found by equating $x_s$ in both, i.e.

$$\frac{1}{2}t + 1 = \frac{3}{2}t$$

The solution is $t = 1$ and $x_s = \frac{3}{2}$. Now the speeds are $u = 2$ and $u = 0$

$$[q] = \frac{1}{2}u^2\Big|_0^2 = \frac{1}{2}(2^2 - 0^2) = 2$$

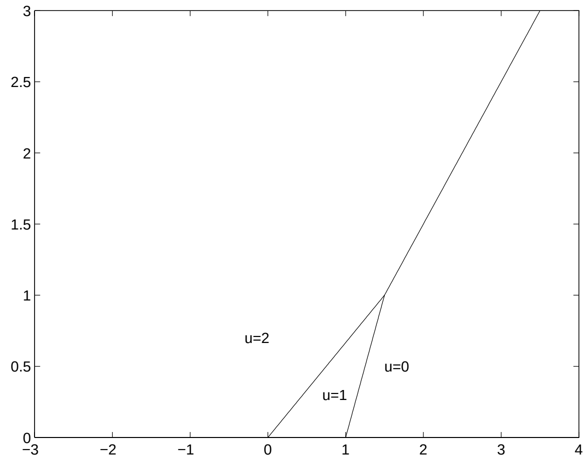


Figure 54: Solution for 7

$$[u] = 2 - 0 = 2$$

Thus

$$\frac{dx_s}{dt} = 1$$

and the characteristic is then

$$x_s = t + C.$$

To find $C$, we substitute the point of intersection $t = 1$ and $x_s = \frac{3}{2}$. Thus

$$\frac{3}{2} = 1 + C$$

or

$$C = \frac{1}{2}$$

The third shock characteristic is then

$$x_s = t + \frac{1}{2}.$$

The shock characteristics and the solutions in each domain are given in the figure above.

## 7.3 Second Order Wave Equation

### 7.3.1 Infinite Domain

**Problems**

1. Suppose that

$$u(x,t) = F(x - ct).$$

Evaluate

a. $\dfrac{\partial u}{\partial t}(x,0)$

b. $\dfrac{\partial u}{\partial x}(0,t)$

2. The general solution of the one dimensional wave equation

$$u_{tt} - 4u_{xx} = 0$$

is given by

$$u(x,t) = F(x - 2t) + G(x + 2t).$$

Find the solution subject to the initial conditions

$$u(x,0) = \cos x \qquad -\infty < x < \infty,$$

$$u_t(x,0) = 0 \qquad -\infty < x < \infty.$$

3. In section 3.1, we suggest that the wave equation can be written as a system of two first order PDEs. Show how to solve

$$u_{tt} - c^2 u_{xx} = 0$$

using that idea.

1a.

$$u(x,\, t) \,=\, F(x \,-\, ct)$$

Use the chain rule:

$$\frac{\partial\, u}{\partial\, t} \,=\, -\, c\, \frac{dF(x \,-\, ct)}{d(x \,-\, ct)}$$

at $t \,=\, 0$

$$\underline{\frac{\partial\, u}{\partial\, t} \,=\, -\, c\, \frac{dF(x)}{dx}}$$

1b.

$$\frac{\partial\, u}{\partial\, x} \,=\, \frac{dF(x \,-\, ct)}{d(x \,-\, ct)} \,\cdot\, 1$$

at $x \,=\, 0$

$$\frac{\partial\, u}{\partial\, x} \,=\, \frac{dF(-\, ct)}{d(-\, ct)} \,=\, -\, \frac{1}{c}\, \frac{dF(-\, ct)}{dt} \,=\, F'\,(-\, ct)$$

$\uparrow$

differentiation with respect to argument

2. $u\,(x,\,t) \,=\, F\,(x\,-2t) \,+\, G(x\,+\,2t)$

$u\,(x,\,0) \,=\, \cos\,x$

$u_t\,(x,\,t) \,=\, 0$

$u\,(x,\,0) \,=\, F(x) \,+\, G(x) \,=\, \cos\,x \qquad (*)$

$u_t\,(x,\,t) \,=\, -\,2F'\,(x\,-\,2t) \,+\, 2G'\,(x\,+\,2t)$

$\Rightarrow u_t\,(x,\,0) \,=\, -\,2F'\,(x) \,+\, 2G'\,(x) \,=\, 0$

Integrate $\Rightarrow -\,F\,(x) \,+\, G\,(x) \,=\, \text{constant} \,=\, k \qquad (\#)$

solve the 2 equations (*) and (#)

$$2G(x) \,=\, \cos x \,+\, k$$

$$G(x) \,=\, \tfrac{1}{2}\,\cos x \,+\, \tfrac{1}{2}k$$

$$2F(x) \,=\, \cos x \,-\, k$$

$$F(x) \,=\, \tfrac{1}{2}\,\cos x \,-\, \tfrac{1}{2}k$$

We need $F(x\,-\,2t) \Rightarrow F(x\,-\,2t) \,=\, \tfrac{1}{2}\,cos\,(x\,-\,2t) \,-\, \tfrac{1}{2}k$

$$G(x\,+\,2t) \,=\, \tfrac{1}{2}\,cos\,(x\,+\,2t) \,+\, \tfrac{1}{2}k$$

$\Rightarrow\; u(x,\,t) \,=\, \tfrac{1}{2}\,\cos\,(x\,-\,2t) \,+\, \tfrac{1}{2}\,\cos\,(x\,+\,2t) \,-\, \tfrac{1}{2}k \,+\, \tfrac{1}{2}k$

$$\underline{u(x,\,t) \,=\, \frac{1}{2}\,\{cos\,(x\,-\,2t) \,+\, \cos\,(x\,+\,2t)\}}$$

3. The wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

can be written as a system of two first order PDEs

$$\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = 0$$

and

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = v.$$

Solving the first for $v$, by rewriting it as a system of ODEs

$$\frac{dv}{dt} = 0$$

$$\frac{dx}{dt} = -c$$

The characteristic equation is solved

$$x = -ct + x_0$$

and then

$$v(x,t) = v(x_0, 0) = V(x + ct)$$

where $V$ is the initial solution for $v$. Now use this solution in the second PDE rewritten as a system of ODEs

$$\frac{du}{dt} = V(x + ct)$$

$$\frac{dx}{dt} = c$$

The characteristic equation is solved

$$x = ct + x_0$$

and then

$$\frac{du}{dt} = V(x + ct) = V(x_0 + 2ct)$$

Integrating

$$u(x_0, t) = \int_0^t V(x_0 + 2c\tau)d\tau + K(x_0)$$

Change variables

$$z = x_0 + 2c\tau$$

then

$$dz = 2cd\tau$$

The limits of integration become $x_0$ and $x_0 + 2ct$. Thus the solution

$$u(x_0, t) \;=\; \int_{x_0}^{x_0+2ct} \frac{1}{2c} V(z)dz \;+\; K(x_0)$$

But $x_0 \;=\; x - ct$

$$u(x, t) \;=\; \int_{x-ct}^{x+ct} \frac{1}{2c} V(z)dz \;+\; K(x - ct)$$

Now break the integral using the point zero.

$$u(x, t) \;=\; K(x - ct) \;-\; \int_{0}^{x-ct} \frac{1}{2c} V(z)dz \;+\; \int_{0}^{x+ct} \frac{1}{2c} V(z)dz$$

The first two terms give a function of $x - ct$ and the last term is a function of $x + ct$, exactly as we expect from D'Alembert solution.

### 7.3.2 Semi-infinite String

### 7.3.3 Semi-infinite String with a Free End

### Problems

1. Solve by the method of characteristics

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \qquad x > 0$$

subject to

$$u(x,0) = 0,$$
$$\frac{\partial u}{\partial t}(x,0) = 0,$$
$$u(0,t) = h(t).$$

2. Solve

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \qquad x < 0$$

subject to

$$u(x,0) = \sin x, \qquad x < 0$$
$$\frac{\partial u}{\partial t}(x,0) = 0, \qquad x < 0$$
$$u(0,t) = e^{-t}, \qquad t > 0.$$

3. a. Solve

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 < x < \infty$$

subject to

$$u(x,0) = \begin{cases} 0 & 0 < x < 2 \\ 1 & 2 < x < 3 \\ 0 & 3 < x \end{cases}$$
$$\frac{\partial u}{\partial t}(x,0) = 0,$$
$$\frac{\partial u}{\partial x}(0,t) = 0.$$

b. Suppose $u$ is continuous at $x = t = 0$, sketch the solution at various times.

4. Solve

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \qquad x > 0, \qquad t > 0$$

subject to

$$u(x,0) = 0,$$

$$\frac{\partial u}{\partial t}(x,0) = 0,$$
$$\frac{\partial u}{\partial x}(0,t) = h(t).$$

5. Give the domain of influence in the case of semi-infinite string.

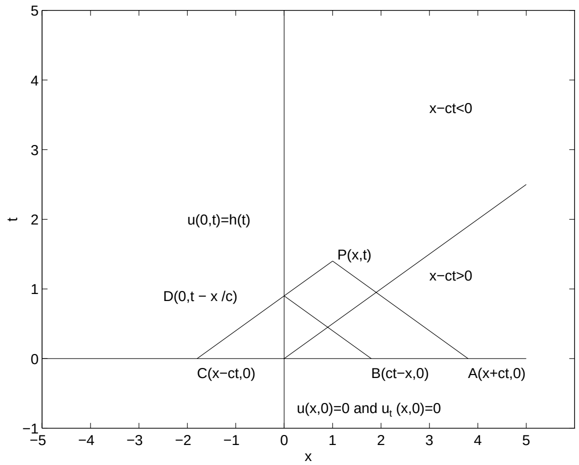


Figure 55: Domain for problem 1

1. $u_{tt} - c^2 u_{xx} = 0$

$u(x,\, 0) = 0$

$u_t(x,\, 0) = 0$

$u(0,\, t) = h(t)$

Solution $\quad u(x,\, t) = F(x - ct) + G(x + ct)$

(*)

$$F(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(\tau)\, d\tau = 0$$

(#)

$$G(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(\tau)\, d\tau = 0$$

since both $f(x)$, $g(x)$ are zero.

Thus for $x - ct > 0$ the solution is zero

(No influence of boundary at $x = 0$)

For $x - ct < 0$ $\quad u(0,\, t) = h(t)$

$$\Downarrow$$

$$F(-ct) + G(ct) = h(t)$$

$u(x,\, t) = F(x - ct) + G(x + ct)$

but argument of F is negative and thus we cannot use (*), instead

$$F(-ct) = h(t) - G(ct)$$

or $F(z) = h(\frac{-z}{c}) - G(-z)$ for $z < 0$

$$F(x - ct) = h(-\tfrac{x-ct}{c}) - G(-(x - ct))$$

$$= h(t - \tfrac{x}{c}) - G(ct - x)$$

$$u(x, t) = F(x - ct) + G(x + ct)$$

$$= h(t - \tfrac{x}{c}) - \underbrace{G(ct - x)}_{\text{zero}} + \underbrace{G(x + ct)}_{\text{zero}}$$

since the arguments are positive and (#) is valid

$$\Rightarrow \quad u(x, t) = h(t - \tfrac{x}{c}) \quad \text{for} \quad 0 < x < ct$$

$$u(x, t) = 0 \qquad \text{for} \quad x - ct > 0$$

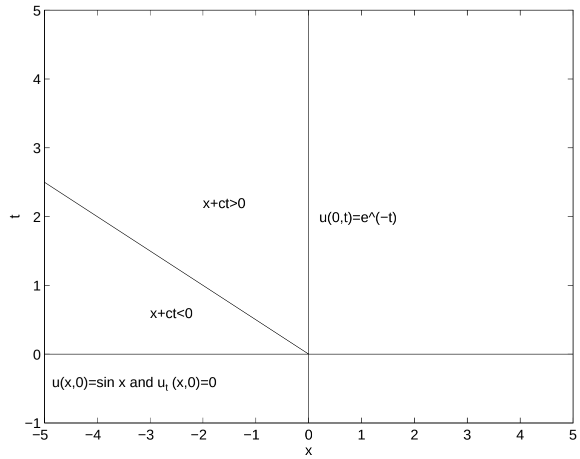


Figure 56: Domain for problem 2

2. $u_{tt} - c^2 u_{xx} = 0 \qquad x < 0$

$u(x, 0) = \sin x \qquad x < 0$

$u_t(x, 0) = 0 \qquad x < 0$

$u(0, t) = e^{-t} \qquad t > 0$

$u(x, t) = F(x - ct) + G(x + ct)$

$F(x) = \frac{1}{2} \sin x$

since $f = \sin x$, $g = 0$

$G(x) = \frac{1}{2} \sin x$

From boundary condition

$u(0, t) = F(-ct) + G(ct) = e^{-t}$

If $x + ct < 0$ no influence of boundary at $x = 0$

$$\Rightarrow \quad u(x, t) = \frac{1}{2} \sin(x - ct) + \frac{1}{2} \sin(x + ct)$$

