MA 1116 — Suggested Homework Problems from Davis & Snider 7^{th} edition

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1.2

Problems: 1–4, 8

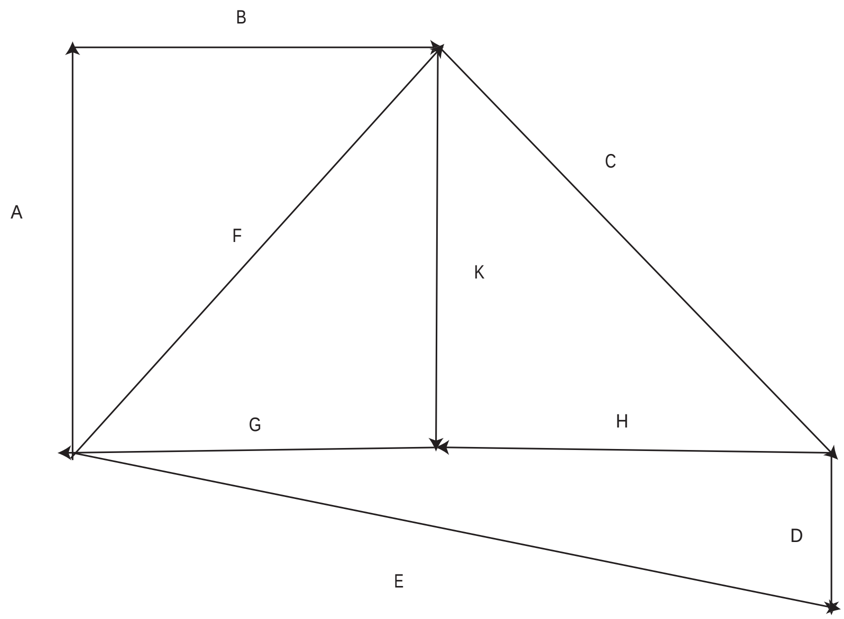

Figure 1: Figure 1.6 for Problems 1-4 on section 1.2

1. C in terms of E, D, F
   Note that

$$F + C + D = E$$

Therefore

$$C = E - D - F$$

2. G in terms of C, D, E, K
   Note that

$$K + G = -F$$

and

$$F + C + D = E$$

or

$$F = E - C - D$$

Therefore

$$K + G = -E + C + D$$

or

$$G = -E + C + D - K$$

3. Given $x + B = F$, solve for x

$$x = F - B$$

based on Figure 1

$$F - B = A$$

Therefore

$$x = A$$

4. Given $x + H = D - E$, solve for x

$$x = D - E - H$$

based on Figure 1

$$D - E = G + H$$

Therefore

$$x = G + H - H = G$$

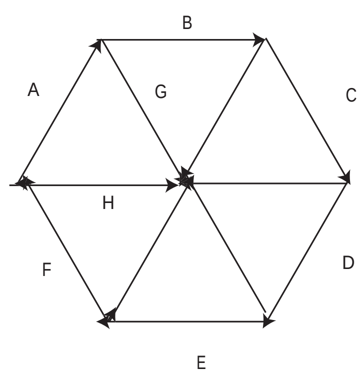

Figure 2: Six equilateral triangles

8. a.

Note that $G = C$ but $G = H - A$ and $H = B$ therefore $G = B - A$ so

$$C = B - A$$

$$D = -A$$

$$E = -B$$

$$F = -C = A - B$$

8. b. Sum of all is zero (vector addition with last terminal point being the same as first initial point)

1.3

Problems: 2, 5, 8, 14

2. If $|\mathbf{A}| = 3$ then $|4\mathbf{A}| = 4|\mathbf{A}| = 4 \times 3 = 12$
   Also $|-2\mathbf{A}| = |-2|\,|\mathbf{A}| = 2 \times 3 = 6$
   For $-2 \leq s \leq 1$ we have $|s\mathbf{A}| = |s||\mathbf{A}| = 3|s| \leq 3 \times 2 = 6$. The minimum value of s is zero, so that $|s\mathbf{A}| \geq 0$.

5.
$$\mathbf{A} = \alpha\mathbf{B}$$

Is it possible that $\mathbf{B} = \gamma\mathbf{A}$?