$$= \sin x \cos ct$$

$\vdots$

after some trigonometric manipulation

If $x + ct > 0$ then the argument of G is positive and thus

$$G(ct) = e^{-t} - F(-ct)$$

or $\quad G(z) = e^{-z/c} - F(-z)$

$$\Rightarrow \quad G(x + ct) = e^{-\frac{x + ct}{c}} - F(-(x + ct))$$

Therefore:

$$u(x,\, t) \;=\; F\,(x\,-\,ct)\,+\,G(x\,+\,ct)$$

$$=\; F\,(x\,-\,ct)\,+\,e^{-\frac{x\,+\,ct}{c}}\,-\,F\,(-\,x\,-\,ct)$$

$$=\; \tfrac{1}{2}\,\sin\,(x\,-\,ct)\,+\,e^{-\frac{x\,+\,ct}{c}}\,-\,\tfrac{1}{2}\,\underbrace{\sin\,(-\,x\,-\,ct)}_{-\,\sin\,(x+ct)}$$

$$=\; \underbrace{\frac{1}{2}\,\sin\,(x\,-\,ct)\,-\,\frac{1}{2}\,\sin\,(x\,+\,ct)}_{\cos\,ct\,\sin\,x}+e^{-\frac{x\,+\,ct}{c}}$$

$$\Rightarrow u(x,\, t) \;=\begin{cases} \sin x \cos ct & x + ct < 0 \\ \\ \sin x \cos ct + e^{-\frac{x\,+\,ct}{c}} & x + ct > 0 \end{cases}$$

3a. $u_{tt} = c^2 u_{xx} \qquad 0 < x < \infty$

$u_x(0, t) = 0$

$$u(x, 0) = \begin{cases} 0 & 0 < x < 2 \\ 1 & 2 < x < 3 \\ 0 & x > 3 \end{cases}$$

$u_t(x, 0) = 0$

$u(x, t) = F(x - ct) + G(x + ct)$

$$F(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(\xi)\, d\xi = \frac{1}{2} f(x) \qquad g \overset{since}{\equiv} 0 \qquad x > 0$$

$$G(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(\xi)\, d\xi = \frac{1}{2} f(x) \qquad g \equiv 0 \qquad x > 0$$

$$u(x, t) = \frac{f(x + ct) + f(x - ct)}{2}, \qquad x > ct$$

$u_x(0, t) = F'(-ct) + G'(ct) = 0$

$\Rightarrow \quad F'(-ct) = -G'(ct)$

$F'(-z) = -G'(z)$

Integrate

$-F(-z) = -G(z) + K$

$$\boxed{F(-z) = G(z) - K}$$

$\Rightarrow \quad F(x - ct) = -G(ct - x) - K \qquad \underline{x - ct < 0}$

$= \frac{1}{2} f(ct - x) - K \qquad \underline{x - ct < 0}$

$\Rightarrow \quad u(x, t) = \frac{1}{2} f(x + ct) + \frac{1}{2} f(ct - x) - K$

To find $K$ we look at $x = 0, t = 0 \qquad u(0, 0) = 0$ from initial condition

but $u(0, 0) = \frac{1}{2} f(0) + \frac{1}{2} f(0) - K = \underbrace{f(0)}_{=0} - K$ from above

$\Rightarrow \quad K = 0$

$$\Rightarrow \quad u(x, t) = \frac{f(x + ct) + f(ct - x)}{2}$$

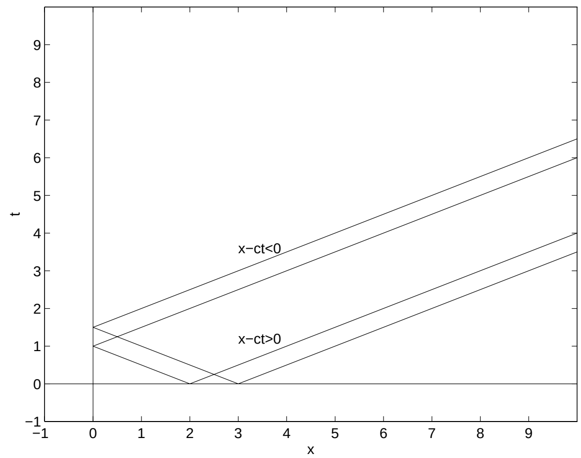


Figure 57: Domain of influence for problem 3

3b.

$$u(x,\, t) \;=\; \begin{cases} \dfrac{f(x\,+\,ct)\;+\;f(x\;-\;ct)}{2} & x > ct \\ \\ \dfrac{f(x\,+\,ct)\;+\;f(ct\;-\;x)}{2} & x < ct \end{cases}$$

where

$$u(x,\, t) \;=\; \begin{cases} 1 & \text{Region I} \\ \\ \dfrac{1}{2} & \text{Region II} \\ \\ 0 & \text{otherwise} \end{cases}$$

In order to find the regions I and II mentioned above, we use the idea of domain of influence. Sketch both characteristics from the end points of the interval (2,3) and remember that when the characteristic curve (line in this case) reaches the $t$ axis, it will be reflected.

As can be seen in the figure, the only region where the solution is 1 is the two triangular regions. Within the three strips (not including the above mentioned triangles), the solution is $\dfrac{1}{2}$ and for the rest, the solution is zero.

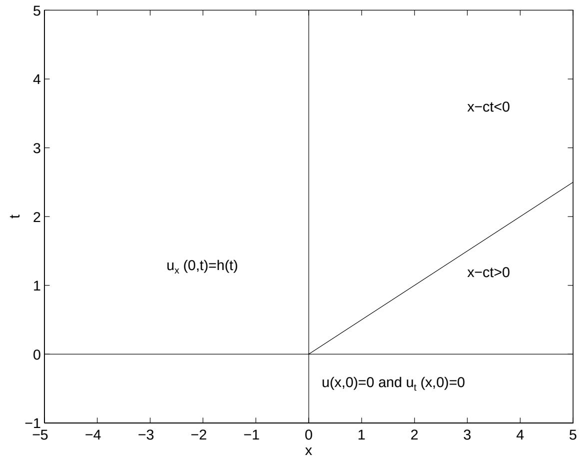


Figure 58: Domain for problem 4

4. $u_{tt} - c^2 u_{xx} = 0$

general solution

$u(x, t) = F(x - ct) + G(x + ct)$
For $x - ct > 0$ $u(x, t) = 0$ since $u = u_t = 0$ on the boundary.

For $x - ct < 0$ we get the influence of the boundary condition

$$u_x(0, t) = h(t)$$

Differentiate the general solution:

$u_x(x, t) = F'(x - ct) \cdot 1 + G'(x + ct) \cdot 1 = \frac{dF(x-ct)}{d(x-ct)} + \frac{dG(x+ct)}{d(x+ct)}$
⋮
chain rule
prime means derivative with respect to argument

As $x = 0$ :

$h(t) = u_x(0, t) = \frac{dF(-ct)}{d(-ct)} + \frac{dG(ct)}{d(ct)} = -\frac{1}{c}\frac{dF(-ct)}{dt} + \frac{1}{c}\frac{dG(ct)}{dt}$

Integrate

$$-\frac{1}{c}\,F(-\,ct)\;+\;\frac{1}{c}\,\underbrace{G(ct)}_{=\,0}+\frac{1}{c}\,\underbrace{F(0)}_{=\,0}-\frac{1}{c}\,\underbrace{G(0)}_{=\,0}\;=\;\int_0^t\,h(\tau)\,d\tau$$

since $f\;=\;g\;=\;0$

$$F(-\,ct)\;=\;-c\,\int_0^t\,h(\tau)\,d\tau$$

$$F(z)\;=\;-c\,\int_0^{-z/c}\,h(\tau)\,d\tau$$

$$\Rightarrow\quad F(x\;-\;ct)\;=\;-c\,\int_0^{-\frac{x\,-\,ct}{c}}\,h(\tau)\,d\tau$$

$$\Rightarrow u(x,\,t)\;=\begin{cases}0 & x-ct>0\\[2ex] -\,c\,\displaystyle\int_0^{-\frac{x\,-\,ct}{c}}\,h(\tau)\,d\tau & x-ct<0\end{cases}$$

5. For the infinite string the domain of influence is a wedge with vertex at the point of interest $(x, 0)$. For the semi infinite string, the left characteristic is reflected by the vertical $t$ axis and one obtains a strip, with one along a characteristic $(x + ct = C)$ reaching the $t$ axis and the other two sides are from the other family of characteristcs $(x - ct = K)$.

### 7.3.4 Finite String

# 8 Finite Differences

## 8.1 Taylor Series

## 8.2 Finite Differences

### Problems

1. Verify that

$$\frac{\partial^3 u}{\partial x^3}\Big|_{i,j} = \frac{\Delta_x^3 u_{i,j}}{(\Delta x)^3} + O(\Delta x).$$

2. Consider the function $f(x) = e^x$. Using a mesh increment $\Delta x = 0.1$, determine $f'(x)$ at $x = 2$ with forward-difference formula, the central-difference formula, and the second order three-point formula. Compare the results with the exact value. Repeat the comparison for $\Delta x = 0.2$. Have the order estimates for truncation errors been a reliable guide? Discuss this point.

3. Develop a finite difference approximation with T.E. of $O(\Delta y)$ for $\partial^2 u/\partial y^2$ at point $(i, j)$ using $u_{i,j}$, $u_{i,j+1}$, $u_{i,j-1}$ when the grid spacing is **not** uniform. Use the Taylor series method. Can you devise a three point scheme with second-order accuracy with unequal spacing? Before you draw your final conclusions, consider the use of compact implicit representations.

4. Establish the T.E. for the following finite difference approximation to $\partial u/\partial y$ at the point $(i, j)$ for a uniform mesh:

$$\frac{\partial u}{\partial y} \approx \frac{-3u_{i,j} + 4u_{i,j+1} - u_{i,j+2}}{2\Delta y}.$$

What is the order?

1. Verify $u_{xxx\,ij} = \dfrac{\Delta_x^3 u_{i\,j}}{(\Delta_x)^3} + O(\Delta x)$

Recall

$$\begin{aligned}
\Delta_x^3 u_{i\,j} &= \Delta_x\left(\Delta_x^2 u_{i\,j}\right) = \Delta_x\left(\Delta_x\left(u_{i+1\,j} - u_{i\,j}\right)\right) \\
&= \Delta_x\left(u_{i+2\,j} - u_{i+1\,j} - u_{i+1\,j} + u_{i\,j}\right) \\
&= \Delta_x\left(u_{i+2\,j} - 2u_{i+1\,j} + u_{i\,j}\right) \\
&= u_{i+3\,j} - u_{i+2\,j} - 2u_{i+2\,j} + 2u_{i+1\,j} + u_{i+1\,j} - u_{i\,j} \\
&= u_{i+3\,j} - 3u_{i+2\,j} + 3u_{i+1\,j} - u_{i\,j}
\end{aligned}$$

Now use Taylor series for each term

$$u_{i+3\,j} = u_{i\,j} + 3\Delta x u_{x\,i\,j} + \frac{(3\Delta x)^2}{2} u_{xx\,i\,j} + \frac{(3\Delta x)^3}{6} u_{xxx\,i\,j} + \frac{(3\Delta x)^4}{24} u_{xxxx\,i\,j} + \cdots$$

$$u_{i+2\,j} = u_{i\,j} + 2\Delta x u_{x\,i\,j} + \frac{(2\Delta x)^2}{2} u_{xx\,i\,j} + \frac{(2\Delta x)^3}{6} u_{xxx\,i\,j} + \frac{(2\Delta x)^4}{24} u_{xxxx\,i\,j} + \cdots$$

$$u_{i+1\,j} = u_{i\,j} + \Delta x u_{x\,i\,j} + \frac{(\Delta x)^2}{2} u_{xx\,i\,j} + \frac{(\Delta x)^3}{6} u_{xxx\,i\,j} + \frac{(\Delta x)^4}{24} u_{xxxx\,i\,j} + \cdots$$

Combine these series we get

$$\begin{aligned}
\Delta_x^3 u_{i\,j} &= u_{i+3\,j} - 3u_{i+2\,j} + 3u_{i+1\,j} - u_{i\,j} \\
&= \underbrace{(1 - 3 + 3 - 1)}_{=0} u_{i\,j} + \underbrace{(3\Delta x - 6\Delta x + 3\Delta x)}_{=0} u_{x\,i\,j} \\
&+ \underbrace{(\frac{9}{2}\Delta x^2 - \frac{12}{2}\Delta x^2 + \frac{3}{2}\Delta x^2)}_{=0} u_{xx\,i\,j} + \underbrace{(\frac{27}{6}\Delta x^3 - \frac{24}{6}\Delta x^3 + \frac{3}{6}\Delta x^3)}_{=\Delta x^3} u_{xxx\,i\,j} \\
&+ \underbrace{(\frac{81}{24}\Delta x^4 - \frac{48}{24}\Delta x^4 + \frac{3}{24}\Delta x^4)}_{=\frac{36}{24}\Delta x^4} u_{xxxx\,i\,j} + \cdots
\end{aligned}$$

$$\Delta_x^3 u_{i\,j} = (\Delta x)^3 u_{xxx\,i\,j} + \frac{3}{2}(\Delta x)^4 u_{xxxx\,i\,j} + \cdots$$

Now divide by $\Delta_x^3$ to get the answer.

2.

$$f(x) = e^x$$

$$\Delta_x = .1$$

Approximate $f'(x)$ at $x = 2$ using forward difference

$$f'(2) \sim \frac{f(2+.1) - f(2)}{.1} + O(.1)$$

$$f'(2) \sim \frac{e^{2.1} - e^2}{.1} = 7.7711381$$

Approximate $f'(x)$ at $x = 2$ using centered difference

$$f'(2) \sim \frac{f(2+.1) - f(2-.1)}{.2} + O(.1^2)$$

$$f'(2) \sim \frac{e^{2.1} - e^{1.9}}{.2} = 7.40137735$$

Approximate $f'(x)$ at $x = 2$ using second order three point

$$f'(2) \sim \frac{-f(2+.2) + 4f(2+.1) - 3f(2)}{.2} + O(.1^2)$$

$$f'(2) \sim \frac{-e^{2.2} + 4e^{2.1} - 3e^2}{.2} = 7.36248927$$

Exact answer

$$f'(2) = e^2 = 7.3890560989\ldots$$

| | $O(.1)$ Forward | $O(.1^2)$ Centered | $O(.1)^2$ 3-point |
|---|---|---|---|
| approximate | 7.7711381 | 7.40137735 | 7.36248927 |
| exact | 7.3890560989 | 7.3890560989 | 7.3890560989 |
| difference | .382082037 | .01232125 | -.0265668 |
| | $\sim .1$ | $\sim .1^2$ | $\sim .1^2$ |

Now use $\Delta x = .2$ Approximate $f'(x)$ at $x = 2$ using forward difference

$$f'(2) \sim \frac{f(2+.2) - f(2)}{.2} + O(.2)$$

$$f'(2) \sim \frac{e^{2.2} - e^2}{.2} = 8.179787$$

Approximate $f'(x)$ at $x = 2$ using centered difference

$$f'(2) \sim \frac{f(2+.2) - f(2-.2)}{.4} + O(.2^2)$$

$$f'(2) \sim \frac{e^{2.2} - e^{1.8}}{.2} = 7.438415087$$

Approximate $f'(x)$ at $x = 2$ using second order three point

$$f'(2) \sim \frac{-f(2+.4) + 4f(2+.2) - 3f(2)}{.4} + O(.1^2)$$

$$f'(2) \sim \frac{-e^{2.4} + 4e^{2.2} - 3e^2}{.4} = 7.2742733$$

| | $O(.2)$ | $O(.2^2)$ | $O(.2)^2$ |
|---|---|---|---|
| | Forward | Centered | 3-point |
| approximate | 8.179787 | 7.438415087 | 7.2742733 |
| exact | 7.3890560989 | 7.3890560989 | 7.3890560989 |
| difference | .7907309 | .0493589 | -.01147827 |
| | $\sim .2$ | $\sim .2^2$ | $\sim .2^2$ |

The ratio of errors in forward difference was cut by a factor of 2 when the $\Delta x$ is halved, for the second order approximation, the error was cut by 4 ($= 2^2$) when $\Delta x$ is halved. So this is a relatively reliable guide.

3. Develop first order approximation for $u_{yy\,i\,j}$ using the points $u_{i\,j}$, $u_{i\,j+1}$, $u_{i\,j+2}$ nonuniformly spaced.

Let $h_1 = y_{j+1} - y_j$ and $h_2 = y_j - y_{j-1}$, then

$$u_{yy\,i\,j} \;=\; Au_{i\,j} + Bu_{i\,j+1} + Cu_{i\,j-1}$$

with $A, B, C$ to be determined. Now take Taylor series expansions

$$u_{i\,j+1} = u_{i\,j} + h_1 u_{y\,i\,j} + \frac{h_1^2}{2} u_{yy\,i\,j} + \frac{h_1^3}{6} u_{yyy\,i\,j} + \cdots$$

$$u_{i\,j-1} = u_{i\,j} - h_2 u_{y\,i\,j} + \frac{h_2^2}{2} u_{yy\,i\,j} - \frac{h_2^3}{6} u_{yyy\,i\,j} + \cdots$$

So

$$\begin{aligned} Au_{i\,j} + Bu_{i\,j+1} + Cu_{i\,j-1} &= (A+B+C)u_{i\,j} + (Bh_1 - Ch_2)u_{y\,i\,j} \\ &+ (B\frac{h_1^2}{2} + C\frac{h_2^2}{2})u_{yy\,i\,j} + (B\frac{h_1^3}{6} - C\frac{h_2^3}{6})u_{yyy\,i\,j} \pm \cdots \end{aligned}$$

Compare coefficients with $u_{yy\,i\,j}$ to get

$$A + B + C = 0$$

$$Bh_1 - Ch_2 = 0$$

$$B\frac{h_1^2}{2} + C\frac{h_2^2}{2} = 1$$

This system of 3 equations can be solved for $A, B, C$ to get from the second

$$B = C\frac{h_2}{h_1}$$

Plugging in the third and solve for $C$

$$C = \frac{2}{h_2(h_1 + h_2)}$$

Thus

$$B = \frac{2}{h_2(h_1 + h_2)}\,\frac{h_2}{h_1} = \frac{2}{h_1(h_1 + h_2)}$$

Now use these two in the first

$$A = -B - C = -\frac{2}{h_1(h_1 + h_2)} - \frac{2}{h_2(h_1 + h_2)}$$

$$A = -\frac{2}{h_1 + h_2}\left(\frac{1}{h_1} + \frac{1}{h_2}\right) = -\frac{2}{h_1 h_2}$$

The error term can be found when computing the next term in Taylor series

$$(B\frac{h_1^3}{6} - C\frac{h_2^3}{6})u_{yyy\,i\,j} = \frac{1}{3}\left(\frac{h_1^2}{h_1+h_2} - \frac{h_2^2}{h_1+h_2}\right)u_{yyy\,i\,j}$$

The denominator is of order $h$ and numerator of order $h^2$ so the method is of order $h$

$$u_{yy\,i\,j} = -\frac{2}{h_1h_2}u_{i\,j} + \frac{2}{h_1(h_1+h_2)}u_{i\,j+1} + \frac{2}{h_2(h_1+h_2)}u_{i\,j-1}$$

Let's check the special case of uniformly spaced points, i.e. $h_1 = h_2 = h$. The above equation becomes the well known

$$u_{yy\,i\,j} = -\frac{2}{h^2}u_{i\,j} + \frac{1}{h^2}u_{i\,j+1} + \frac{1}{h^2}u_{i\,j-1}$$

The error term listed above is

$$\frac{1}{3}\left(\frac{h^2}{2h} - \frac{h^2}{2h}\right)u_{yyy\,i\,j} = 0$$

and so we have to go to the next term in Taylor and thus we get a second order (as expected for uniformly spaced points).

4.

Find the truncation error and order for

$$u_{y\,i\,j} \sim \frac{-3u_{i\,j} + 4u_{i\,j+1} - u_{i\,j+2}}{2\Delta y}$$

The Taylor series are

$$u_{i\,j+2} = u_{i\,j} + 2\Delta y u_{y\,i\,j} + \frac{(2\Delta y)^2}{2} u_{yy\,i\,j} + \frac{(2\Delta y)^3}{6} u_{yyy\,i\,j} + \cdots$$

$$u_{i\,j+1} = u_{i\,j} + \Delta y u_{y\,i\,j} + \frac{(\Delta y)^2}{2} u_{yy\,i\,j} + \frac{(\Delta y)^3}{6} u_{yyy\,i\,j} + \cdots$$

$$\begin{aligned} -3u_{i\,j} + 4u_{i\,j+1} - u_{i\,j+2} &= \underbrace{(-1 + 4 - 3)}_{=0} u_{i\,j} + \underbrace{(-2\Delta y + 4\Delta y)}_{=2\Delta y} u_{y\,i\,j} \\ &+ \underbrace{\left(-\frac{4\Delta y^2}{2} + \frac{4\Delta y^2}{2}\right)}_{=0} u_{yy\,i\,j} \\ &+ \underbrace{\left(-\frac{8\Delta y^3}{6} + \frac{4\Delta y^3}{6}\right)}_{=-\frac{2}{3}\Delta y^3} u_{yyy\,i\,j} + \cdots \end{aligned}$$

Therefore

$$u_{y\,i\,j} = \frac{-3u_{i\,j} + 4u_{i\,j+1} - u_{i\,j+2}}{2\Delta y} + \underbrace{\frac{1}{3}(\Delta y)^2 u_{yyy\,i\,j} + \cdots}_{\text{truncation error}}$$

Therefore the approximation is of second order.

## 8.3 Irregular Mesh

### Problems

1. Develop a finite difference approximation with T.E. of $O(\Delta y)^2$ for $\partial T/\partial y$ at point $(i,j)$ using $T_{i,j}$, $T_{i,j+1}$, $T_{i,j+2}$ when the grid spacing is **not** uniform.