If $\alpha \neq = 0$ then we can divide and so $\gamma = \dfrac{1}{\alpha}$.

If $\alpha = 0$ then $|\mathbf{A}| = 0$ and one cannot have any **nonzero** vector as a multiple of zero.

8. Two vectors, one pointing up from plane and one is pointing downward.

14. Given the vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ (see Figure 3)

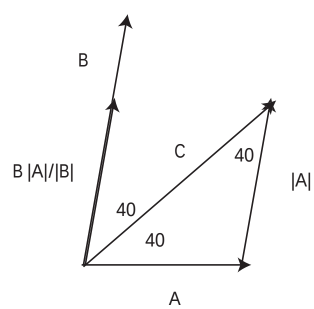

Figure 3: For problem 14 section 1.2

Let us sketch the vector $\mathbf{B}$ from the terminal point of the vector $\mathbf{A}$. In order to get an isosceles triangle, we make this new vector the same length of $\mathbf{A}$, i.e. the new vecor is $\mathbf{B}\frac{|\mathbf{A}|}{|\mathbf{B}|}$.

Note that I used 80 degree-angle between the vectors $\mathbf{A}$ and $\mathbf{B}$.

$$\mathbf{C} = \mathbf{A} + \underbrace{\mathbf{B}\frac{|\mathbf{A}|}{|\mathbf{B}|}}_{\text{vector of length } |\mathbf{A}| \text{ in the direction of } \mathbf{B} \text{ so that the angle is bisected}}$$

1.4

Problems: 4, 6, 9, 12

4. $|3\mathbf{i} - 4\mathbf{j}| = \sqrt{3^2 + (-4)^2} = \sqrt{9+16} = 5$

6. Given $P_1(4,2)$, $P_2(5,-1)$

$$\vec{P_1P_2} = (5-4)\mathbf{i} + (-1-2)\mathbf{j} = \mathbf{i} - 3\mathbf{j}$$

9. a. $\mathbf{A} = \alpha\mathbf{i} + \beta\mathbf{j}$ with $|\mathbf{A}| = \sqrt{\alpha^2+\beta^2} = 1$ and $\mathbf{A}$ makes an angle of $60°$ with the x axis. Clearly, $\sin(60°) = \dfrac{\beta}{|\mathbf{A}|} = \beta$, so

$$\beta = \frac{\sqrt{3}}{2}$$

and thus

$$\alpha = \sqrt{1-\beta^2} = \sqrt{1-\frac{3}{4}} = \frac{1}{2}$$

Therefore

$$\mathbf{A} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

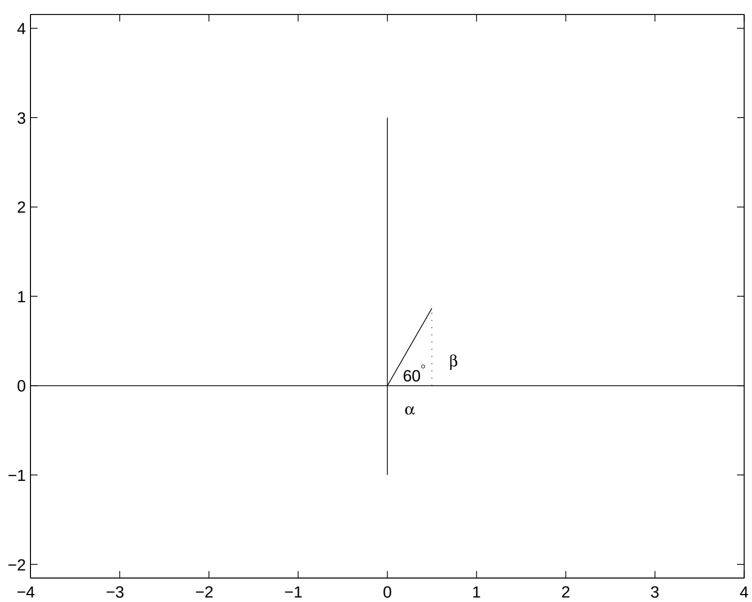

Figure 4: For Problem 9a of section 1.4

9. b. $\mathbf{A} = \alpha\mathbf{i} + \beta\mathbf{j}$ with $|\mathbf{A}| = \sqrt{\alpha^2+\beta^2} = 1$ and $\mathbf{A}$ makes an angle of $-30°$ with the x axis. Clearly, $\sin(-30°) = \dfrac{\beta}{|\mathbf{A}|} = \beta$, so

$$\beta = -\frac{1}{2}$$

and thus

$$\alpha = \sqrt{1 - \beta^2} = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$$