2. Determine the T.E. of the following finite difference approximation for $\partial u/\partial x$ at point $(i,j)$ when the grid space is **not** uniform:

$$\frac{\partial u}{\partial x}\Big|_{i,j} \approx \frac{u_{i+1,j} - (\Delta x_+/\Delta x_-)^2 u_{i-1,j} - \left[1 - (\Delta x_+/\Delta x_-)^2\right] u_{i,j}}{\Delta x_-(\Delta x_+/\Delta x_-)^2 + \Delta x_+}$$

1. Let $h_1 = y_{j+2} - y_{j+1}$ and $h_2 = y_{j+1} - y_j$, then

$$T_{y\,i\,j} \;=\; AT_{i\,j} + BT_{i\,j+1} + CT_{i\,j+2} + O(\Delta y^2)$$

with $A, B, C$ to be determined. Now take Taylor series expansions

$$T_{i\,j+1} = T_{i\,j} + h_2 T_{y\,i\,j} + \frac{h_2^2}{2} T_{yy\,i\,j} + \frac{h_2^3}{6} T_{yyy\,i\,j} + \cdots$$

$$T_{i\,j+2} = T_{i\,j} + (h_1 + h_2) T_{y\,i\,j} + \frac{(h_1 + h_2)^2}{2} T_{yy\,i\,j} + \frac{(h_1 + h_2)^3}{6} T_{yyy\,i\,j} + \cdots$$

So

$$\begin{aligned} AT_{i\,j} + BT_{i\,j+1} + CT_{i\,j+2} \;&=\; (A + B + C)T_{i\,j} + (Bh_2 + C(h_1 + h_2))T_{y\,i\,j} \\ &\quad (B\frac{h_2^2}{2} + C\frac{(h_1 + h_2)^2}{2})T_{yy\,i\,j} + (B\frac{h_2^3}{6} + C\frac{(h_1 + h_2)^3}{6})T_{yyy\,i\,j} \pm \cdots \end{aligned}$$

Compare coefficients with $T_{y\,i\,j}$ to get

$$A + B + C = 0$$

$$Bh_2 + C(h_1 + h_2) = 1$$

$$B\frac{h_2^2}{2} + C\frac{(h_1 + h_2)^2}{2} = 0$$

This system of 3 equations can be solved for $A, B, C$ to get from the third

$$B = -C\left(\frac{h_1 + h_2}{h_2}\right)^2$$

Plugging in the second and solve for $C$

$$C\left(h_1 + h_2 - \left(\frac{h_1 + h_2}{h_2}\right)^2\right) = 1$$

$$C = \frac{1}{h_1 + h_2 - \left(\frac{h_1 + h_2}{h_2}\right)^2}$$

Thus

$$B = -\frac{\left(\frac{h_1 + h_2}{h_2}\right)^2}{h_1 + h_2 - \left(\frac{h_1 + h_2}{h_2}\right)^2}$$

Now use these two in the first

$$A = -B - C = \frac{\left(\frac{h_1 + h_2}{h_2}\right)^2}{h_1 + h_2 - \left(\frac{h_1 + h_2}{h_2}\right)^2} - \frac{1}{h_1 + h_2 - \left(\frac{h_1 + h_2}{h_2}\right)^2}$$

$$A = \frac{\left(\frac{h_1+h_2}{h_2}\right)^2 - 1}{h_1 + h_2 - \left(\frac{h_1+h_2}{h_2}\right)^2}$$

The error term can be found when computing the next term in Taylor series

$$(B\frac{h_2^3}{6} + C\frac{(h_1+h_2)^3}{6})T_{yyy\,i\,j} = \frac{1}{6}\frac{-\left(\frac{h_1+h_2}{h_2}\right)^2 h_2^3 + (h_1+h_2)^3}{h_1 + h_2 - \left(\frac{h_1+h_2}{h_2}\right)^2}T_{yyy\,i\,j}$$

The denominator is of order $h$ and numerator of order $h^3$ so the method is of order $h^2$

2. Let $h_- = x_i - x_{i-1}$ and $h_+ = x_{i+1} - x_i$, then

$$u_{x\,i\,j} \;=\; \frac{u_{i+1\,j} - \left(\frac{h_+}{h_-}\right)^2 u_{i\,j-1} - \left(1 - \left(\frac{h_+}{h_-}\right)^2\right) u_{i\,j}}{h_- \left(\frac{h_+}{h_-}\right)^2 + h_+}$$

Now take Taylor series expansions

$$u_{i+1\,j} = u_{i\,j} + h_+ u_{x\,i\,j} + \frac{h_+^2}{2} u_{xx\,i\,j} + \frac{h_+^3}{6} u_{xxx\,i\,j} + \frac{h_+^4}{24} u_{xxxx\,i\,j} + \cdots$$

$$u_{i\,j-1} = u_{i\,j} - h_- u_{x\,i\,j} + \frac{h_-^2}{2} u_{xx\,i\,j} - \frac{h_-^3}{6} u_{xxx\,i\,j} + \frac{h_-^4}{24} u_{xxxx\,i\,j} + \cdots$$

So

$$\begin{aligned}
\text{numerator} \;&=\; \underbrace{\left(1 - \left(\frac{h_+}{h_-}\right)^2 - 1 + \left(\frac{h_+}{h_-}\right)^2\right)}_{=0} u_{i\,j} \\
&\quad + \left(h_+ + h_- \left(\frac{h_+}{h_-}\right)^2\right) u_{x\,i\,j} + \underbrace{\left(\frac{h_+^2}{2} - \frac{h_-^2}{2}\left(\frac{h_+}{h_-}\right)^2\right)}_{=0} u_{xx\,i\,j} \\
&\quad + \underbrace{\left(\frac{h_+^3}{6} + \frac{h_-^3}{6}\left(\frac{h_+}{h_-}\right)^2\right)}_{=\frac{h_+^2(h_+ + h_-)}{6}} u_{xxx\,i\,j} \pm \cdots
\end{aligned}$$

Divide by the coefficient of $u_{x\,i\,j}$ we get

$$u_{x\,i\,j} \;=\; \frac{u_{i+1\,j} - \left(\frac{h_+}{h_-}\right)^2 u_{i-1\,j} - \left(1 - \left(\frac{h_+}{h_-}\right)^2\right) u_{i\,j}}{h_+ + h_- \left(\frac{h_+}{h_-}\right)^2} - \frac{h_+^2(h_+ + h_-)}{6(h_+ + h_- \left(\frac{h_+}{h_-}\right)^2)} u_{xxx\,i\,j}$$

The error term can be manipulated

$$-\frac{h_+^2(h_+ + h_-)}{6(h_+ + h_- \left(\frac{h_+}{h_-}\right)^2)} = -\frac{h_+^2(h_+ + h_-)}{6\frac{h_- h_+ + h_+^2}{h_-}} = -\frac{h_- h_+}{6}$$

## 8.4 Thomas Algorithm

#### Problems

## 8.5 Methods for Approximating PDEs

### 8.5.1 Undetermined coefficients

### 8.5.2 Polynomial Fitting

### 8.5.3 Integral Method

#### Problems

## 8.6 Eigenpairs of a Certain Tridiagonal Matrix

#### Problems

# 9 Finite Differences

## 9.1 Introduction

## 9.2 Difference Representations of PDEs

### Problems

1. Utilize Taylor series expansions about the point $(n + \frac{1}{2}, j)$ to determine the T.E. of the Crank Nicolson representation of the heat equation. Compare these results with the T.E. obtained from Taylor series expansion about the point $(n, j)$.

2. The DuFort Frankel method for solving the heat equation requires solution of the difference equation

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = \frac{\alpha}{(\Delta x)^2}\left(u_{j+1}^n - u_j^{n+1} - u_j^{n-1} + u_{j-1}^n\right)$$

Develop the stability requirements necessary for the solution of this equation.

1. Utilize Taylor series expansions about the point $(n+\frac{1}{2},j)$ to determine the T.E. of the Crank Nicolson representation of the heat equation.

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}=\frac{\alpha}{2(\Delta x)^2}\left[u_{j+1}^{n+1}+u_{j+1}^n-2(u_j^{n+1}+u_j^n)+u_{j-1}^{n+1}+u_{j-1}^n\right]$$

Expand about $(n+1/2,j)$

$$u_j^{n+1}=u+\frac{\Delta t}{2}u_t+\frac{1}{2}(\frac{\Delta t}{2})^2u_{tt}+\frac{1}{6}(\frac{\Delta t}{2})^3u_{ttt}+\frac{1}{24}(\frac{\Delta t}{2})^4u_{tttt}+\cdots$$

All terms on the right are at $(n+1/2,j)$

$$u_j^n=u-\frac{\Delta t}{2}u_t+\frac{1}{2}(\frac{\Delta t}{2})^2u_{tt}-\frac{1}{6}(\frac{\Delta t}{2})^3u_{ttt}+\cdots$$

$$\text{LHS}=\underbrace{u_t}_{\text{term from PDE}}+\frac{(\Delta t)^2}{24}u_{ttt}+\cdots$$

$$\begin{aligned}
u_{j+1}^{n+1} &= u+\frac{\Delta t}{2}u_t+\Delta x u_x+\frac{1}{2}(\frac{\Delta t}{2})^2u_{tt}+\frac{\Delta t}{2}\Delta x u_{tx}\\
&+ \frac{1}{2}(\Delta x)^2u_{xx}+\frac{1}{6}(\frac{\Delta t}{2})^3u_{ttt}+3(\frac{\Delta t}{2})^2\frac{\Delta x}{6}u_{ttx}+3\frac{\Delta t}{2}\frac{(\Delta x)^2}{6}u_{txx}\\
&+ \frac{(\Delta x)^3}{6}u_{xxx}+\frac{1}{24}(\frac{\Delta t}{2})^4u_{tttt}+4(\frac{\Delta t}{2})^3\frac{\Delta x}{24}u_{tttx}+6(\frac{\Delta t}{2})^2\frac{(\Delta x)^2}{24}u_{ttxx}\\
&+ 4\frac{\Delta t}{2}\frac{(\Delta x)^3}{24}u_{txxx}+\frac{(\Delta x)^4}{24}u_{xxxx}+\cdots
\end{aligned}$$

$$\begin{aligned}
u_{j-1}^{n+1} &= u+\frac{\Delta t}{2}u_t-\Delta x u_x+\frac{1}{2}(\frac{\Delta t}{2})^2u_{tt}-\frac{\Delta t}{2}\Delta x u_{tx}\\
&+ \frac{1}{2}(\Delta x)^2u_{xx}+\frac{1}{6}(\frac{\Delta t}{2})^3u_{ttt}-3(\frac{\Delta t}{2})^2\frac{\Delta x}{6}u_{ttx}+3\frac{\Delta t}{2}\frac{(\Delta x)^2}{6}u_{txx}\\
&- \frac{(\Delta x)^3}{6}u_{xxx}+\frac{1}{24}(\frac{\Delta t}{2})^4u_{tttt}-4(\frac{\Delta t}{2})^3\frac{\Delta x}{24}u_{tttx}+6(\frac{\Delta t}{2})^2\frac{(\Delta x)^2}{24}u_{ttxx}\\
&- 4\frac{\Delta t}{2}\frac{(\Delta x)^3}{24}u_{txxx}+\frac{(\Delta x)^4}{24}u_{xxxx}\pm\cdots
\end{aligned}$$

Now collect terms to compute the terms on the right at time $n+1$

$$\begin{aligned}
u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1} &= 0 \cdot u + 0 \cdot u_t + 0 \cdot u_x + 0 \cdot u_{tt} + 0 \cdot u_{tx} \\
&+ (\Delta x)^2 u_{xx} + 0 \cdot u_{ttt} + 0 \cdot u_{ttx} + \frac{\Delta t}{2}(\Delta x)^2 u_{txx} \\
&+ 0 \cdot u_{xxx} + (\frac{\Delta t}{2})^2 \frac{(\Delta x)^2}{2} u_{ttxx} + \frac{(\Delta x)^4}{12} u_{xxxx} + \cdots
\end{aligned}$$

Divide by $2(\Delta x)^2$ we get

$$\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{2(\Delta x)^2} = \frac{1}{2}u_{xx} + \frac{\Delta t}{4}u_{txx} + \frac{(\Delta t)^2}{16}u_{ttxx} + \frac{(\Delta x)^2}{24}u_{xxxx} + \cdots$$

Now to terms at time $n$

$$\begin{aligned}
u_{j+1}^n &= u - \frac{\Delta t}{2}u_t + \Delta x u_x + \frac{1}{2}(\frac{\Delta t}{2})^2 u_{tt} - \frac{\Delta t}{2}\Delta x u_{tx} \\
&+ \frac{1}{2}(\Delta x)^2 u_{xx} - \frac{1}{6}(\frac{\Delta t}{2})^3 u_{ttt} + 3(\frac{\Delta t}{2})^2 \frac{\Delta x}{6} u_{ttx} - 3\frac{\Delta t}{2}\frac{(\Delta x)^2}{6} u_{txx} \\
&+ \frac{(\Delta x)^3}{6} u_{xxx} + \frac{1}{24}(\frac{\Delta t}{2})^4 u_{tttt} - 4(\frac{\Delta t}{2})^3 \frac{\Delta x}{24} u_{tttx} + 6(\frac{\Delta t}{2})^2 \frac{(\Delta x)^2}{24} u_{ttxx} \\
&- 4\frac{\Delta t}{2}\frac{(\Delta x)^3}{24} u_{txxx} + \frac{(\Delta x)^4}{24} u_{xxxx} + \cdots
\end{aligned}$$

$$\begin{aligned}
u_{j-1}^n &= u - \frac{\Delta t}{2}u_t - \Delta x u_x + \frac{1}{2}(\frac{\Delta t}{2})^2 u_{tt} + \frac{\Delta t}{2}\Delta x u_{tx} \\
&+ \frac{1}{2}(\Delta x)^2 u_{xx} - \frac{1}{6}(\frac{\Delta t}{2})^3 u_{ttt} - 3(\frac{\Delta t}{2})^2 \frac{\Delta x}{6} u_{ttx} - 3\frac{\Delta t}{2}\frac{(\Delta x)^2}{6} u_{txx} \\
&- \frac{(\Delta x)^3}{6} u_{xxx} + \frac{1}{24}(\frac{\Delta t}{2})^4 u_{tttt} + 4(\frac{\Delta t}{2})^3 \frac{\Delta x}{24} u_{tttx} + 6(\frac{\Delta t}{2})^2 \frac{(\Delta x)^2}{24} u_{ttxx} \\
&+ 4\frac{\Delta t}{2}\frac{(\Delta x)^3}{24} u_{txxx} + \frac{(\Delta x)^4}{24} u_{xxxx} \pm \cdots
\end{aligned}$$

Now collect terms to compute the terms on the right at time $n+1$

$$\begin{aligned}
u_{j+1}^n - 2u_j^n + u_{j-1}^n &= 0 \cdot u + 0 \cdot u_t + 0 \cdot u_x + 0 \cdot u_{tt} + 0 \cdot u_{tx} \\
&+ (\Delta x)^2 u_{xx} + 0 \cdot u_{ttt} + 0 \cdot u_{ttx} - \frac{\Delta t}{2}(\Delta x)^2 u_{txx} \\
&+ 0 \cdot u_{xxx} + (\frac{\Delta t}{2})^2 \frac{(\Delta x)^2}{2} u_{ttxx} + \frac{(\Delta x)^4}{12} u_{xxxx} + \cdots
\end{aligned}$$

Divide by $2(\Delta x)^2$ we get

$$\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2(\Delta x)^2} = \frac{1}{2}u_{xx} - \frac{\Delta t}{4}u_{txx} + \frac{(\Delta t)^2}{16}u_{ttxx} + \frac{(\Delta x)^2}{24}u_{xxxx} + \cdots$$

Now the right hand side become

$$\alpha(u_{xx} + \frac{(\Delta t)^2}{8}u_{ttxx} + \frac{1}{12}(\Delta x)^2 u_{xxxx} + \cdots)$$

Combine LHS and RHS

$$u_t + \frac{(\Delta t)^2}{24}u_{ttt} + \cdots = \alpha(u_{xx} + \frac{(\Delta t)^2}{8}u_{ttxx} + \frac{1}{12}(\Delta x)^2 u_{xxxx} + \cdots)$$

Thus the truncation error is $O\left((\Delta t)^2, (\Delta x)^2\right)$

Compare these results with the T.E. obtained from Taylor series expansion about the point $(n, j)$. To do that we need to exapnd about $n, j$

$$u_j^{n+1} = u + \Delta t u_t + \frac{(\Delta t)^2}{2}u_{tt} + \cdots$$

All terms on the right are now at $n, j$.

$$\text{LHS} = u_t + \frac{\Delta t}{2}u_{tt} + \cdots$$

For the right hand side

$$\begin{aligned} u_{j+1}^{n+1} &= u + \Delta t u_t + \Delta x u_x + \frac{(\Delta t)^2}{2}u_{tt} + \Delta t \Delta x u_{tx} + \frac{(\Delta x)^2}{2}u_{xx} \\ &+ \frac{(\Delta t)^3}{6}u_{ttt} + 3\frac{(\Delta t)^2}{6}\Delta x u_{ttx} + 3\frac{(\Delta x)^2}{6}\Delta t u_{txx} + \frac{(\Delta x)^3}{6}u_{xxx} + \cdots \end{aligned}$$

$$u_j^{n+1} = u + \Delta t u_t + \frac{(\Delta t)^2}{2}u_{tt} + \frac{(\Delta t)^3}{6}u_{ttt} + \cdots$$

$$\begin{aligned} u_{j-1}^{n+1} &= u + \Delta t u_t - \Delta x u_x + \frac{(\Delta t)^2}{2}u_{tt} - \Delta t \Delta x u_{tx} + \frac{(\Delta x)^2}{2}u_{xx} \\ &+ \frac{(\Delta t)^3}{6}u_{ttt} - 3\frac{(\Delta t)^2}{6}\Delta x u_{ttx} + 3\frac{(\Delta x)^2}{6}\Delta t u_{txx} - \frac{(\Delta x)^3}{6}u_{xxx} \pm \cdots \end{aligned}$$

$$u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1} = (\Delta x)^2 u_{xx} + \Delta t(\Delta x)^2 u_{txx} + \cdots$$

So

$$\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{2(\Delta x)^2} = \frac{1}{2}u_{xx} + \frac{\Delta t}{2}u_{txx} + \cdots$$

$$u_{j+1}^n = u + \Delta x u_x + \frac{(\Delta x)^2}{2} u_{xx} + \frac{(\Delta x)^3}{6} u_{xxx} + \frac{(\Delta x)^4}{24} u_{xxxx} \cdots$$

$$u_{j-1}^n = u - \Delta x u_x + \frac{(\Delta x)^2}{2} u_{xx} - \frac{(\Delta x)^3}{6} u_{xxx} + \frac{(\Delta x)^4}{24} u_{xxxx} \cdots$$

So

$$\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2(\Delta x)^2} = \frac{1}{2} u_{xx} + \frac{(\Delta x)^2}{24} u_{xxxx} + \cdots$$

The RHS now becomes

$$\alpha(u_{xx} + \frac{\Delta t}{2} u_{txx} + \frac{1}{24} (\Delta x)^2 u_{xxxx} + \cdots)$$

Combine LHS and RHS

$$u_t + \frac{\Delta t}{2} u_{tt} + \cdots = \alpha(u_{xx} + \frac{\Delta t}{2} u_{txx} + \frac{1}{24} (\Delta x)^2 u_{xxxx} + \cdots)$$

Thus the truncation error is $O\left(\Delta t, (\Delta x)^2\right)$ Notice that it is **only** first order in time.

2. The DuFort Frankel method for solving the heat equation requires solution of the difference equation

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = \frac{\alpha}{(\Delta x)^2}\left(u_{j+1}^n - u_j^{n+1} - u_j^{n-1} + u_{j-1}^n\right)$$

Develop the stability requirements necessary for the solution of this equation.

Let $r = \dfrac{\alpha \Delta t}{(\Delta x)^2}$ then DuFort Frankel method can be written as

$$u_j^{n+1} - u_j^{n-1} = 2r\left(u_{j+1}^n - u_j^{n+1} - u_j^{n-1} + u_{j-1}^n\right)$$

or

$$(1+2r)u_j^{n+1} = 2r\left(u_{j+1}^n + u_{j-1}^n\right) + (1-2r)u_j^{n-1}$$

Use $\lambda^n e^{ik_m j\Delta x}$

$$\lambda^{n+1}(1+2r) - 2r\left(\underbrace{e^{ik_m\Delta x} - e^{-ik_m\Delta x}}_{2\cos\beta}\right)\lambda^n - (1-2r)\lambda^{n-1} = 0$$

where $\beta = k_m\Delta x$. This leads to a quadratic equation for $\lambda$

$$(1+2r)\lambda^2 - 4r\cos\beta\lambda - (1-2r) = 0$$

and the solutions are

$$\lambda = \frac{4r\cos\beta \pm \sqrt{\Delta}}{2(1+2r)}$$

where

$$\begin{aligned}\Delta &= (4r\cos\beta)^2 + 4(1-2r)(1+2r)\\ &= 16r^2\cos^2\beta + 4 - 16r^2\\ &= 16r^2(\cos^2\beta - 1) + 4\\ &= 4\left[1 - 4r^2(1-\cos^2\beta)\right]\end{aligned}$$

Thus

$$\lambda = \frac{2r\cos\beta \pm \sqrt{1 - 4r^2(1-\cos^2\beta)}}{1+2r}$$

Consider the 2 cases:

**case 1** $1 - 4r^2(1-\cos^2\beta) \geq 0$

Then $4r^2\underbrace{(1-\cos^2\beta)}_{\sin^2\beta} \leq 1$

Taking square roots

$$|r| \leq \frac{1}{2\sin\beta}$$

In this case the discriminant is nonnegative and we have two real $\lambda$

$$\lambda = \frac{2r\cos\beta \pm \sqrt{1-4r^2\sin^2\beta}}{1+2r}$$

Note that the terms under radical are less than or equal 1 and therefore the numerator is less than or equal the denominator

$$|\lambda| \le \left|\frac{2r\cos\beta + 1}{1+2r}\right| \le 1 \text{ for all } r$$

Thus the method is unconditionally stable in this case.

**case 2** $1 - 4r^2(1-\cos^2\beta) < 0$
Then $4r^2 \underbrace{(1-\cos^2\beta)}_{\sin^2\beta} > 1$
In this case the discriminant is negative and we have two complex conjugate $\lambda$

$$\lambda = \frac{2r\cos\beta \pm i\sqrt{4r^2\sin^2\beta - 1}}{1+2r}$$

$$|\lambda|^2 = (Re(\lambda))^2 + (Im(\lambda))^2$$

$$|\lambda|^2 = \left(\frac{2r\cos\beta}{1+2r}\right)^2 + \frac{4r^2\sin^2\beta - 1}{(1+2r)^2}$$

$$|\lambda|^2 = \frac{4r^2\cos^2\beta + 4r^2\sin^2\beta - 1}{(1+2r)^2}$$

$$|\lambda|^2 = \frac{4r^2 - 1}{4r^2 + 4r + 1}$$

Again the numerator is smaller than denominator for all $r$,

$$|\lambda|^2 \le 1$$

Thus the method is unconditionally stable in this case.

## 9.3 Heat Equation in One Dimension

### Problems

1. Use the simple explicit method to solve the 1-D heat equation on the computational grid (figure 59) with the boundary conditions

$$u_1^n = 2 = u_3^n$$

and initial conditions

$$u_1^1 = 2 = u_3^1, \qquad u_2^1 = 1.$$

Show that if $r = \frac{1}{4}$, the steady state value of $u$ along $j = 2$ becomes

$$u_2^{steadystate} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^{k-1}}$$

Note that this infinite series is geometric that has a known sum.

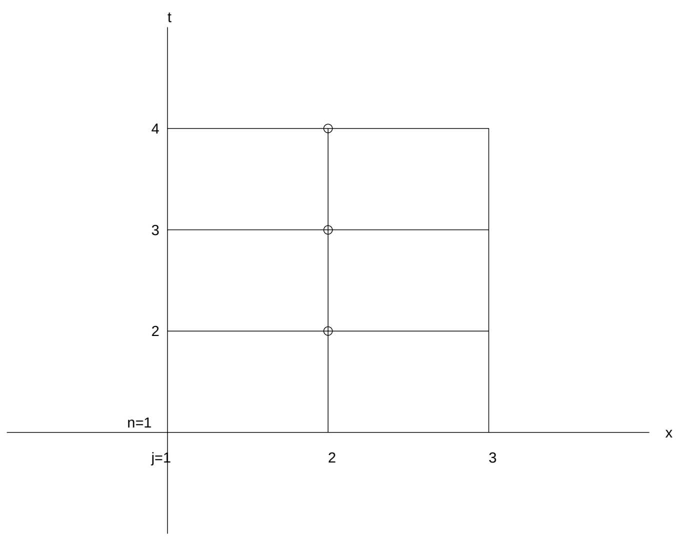


Figure 59: domain for problem 1 section 9.3

1. Use the simple explicit method to solve the 1-D heat equation on the computational grid (figure 59) with the boundary conditions

$$u_1^n \ = \ 2 \ = \ u_3^n$$

and initial conditions

$$u_1^1 \ = \ 2 \ = \ u_3^1, \qquad u_2^1 \ = \ 1.$$

$$u_j^{n+1} = u_j^n + r(u_{j+1}^n - 2u_j^n + u_{j-1}^n) = (1-2r)u_j^n + r(u_{j+1}^n + u_{j-1}^n)$$

$$u_1^1,\ u_2^1,\ u_3^1 \text{ are known - initial condition}$$

$$u_1^2,\ u_3^2 \ \text{ are known - boundary conditions}$$

Therefore

$$u_2^2 = (1-2r)u_2^1 + r(u_3^1 + u_1^1) \qquad j = 2,\, n = 1$$

$$u_2^2 = 1 - 2r + r(2+2) = 1 + 2r$$

So

$$u_2^2 = 1 + 2r$$

$$u_2^3 = (1-2r)\underbrace{u_2^2}_{1+2r} + r(u_3^2 + u_1^2) \qquad j = 2,\, n = 2$$

$$u_2^3 = (1-2r)(1+2r) + r(2+2)$$

So

$$u_2^3 = 1 + 4r - 4r^2$$

$$u_2^4 = (1-2r)\underbrace{u_2^3}_{1+4r-4r^2} + r(u_3^3 + u_1^3) \qquad j = 2,\, n = 3$$

$$u_2^4 = (1-2r)(1+4r-4r^2) + r(2+2)$$

So

$$u_2^4 = 1 + 6r - 12r^2 + 8r^3$$

and in general,

$$u_2^n = (1-2r)u_2^{n-1} + 4r$$

Show that if $r = \frac{1}{4}$, the steady state value of $u$ along $j = 2$ becomes

$$u_2^{steadystate} \ = \ \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^{k-1}}$$