Therefore

$$\mathbf{A} = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$$

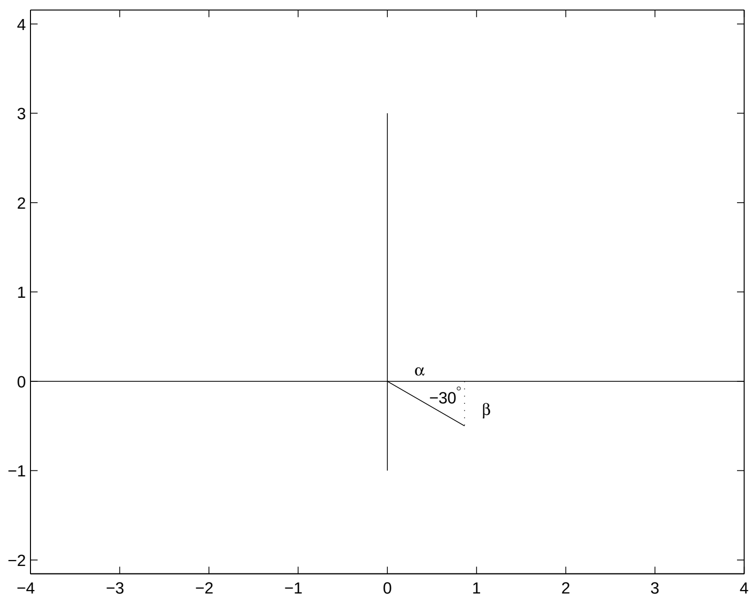

Figure 5: For Problem 9b of section 1.4

9. c. $|3\mathbf{i} + 4\mathbf{j}| = 5$. The vector $\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$ is in the same direction as the original and of unit length (dividing by the length of the original vector)

9. d. $\mathbf{A} = \frac{1}{2}\mathbf{i} + \beta\mathbf{j}$. The length of the vector is unity, implying $\beta^2 + \left(\frac{1}{2}\right)^2 = 1$ or $\beta^2 = 1 - \frac{1}{4} = \frac{3}{4}$, so $\beta = \pm\frac{\sqrt{3}}{2}$ and

$$\mathbf{A} = \frac{1}{2}\mathbf{i} \pm \frac{\sqrt{3}}{2}\mathbf{j}$$

9. e. $\mathbf{A}$ is a unit vector perpendicular to the line $x + y = 0$, so $\mathbf{A} = \alpha\mathbf{i} + \beta\mathbf{j}$ with $\alpha^2 + \beta^2 = 1$. Therefore (see figure 6) $\alpha = \beta = \pm\frac{\sqrt{2}}{2}$ and

$$\mathbf{A} = \pm\frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$$

(Only two vectors!!)