In this case

$$u_2^2 = 1 + \frac{1}{2} = \frac{3}{2}, \qquad \text{compare this to } n = 2 \text{ in the sum}$$

$$u_2^3 = 1 + 1 - \frac{1}{4} = \frac{7}{4}, \qquad \text{compare this to } n = 3 \text{ in the sum}$$

$$u_2^4 = 1 + \frac{3}{2} - \frac{3}{4} + \frac{1}{8} = \frac{15}{8}, \qquad \text{compare this to } n = 4 \text{ in the sum}$$

In general for $r = \frac{1}{4}$ we have

$$u_2^n = \frac{1}{2}u_2^{n-1} + 1$$

This is a nonhomogeneous difference equation. The general solution of the homogeneous is

$$u_2^n = \left(\frac{1}{2}\right)^n$$

A particular solution for the nonhomogeneous is a constant, which when substituted in the equation turn out to be 2, so

$$u_2^n = \left(\frac{1}{2}\right)^n + 2$$

Upon letting $n \to \infty$ we get the steady state

$$u_2 = 2$$

If we look at the limit for the infinite series, note that this infinite series is geometric with first term 1 and quotient $\frac{1}{2}$. Thus the sum is

$$\frac{1}{1 - \frac{1}{2}} = 2$$

the same as before.

### 9.3.1 Implicit method

### 9.3.2 DuFort Frankel method

### 9.3.3 Crank-Nicolson method

### 9.3.4 Theta ($\theta$) method

### 9.3.5 An example

### 9.3.6 Unbounded Region - Coordinate Transformation

## 9.4 Two Dimensional Heat Equation

### 9.4.1 Explicit

### 9.4.2 Crank Nicolson

### 9.4.3 Alternating Direction Implicit

**Problems**

1. Apply the ADI scheme to the 2-D heat equation and find $u^{n+1}$ at the internal grid points in the mesh shown in figure 60 for $r_x = r_y = 2$. The initial conditions are

$$u^n = 1 - \frac{x}{3\Delta x} \qquad \text{along } y = 0$$

$$u^n = 1 - \frac{y}{2\Delta y} \qquad \text{along } x = 0$$

$$u^n = 0 \qquad \text{everywhere else}$$

and the boundary conditions remain fixed at their initial values.

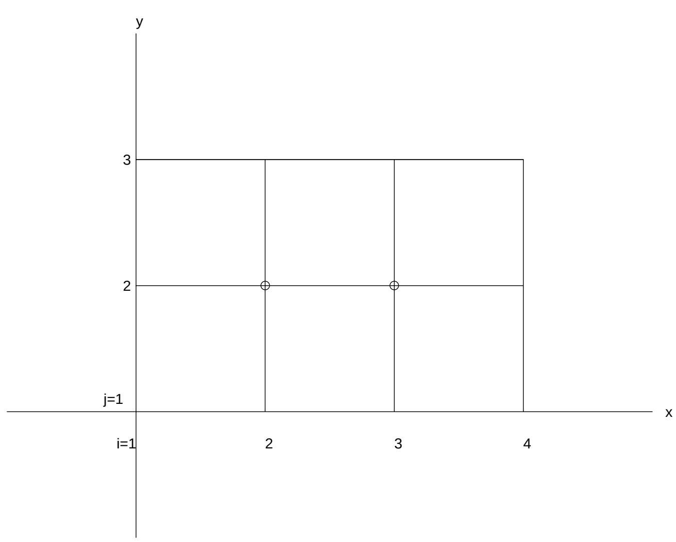


Figure 60: domain for problem 1 section 9.4.2

1. Apply the ADI scheme to the 2-D heat equation and find $u^{n+1}$ at the internal grid points in the mesh shown in figure 60 for $r_x = r_y = 2$. The initial conditions are

$$u^n = 1 - \frac{x}{3\Delta x} \qquad \text{along } y = 0$$

$$u^n = 1 - \frac{y}{2\Delta y} \qquad \text{along } x = 0$$

$$u^n = 0 \qquad \text{everywhere else}$$

and the boundary conditions remain fixed at their initial values.

Step 1:

$$u_{ij}^{n+1/2} - u_{ij}^n = \underbrace{\frac{r_x}{2}}_{=1} \left( u_{i+1\,j}^{n+1/2} - 2u_{ij}^{n+1/2} + u_{i-1\,j}^{n+1/2} \right) + \underbrace{\frac{r_y}{2}}_{=1} \left( u_{i\,j+1}^n - 2u_{ij}^n + u_{i\,j-1}^n \right)$$

Step 2:

$$u_{ij}^{n+1} - u_{ij}^{n+1/2} = \underbrace{\frac{r_x}{2}}_{=1} \left( u_{i+1\,j}^{n+1/2} - 2u_{ij}^{n+1/2} + u_{i-1\,j}^{n+1/2} \right) + \underbrace{\frac{r_y}{2}}_{=1} \left( u_{i\,j+1}^{n+1} - 2u_{ij}^{n+1} + u_{i\,j-1}^{n+1} \right)$$

For $y = 0$, bottom boundary, the index $j = 1$,

$$u_{i\,1}^n = 1 - \frac{x}{3\Delta x}$$

$$u_{2\,1}^n = \frac{2}{3}$$

$$u_{3\,1}^n = \frac{1}{3}$$

$u_{1\,1}^n$ and $u_{4\,1}^n$ are not needed

For $x = 0$, left boundary $i = 1$

$$u_{1\,j}^n = 1 - \frac{y}{2\Delta y}$$

$$u_{1\,2}^n = \frac{1}{2}$$

$u_{1\,3}^n$ is not needed

$$u_{4\,j}^n = u_{i\,3}^n = 0, \text{ given}$$

$$u_{2\,2}^0 = u_{3\,2}^0 = 0$$

Write the system of equations resulting from step 1 with $i = j = 2$ and $i = 3,\ j = 2$

$$u_{2\,2}^{n+1/2} = u_{3\,2}^{n+1/2} - 2u_{2\,2}^{n+1/2} + \frac{1}{2} + \frac{2}{3}$$

$$u_{3\,2}^{n+1/2} = -2u_{3\,2}^{n+1/2} + u_{2\,2}^{n+1/2} + \frac{1}{3}$$

$$\begin{pmatrix} 3 & -1 \\ & \\ -1 & 3 \end{pmatrix} \begin{pmatrix} u_{2\,2}^{n+1/2} \\ \\ u_{3\,2}^{n+1/2} \end{pmatrix} = \begin{pmatrix} \frac{7}{6} \\ \\ \frac{1}{3} \end{pmatrix}$$

The solution is

$$\begin{pmatrix} u_{2\,2}^{n+1/2} \\ \\ u_{3\,2}^{n+1/2} \end{pmatrix} = \begin{pmatrix} \frac{23}{48} \\ \\ \frac{13}{48} \end{pmatrix}$$

For step 2 with $i = j = 2$

$$u_{2\,2}^{n+1} - \frac{23}{48} = \frac{13}{48} - 2\left(\frac{23}{48}\right) + \frac{1}{2} - 2u_{2\,2}^{n+1} + \frac{2}{3}$$

so

$$u_{2\,2}^{n+1} = \frac{23}{72}$$

For $i = 3,\ j = 2$ step 2 gives

$$u_{3\,2}^{n+1} - \frac{13}{48} = -2\left(\frac{13}{48}\right) + \frac{23}{48} - 2u_{3\,2}^{n+1} + \frac{1}{3}$$

so

$$u_{3\,2}^{n+1} = \frac{13}{72}$$

### 9.4.4 Alternating Direction Implicit for Three Dimensional Problems

## 9.5 Laplace's Equation

### 9.5.1 Iterative solution

## 9.6 Vector and Matrix Norms

**Problems**

1. Find the one-, two-, and infinity norms of the following vectors and matrices:

$$\text{(a)} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 9 \end{pmatrix} \quad \text{(b)} \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \quad \text{(c)} \begin{pmatrix} 1 & 6 \\ 7 & 3 \end{pmatrix}$$

a. The 1-norm (column sum) for the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

The sum of elements in each column:

$$1 + 2 + 3 = 6$$

$$2 + 5 + 6 = 13$$

$$3 + 6 + 9 = 18$$

The maximum is 18, so

$$||A||_1 = 18$$

The $\infty$-norm (row sum): The sum of elements in each row is the same since the matrix is symmetric, so

$$||A||_\infty = 18$$

The 2-norm is the same as the spectral norm since the matrix is symmetric. We need to find the eigenvalues of $A$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 5-\lambda & 6 \\ 3 & 6 & 9-\lambda \end{vmatrix} = -\lambda(\lambda^2 + 15\lambda + 10)$$

To get the eigenvalues we have the make the determinant zero, so

$$\lambda = 0$$

and

$$\lambda^2 + 15\lambda + 10 = 0$$

which gives

$$\lambda = -\frac{15}{2} \pm \frac{1}{2}\sqrt{185}$$

The spectral radius is the largest eigenvalue in absolute value

$$\rho(A) = \left| -\frac{15}{2} - \frac{1}{2}\sqrt{185} \right| \sim 14.3$$

$$||A||_2 \sim 14.3$$

b. Given the vector

$$b = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

The sum of elements in the column:

$$3 + 4 + 5 = 12$$

so

$$||b||_1 = 12$$

The $\infty$-norm (row sum): The sum of elements in each row is 3,4,5, so

$$||b||_\infty = 5$$

The 2-norm is given by

$$||b||_2 = \sqrt{9 + 16 + 25} = \sqrt{50} = 5\sqrt{2}$$

c. here the matrix is **not** symmetric

$$A = \begin{pmatrix} 1 & 6 \\ 7 & 3 \end{pmatrix}$$

The sum of elements in each column:

$$1 + 7 = 8$$

$$6 + 3 = 9$$

The maximum is 9, so

$$||A||_1 = 9$$

The $\infty$-norm (row sum): The sum of elements in each row is

$$1 + 6 = 7$$

$$7 + 3 = 10$$

The maximum is 10, so

$$||A||_\infty = 10$$

For the 2-norm, since the matrix is not symmetric, we need

$$||A||_2 = \sqrt{\rho(A^T A)}$$

Now the matrix $A^T A$ is

$$A^T A = \begin{pmatrix} 1 & 7 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} 1 & 6 \\ 7 & 3 \end{pmatrix} = \begin{pmatrix} 50 & 27 \\ 27 & 45 \end{pmatrix}$$

Now find the eigenvalues

$$\begin{vmatrix} 50-\lambda & 27 \\ 27 & 45-\lambda \end{vmatrix} = \lambda^2 - 95\lambda + 1521 = 0$$

So

$$\lambda = \frac{1}{2}\left(95 \pm \sqrt{2941}\right)$$

The spectral radius is

$$\rho(A^T A) = \frac{1}{2}\left(95 + \sqrt{2941}\right)$$

and the spectral norm is

$$||A||_2 \sim 8.638$$

## 9.7 Matrix Method for Stability

## 9.8 Derivative Boundary Conditions

## 9.9 Hyperbolic Equations

### 9.9.1 Stability

**Problems**

1. Use a von Neumann stability analysis to show for the wave equation that a simple explicit Euler predictor using central differencing in space is unstable. The difference equation is

$$u_j^{n+1} = u_j^n - c\frac{\Delta t}{\Delta ax}\left(\frac{u_{j+1}^n - u_{j-1}^n}{2}\right)$$

Now show that the same difference method is stable when written as the implicit formula

$$u_j^{n+1} = u_j^n - c\frac{\Delta t}{\Delta x}\left(\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2}\right)$$

2. Prove that the CFL condition is the stability requirement when the Lax Wendroff method is applied to solve the simple 1-D wave equation. The difference equation is of the form:

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{c^2\left(\Delta t\right)^2}{2\left(\Delta x\right)^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

3. Determine the stability requirement to solve the 1-D heat equation with a source term

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2} + ku$$

Use the central-space, forward-time difference method. Does the von Neumann necessary condition make physical sense for this type of computational problem?

4. In attempting to solve a simple PDE, a system of finite-difference equations of the form

$$u_j^{n+1} = \begin{bmatrix} 1+\nu & 1+\nu & 0 \\ 0 & 1+\nu & \nu \\ -\nu & 0 & 1+\nu \end{bmatrix} u_j^n.$$

Investigate the stability of the scheme.

1. Use a von Neumann stability analysis to show for the wave equation that a simple explicit Euler predictor using central differencing in space is unstable. The difference equation is

$$u_j^{n+1} \ = \ u_j^n - c\frac{\Delta t}{\Delta a x}\left(\frac{u_{j+1}^n - u_{j-1}^n}{2}\right)$$

Substitute a Fourier mode

$$\lambda^n e^{ik_m j\Delta x}$$

where

$$k_m = \frac{m\pi}{L},\ \ \Delta x = \frac{L}{N},\ \ m = 0, 1, \ldots, N$$

we get

$$\lambda^{n+1} e^{ik_m j\Delta x} = \lambda^n e^{ik_m j\Delta x} - \frac{\nu}{2}\lambda^n\left(e^{ik_m(j+1)\Delta x} - e^{-ik_m(j-1)\Delta x}\right)$$

or

$$\lambda = 1 - \frac{\nu}{2}\left(\underbrace{e^{ik_m\Delta x} - e^{-ik_m\Delta x}}_{2i\sin k_m\Delta x}\right)$$

Taking absolute value

$$|\lambda| = \sqrt{Re(\lambda)^2 + Im(\lambda)^2} = \sqrt{1 + \nu^2\sin^2 k_m\Delta x} > 1$$

This is always greater than 1 since the second term under the radical is positive. Therefore the method is unstable.

Now show that the same difference method is stable when written as the implicit formula

$$u_j^{n+1} \ = \ u_j^n - c\frac{\Delta t}{\Delta x}\left(\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2}\right)$$

As before

$$\lambda = 1 - i\nu\lambda\sin k_m\Delta x$$

or

$$\lambda = \frac{1}{1 + i\nu\sin k_m\Delta x} = \frac{1 - i\nu\sin k_m\Delta x}{1 + \nu^2\sin^2 k_m\Delta x}$$

Taking absolute value

$$|\lambda| = \sqrt{Re(\lambda)^2 + Im(\lambda)^2} = \sqrt{\frac{1}{(1 + \nu^2\sin^2 k_m\Delta x)^2} + \frac{\nu^2\sin^2 k_m\Delta x}{(1 + \nu^2\sin^2 k_m\Delta x)^2}}$$

so

$$|\lambda| = \sqrt{\frac{1}{1 + \nu^2\sin^2 k_m\Delta x}} \leq 1$$

This is always less than or equal 1 since the denominator is larger than numerator. Therefore the method is always stable.

2. Prove that the CFL condition is the stability requirement when the Lax Wendroff method is applied to solve the simple 1-D wave equation. The difference equation is of the form:

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{c^2\left(\Delta t\right)^2}{2\left(\Delta x\right)^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

Substitute a Fourier mode as before we get

$$\lambda = 1 - \frac{\nu}{2}\left(\underbrace{e^{i\beta} - e^{-i\beta}}_{2i\sin\beta}\right) + \frac{\nu^2}{2}\left(\underbrace{e^{i\beta} - 2 + e^{-i\beta}}_{2\cos\beta - 2}\right)$$

Take the absolute value

$$|\lambda|^2 = \left[\underbrace{1 + \nu^2(\cos\beta - 1)}_{=Re(\lambda)}\right]^2 + \left(\underbrace{-\nu\sin\beta}_{=Im(\lambda)}\right)^2$$

$$|\lambda|^2 = 1 + 2\nu^2(\underbrace{\cos\beta - 1}_{=-2\sin^2\frac{\beta}{2}}) + \nu^4(\underbrace{\cos\beta - 1}_{=-2\sin^2\frac{\beta}{2}})^2 + \nu^2\sin^2\beta$$

$$|\lambda|^2 = 1 + 4\nu^2\sin^4\frac{\beta}{2}(\nu^2 - 1)$$

In order to get stability, we must have

$$\nu^2 - 1 \leq 0$$

or

$$\nu^2 \leq 1$$

which is the CFL condition.

3. Determine the stability requirement to solve the 1-D heat equation with a source term

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + ku$$

Use the central-space, forward-time difference method. Does the von Neumann necessary condition make physical sense for this type of computational problem?

The method is

$$u_j^{n+1} = (1 + k\Delta t - 2r)\, u_j^n + r\left(u_{j+1}^n + u_{j-1}^n\right)$$

Substitute a Fourier mode and we get the following equation for $\lambda$

$$\lambda = 1 + k\Delta t - 2r + 2r\cos\beta = 1 + 2r\underbrace{(\cos\beta - 1)}_{=-2\sin^2\frac{\beta}{2}} + k\Delta t$$

$$\lambda = 1 - 4r\sin^2\frac{\beta}{2} + k\Delta t$$

If $r \le \frac{1}{2}$ then

$$\lambda \le 1 + O(\Delta t)$$

The $\Delta t$ term makes sense since $ku$ term allows the solution to grow in time and thus $\lambda$ (and the numerical solution) must be allowed to grow.

4. In attempting to solve a simple PDE, a system of finite-difference equations of the form

$$u_j^{n+1} = \begin{bmatrix} 1+\nu & \nu & 0 \\ 0 & 1+\nu & \nu \\ -\nu & 0 & 1+\nu \end{bmatrix} u_j^n .$$

Investigate the stability of the scheme.

For the stability, we need $||A|| \leq 1$ or that $|eig(A)| \leq 1$. To compute the eigenvalues we need to solve

$$\begin{vmatrix} 1+\nu-\lambda & \nu & 0 \\ 0 & 1+\nu-\lambda & \nu \\ -\nu & 0 & 1+\nu-\lambda \end{vmatrix} = 0$$

or

$$(1+\nu-\lambda)^3 - \nu^3 = 0$$

So

$$1+\nu-\lambda = \begin{cases} \nu \\ \nu e^{2\pi i/3} \\ \nu e^{4\pi i/3} \end{cases}$$

$$\lambda = \begin{cases} 1 \\ 1+\nu\left(1-e^{2\pi i/3}\right) \\ 1+\nu\left(1-e^{4\pi i/3}\right) \end{cases}$$

Note that the last two eigenvalues are complex conjugate of each other. Now clearly the absolute value of the first is 1. The absolute value of the other two is the same and it is

$$\sqrt{\left[1+\nu\left(1-\cos\frac{2\pi}{3}\right)\right]^2 + \left(-\nu\sin\frac{2\pi}{3}\right)^2}$$

or

$$\sqrt{1+2\nu+\nu^2-2\nu(1+\nu)\underbrace{\cos\frac{2\pi}{3}}_{-\frac{1}{2}}+\nu^2\left(\underbrace{\cos^2\frac{2\pi}{3}+\sin^2\frac{2\pi}{3}}_{=1}\right)}$$

So

$$|\lambda| = \sqrt{1+3\nu+3\nu^2}$$

In order for this to be less than or equal 1, we have

$$1+3\nu+3\nu^2 \leq 1$$

$$3\nu+3\nu^2 \leq 0$$

$$\nu(1+\nu) \leq 0$$

Therefore

$$-1 \leq \nu \leq 0$$

### 9.9.2 Euler Explicit Method

### 9.9.3 Upstream Differencing

### 9.9.4 Lax Wendroff method

#### Problems

1. Derive the modified equation for the Lax Wendroff method.

1. Derive the modified equation for the Lax Wendroff method.

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = \frac{c^2 \Delta t}{2\Delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) \tag{1}$$

Expand each fraction in Taylor series (all terms on the right side of the following equations are evaluated at the point $(j, n)$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = u_t + \frac{\Delta t}{2}u_{tt} + \frac{\Delta t^2}{6}u_{ttt} + O(\Delta t^3)$$

$$c\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = cu_x + c\frac{\Delta x^2}{6}u_{xxx} + O(\Delta x^4)$$

$$\frac{c^2}{2}\Delta t\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} = \frac{c^2}{2}\Delta t u_{xx} + \frac{c^2}{24}\Delta t\Delta x^2 u_{xxxx}$$

Now substitute these expansions in the Lax Wendroff scheme (1)

$$u_t + \frac{\Delta t}{2}u_{tt} + \frac{\Delta t^2}{6}u_{ttt} + cu_x + c\frac{\Delta x^2}{6}u_{xxx} = \frac{c^2}{2}\Delta t u_{xx} + \frac{c^2}{24}\Delta t\Delta x^2 u_{xxxx}$$

Reorganize

$$u_t = -cu_x + \frac{\Delta t}{2}\left(c^2 u_{xx} - u_{tt}\right) - \frac{\Delta t^2}{6}u_{ttt} - c\frac{\Delta x^2}{6}u_{xxx} + \text{higher order terms} \tag{2}$$

Differentiate this (2) with respect to $t$

$$u_{tt} = -cu_{xt} + \frac{\Delta t}{2}\left(c^2 u_{xxt} - u_{ttt}\right) + \text{quadratic terms} \tag{3}$$

Substitute in (2)

$$u_t = -cu_x + \frac{\Delta t}{2}\left(c^2 u_{xx} + cu_{xt} - \frac{\Delta t}{2}\left(c^2 u_{xxt} - u_{ttt}\right)\right) - \frac{\Delta t^2}{6}u_{ttt} - c\frac{\Delta x^2}{6}u_{xxx}$$

Collect terms

$$u_t = -cu_x + c\frac{\Delta t}{2}\left(cu_{xx} + u_{xt}\right) + \Delta t^2\left(-\frac{c^2}{4}u_{xxt} + \frac{1}{12}u_{ttt}\right) - c\frac{\Delta x^2}{6}u_{xxx} + \text{cubic terms} \tag{4}$$

Differentiate (2) with respect to $x$

$$u_{tx} = -cu_{xx} + \frac{\Delta t}{2}\left(c^2 u_{xxx} - u_{ttx}\right) + \text{quadratic terms} \tag{5}$$

Substitute in (4)

$$\begin{aligned} u_t &= -cu_x + c\frac{\Delta t}{2}\left(cu_{xx} - cu_{xx} + \frac{\Delta t}{2}\left(c^2 u_{xxx} - u_{ttx}\right)\right) + \\ &+ \Delta t^2\left(-\frac{c^2}{4}u_{xxt} + \frac{1}{12}u_{ttt}\right) - c\frac{\Delta x^2}{6}u_{xxx} + \text{cubic terms} \end{aligned} \tag{6}$$

Differentiate (2) with respect to $x$ twice

$$u_{txx} = -cu_{xxx} + \text{linear terms}$$

Linear terms are enough because we have $\Delta t^2$ in front. Also get

$$\begin{aligned} u_{ttx} &= -cu_{xxt} + \text{linear terms} \\ &= -c\left(-cu_{xxx}\right) + \text{linear terms} \\ &= c^2 u_{xxx} + \text{linear terms} \end{aligned}$$

$$\begin{aligned} u_{ttt} &= -cu_{xtt} + \text{linear terms} \\ &= -c^3 u_{xxx} + \text{linear terms} \end{aligned}$$

Substitute in (6)

$$\begin{aligned} u_t &= -cu_x + c\frac{\Delta t^2}{4}\left(c^2 u_{xxx} - c^2 u_{xxx} + \text{linear terms}\right) + \\ &+ \Delta t^2 \left(-\frac{c^2}{4}\left(-cu_{xxx} + \text{linear terms}\right) + \frac{1}{12}\left(-c^3 u_{xxx}\right)\right) \\ &- c\frac{\Delta x^2}{6} u_{xxx} + \text{cubic terms} \end{aligned}$$

Collect terms:

$$u_t = -cu_x + \Delta t^2 \left(\frac{c^3}{4} u_{xxx} - \frac{c^3}{12} u_{xxx}\right) - c\frac{\Delta x^2}{6} u_{xxx} + \text{cubic terms}$$

or

$$u_t = -cu_x + \Delta t^2 \frac{c^3}{6} u_{xxx} - c\frac{\Delta x^2}{6} u_{xxx} + \text{cubic terms}$$

In terms of $\nu$

$$u_t = -cu_x + \frac{c}{6}\left(\nu^2 - 1\right)\Delta x^2 u_{xxx} + \text{cubic terms}$$

### 9.9.5 MacCormack Method

## 9.10 Inviscid Burgers' Equation

### 9.10.1 Lax Method

### 9.10.2 Lax Wendroff Method

### 9.10.3 MacCormack Method

### Problems

1. Determine the errors in amplitude and phase for $\beta = 90^\circ$ if the MacCormack scheme is applied to the wave equation for 10 time steps with $\nu = .5$.

1. Determine the errors in amplitude and phase for $\beta = 90°$ if the MacCormack scheme is applied to the wave equation for 10 time steps with $\nu = .5$.

Recall that

$$\begin{aligned} |G| &= \left|1 - \nu^2(1-\cos\beta) - i\nu\sin\beta\right| \\ &= \sqrt{\left[1-\nu^2(1-\cos\beta)\right]^2 + (-\nu\sin\beta)^2} \end{aligned}$$

Given that $\beta = 90°$ then $\cos\beta = 0$ and $\sin\beta = 1$. Substitute these values and the given $\nu = .5$ in $|G|$, we have

$$\begin{aligned} |G| &= \sqrt{\left[1 - .5^2(1-0)\right]^2 + (-.5\cdot 1)^2} \\ &= \sqrt{\left[1-.25\right]^2 + .25} = \sqrt{\frac{9}{16} + \frac{1}{4}} = \sqrt{\frac{13}{16}} \end{aligned}$$

After 10 steps

$$|G|^{10} = \left[\sqrt{\frac{13}{16}}\right]^{10} = \left[\frac{13}{16}\right]^5 \sim .354$$

The error in amplitude is $1 - .354 = .646$

Phase error is given by $10(\phi_e - \phi)$ where $\phi_e = -\beta\nu = -\frac{\pi}{4}$ Now

$$\tan\phi = \frac{-\nu\sin\beta}{1-\nu^2(1-\cos\beta)} = -\frac{\nu}{1-\nu^2} = -\frac{\frac{1}{2}}{\frac{3}{4}} = -\frac{2}{3}$$

Therefore

$$\phi \sim -.588$$

So the phase error is

$$10(-\frac{\pi}{4} + .588) \sim 1.974 \text{ radians}$$

### 9.10.4 Implicit Method

**Problems**

1. Apply the two-step Lax Wendroff method to the PDE

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} + u\frac{\partial^3 u}{\partial x^3} \;=\; 0$$

where $F = F(u)$. Develop the final finite difference equations.

2. Apply the Beam-Warming scheme with Euler implicit time differencing to the linearized Burgers' equation on the computational grid given in Figure 61 (use $c = 2$, $\mu = 2$, $\Delta x = 1$) and determine the steady state values of $u$ at $j = 2$ and $j = 3$. the boundary conditions are

$$u_1^n = 1, \qquad u_4^n = 4$$

and the initial conditions are

$$u_2^1 = 0, \qquad u_3^1 = 0$$

Do not use a computer to solve this problem.

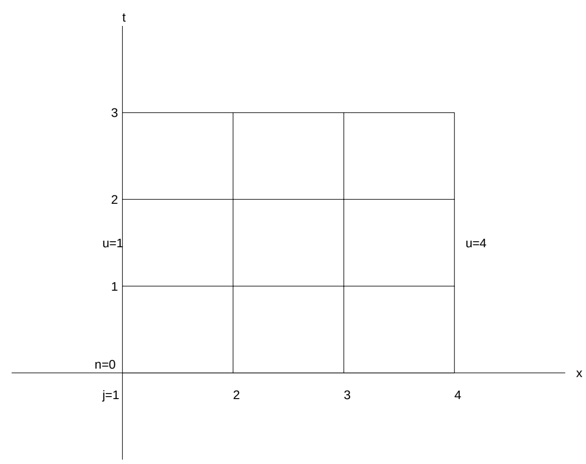


Figure 61: Computational Grid for Problem 2

1. Apply the two-step Lax Wendroff method to the PDE

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} + u\frac{\partial^3 u}{\partial x^3} \;=\; 0$$

where $F = F(u)$. Develop the final finite difference equations.

Recall that for

$$u_t + F_x = \mu u_{xx}$$

we had

Step 1:

$$\frac{u_j^{n+1/2} - \frac{1}{2}\left(u_{j+1/2}^n + u_{j-1/2}^n\right)}{\frac{1}{2}\Delta t} + \frac{F_{j+1/2}^n - F_{j-1/2}^n}{\Delta x} = \mu\frac{u_{xx}\Big|_{j+1/2}^n + u_{xx}\Big|_{j-1/2}^n}{2}$$

Step 2:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{F_{j+1/2}^{n+1/2} - F_{j-1/2}^{n+1/2}}{\Delta x} = \mu u_{xx}\Big|_j^n$$

In our case we have $uu_{xxx}$ instead of $\mu u_{xx}$, so

Step 1:

$$\frac{u_j^{n+1/2} - \frac{1}{2}\left(u_{j+1/2}^n + u_{j-1/2}^n\right)}{\frac{1}{2}\Delta t} + \frac{F_{j+1/2}^n - F_{j-1/2}^n}{\Delta x} + \frac{uu_{xxx}\Big|_{j+1/2}^n + uu_{xxx}\Big|_{j-1/2}^n}{2} = 0$$

Step 2:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{F_{j+1/2}^{n+1/2} - F_{j-1/2}^{n+1/2}}{\Delta x} = uu_{xxx}\Big|_j^n = 0$$

We need $uu_{xxx}$ approximated to $O(\Delta x^2)$, one can show

$$u_{xxx}\Big|_j^n = \frac{4u_{j+1}^n - 8u_{j+1/2}^n + 8u_{j-1/2}^n - 4u_{j-1}^n}{\Delta x^3} + O(\Delta x^2)$$

Using this approximation shifted to $j \pm 1/2$, we get

$$u_{xxx}\Big|_{j+1/2}^n = \frac{4u_{j+3/2}^n - 8u_{j+1}^n + 8u_j^n - 4u_{j-1/2}^n}{\Delta x^3}$$

$$u_{xxx}\Big|_{j-1/2}^n = \frac{4u_{j+1/2}^n - 8u_j^n + 8u_{j-1}^n - 4u_{j-3/2}^n}{\Delta x^3}$$

Substitute these in step 1

Step 1:

$$\frac{u_j^{n+1/2} - \frac{1}{2}\left(u_{j+1/2}^n + u_{j-1/2}^n\right)}{\frac{1}{2}\Delta t} + \frac{F_{j+1/2}^n - F_{j-1/2}^n}{\Delta x}$$

$$+\frac{1}{2}u_{j-1/2}^n\left(\frac{4u_{j+1/2}^n - 8u_j^n + 8u_{j-1}^n - 4u_{j-3/2}^n}{\Delta x^3}\right)$$

$$+\frac{1}{2}u_{j+1/2}^n\left(\frac{4u_{j+3/2}^n - 8u_{j+1}^n + 8u_j^n - 4u_{j-1/2}^n}{\Delta x^3}\right) = 0$$

Multiply through by $\frac{1}{2}\Delta t$ and collect terms

Step 1:

$$\begin{aligned} u_j^{n+1/2} &- \frac{1}{2}\left(u_{j+1/2}^n + u_{j-1/2}^n\right) + \frac{1}{2}\frac{\Delta t}{\Delta x}\left(F_{j+1/2}^n - F_{j-1/2}^n\right) \\ &+ \frac{\Delta t}{\Delta x^3}\left[u_{j-1/2}^n\left(-8u_j^n + 8u_{j-1}^n - 4u_{j-3/2}^n\right)\right. \\ &+ \left. u_{j+1/2}^n\left(4u_{j+3/2}^n - 8u_{j+1}^n + 8u_j^n\right)\right] = 0 \end{aligned}$$

Step 2:

$$\begin{aligned} u_j^{n+1} &- u_j^n + \frac{\Delta t}{\Delta x}\left(F_{j+1/2}^{n+1/2} - F_{j-1/2}^{n+1/2}\right) \\ &+ \frac{\Delta t}{\Delta x^3}u_j^n\left(4u_{j+1}^n - 8u_{j+1/2}^n + 8u_{j-1/2}^n - 4u_{j-1}^n\right) = 0 \end{aligned}$$

Another possibility is to include $uu_{xxx}$ term into $F$. Let

$$G = uu_{xx} - \frac{1}{2}u_x^2$$

then

$$\frac{\partial G}{\partial x} = u_x u_{xx} + uu_{xxx} - \frac{1}{2}2u_x u_{xx} = uu_{xxx}$$

So now the equation is

$$u_t + F_x + G_x = 0$$

or

$$u_t + (F + G)_x = 0$$

For this equation we can use Lax Wendroff method as in class.

2. Apply the Beam-Warming scheme with Euler implicit time differencing to the linearized Burgers' equation on the computational grid given in Figure 61 (use $c = 2$, $\mu = 2$, $\Delta x = 1$) and determine the steady state values of $u$ at $j = 2$ and $j = 3$. the boundary conditions are

$$u_1^n = 1, \qquad u_4^n = 4$$

and the initial conditions are

$$u_2^1 = 0, \qquad u_3^1 = 0$$

Do not use a computer to solve this problem.

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c u_x \Big|_j^{n+1} = \mu u_{xx} \Big|_j^{n+1}$$

Let $\nu = c\dfrac{\Delta t}{\Delta x}$ and $r = \mu\dfrac{\Delta t}{\Delta x^2}$ then

$$u_j^{n+1} = u_j^n - \frac{\nu}{2}\left(u_{j+1}^{n+1} - u_{j-1}^{n+1}\right) + r\left(u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}\right)$$

or

$$-\left(\frac{\nu}{2} + r\right) u_{j-1}^{n+1} + (2r+1)u_j^{n+1} + \left(\frac{\nu}{2} - r\right) u_{j+1}^{n+1} = u_j^n$$

In our case $c = 2$, $\mu = 2$, $\Delta x = 1$ and so if we let $\Delta t = 1$, then $r = \nu = 2$. Using these values in the above equation, we get

$$-3u_{j-1}^{n+1} + 5u_j^{n+1} - u_{j+1}^{n+1} = u_j^n$$

For $n = 1$ (don't forget to employ the boundary conditions)

$$\begin{aligned} -3 \cdot 1 + 5u_2^2 - u_3^2 &= 0 \\ -3u_2^2 + 5u_3^2 - 4 &= 0 \end{aligned}$$

The solution of this system of two equations is

$$u_2^2 = \frac{19}{22}$$

$$u_3^2 = \frac{29}{22}$$

Now go to the next time step $n = 2$

$$\begin{aligned} -3 + 5u_2^3 - u_3^3 &= \frac{19}{22} \\ -3u_2^3 + 5u_3^3 - 4 &= \frac{29}{22} \end{aligned}$$

The solution of this system of two equations is

$$u_2^3 = \frac{542}{484} \sim 1.1198$$

$$u_3^3 \sim 1.7355$$

The next time step $n = 4$, we have

$$u_2^4 \sim 1.1970$$

$$u_3^4 \sim 1.8653$$

The next time step $n = 5$, we have

$$u_2^5 \sim 1.2205$$

$$u_3^5 \sim 1.9053$$

The next time step $n = 6$, we have

$$u_2^6 \sim 1.2276$$

$$u_3^6 \sim 1.9176$$

Analytic steady state solution:

$$cu_x = \mu u_{xx}$$

$$u(0) = 1$$

$$u(3) = 4$$

$$u = 1 + \frac{3}{1 - e^3} (1 - e^x)$$

so

$$u(1) = u_2 \sim 1.2701$$

$$u(2) = u_3 \sim 2.0043$$

We are getting there, it may require few more steps. Since the method is unconditionally stable, we can choose a larger time step.

## 9.11 Viscous Burgers' Equation

### 9.11.1 FTCS method

### 9.11.2 Lax Wendroff method

### 9.11.3 MacCormack method

### 9.11.4 Time-Split MacCormack method