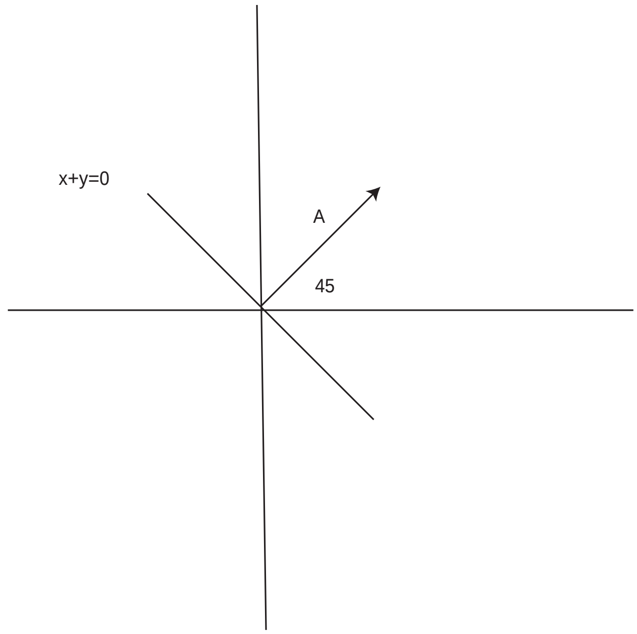

Figure 6: For Problem 9e of section 1.4

12. given the vector $\mathbf{V}$ whose endpoints are A and B, so that $\mathbf{V} = 2\mathbf{i} + 3\mathbf{j}$. Given the midpoint $M = (2, 1)$ we have to find the coordinates of the endpoints. Suppose $A = (x_1, y_1)$ and $B = (x_2, y_2)$ then

$$x_2 - x_1 = 2,\ y_2 - y_1 = 3$$

and from midpoint formula

$$\frac{x_1 + x_2}{2} = 2,\ \frac{y_1 + y_2}{2} = 1$$

solve the set of two equations for the x coordinates, we get $x_1 = 1$ and $x_2 = 3$. Solve the other set for the y coordinates and we have $y_1 = -\frac{1}{2}$ and $y_2 = \frac{5}{2}$. Therefore the endpoints $A = (1, -.5),\ B = (3, 2.5)$

1.5

Problems: 4–6, 13, 15

4. $\mathbf{B} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, so $||\mathbf{B}|| = \sqrt{4+4+1} = 3$.
   $||s\mathbf{B}|| = 1$ implies $|s| = \dfrac{1}{||\mathbf{B}||}$. Therefore $|s| = \frac{1}{3}$ or $s = \pm\frac{1}{3}$.

5. $\mathbf{A} = 3\mathbf{i} + 4\mathbf{j}$, so $||\mathbf{A}|| = \sqrt{9+16} = 5$ and the unit vector

$$\frac{\mathbf{A}}{||\mathbf{A}||} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

6. $\mathbf{A} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{C} = 3\mathbf{i} - 4\mathbf{k}$
   We have $\mathbf{C} + \mathbf{D} = \mathbf{A}$, so $\mathbf{D}$=$\mathbf{A}$-$\mathbf{C} = 4\mathbf{j} + 4\mathbf{k}$
   a. $||\mathbf{D}|| = \sqrt{16+16} = 4\sqrt{2}$
   b. The vector is parallel to yz plane since there is no component in x direction.

13. $\mathbf{A} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ therefore $||\mathbf{A}|| = \sqrt{4+4+1} = \sqrt{9} = 3$
   Therefore $\cos\alpha = \dfrac{2}{3},\ \cos\beta = -\dfrac{2}{3},\ \cos\gamma = \dfrac{1}{3}$

15. $\cos\alpha = \frac{1}{2}$ describes a cone with vertex at the origin and base perpendicular to x axis.

1.6

Problems: 3, 5

3. $\mathbf{R}_1(t=1) = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, and $\mathbf{R}_2(t=3) = 3\mathbf{i} + 36\mathbf{j} - 27\mathbf{k}$,
   then the displacement is $\mathbf{R}_2 - \mathbf{R}_1 = 32\mathbf{j} - 26\mathbf{k}$.

5. $\mathbf{F}_1 = 3\mathbf{k}$, $\mathbf{F}_2 = 6\mathbf{i}$, $\mathbf{F}_3 = 2\mathbf{j}$, what is $\mathbf{F}_4$, such that $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 = 0$?
   $\mathbf{F}_4 = -\mathbf{F}_1 - \mathbf{F}_2 - \mathbf{F}_3 = -6\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$.
   The magintude of $\mathbf{F}_4$ is $||\mathbf{F}_4|| = \sqrt{36+4+9} = \sqrt{49} = 7\ \ell b$

1.7

Problems: 1, 3, 7, 9, 17, 19

1. $\mathbf{A} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k},\ \mathbf{B} = 3\mathbf{i} - 4\mathbf{k}$
   $||\mathbf{A}|| = \sqrt{4+1+4} = \sqrt{9} = 3$, and $||\mathbf{B}|| = \sqrt{9+0+16} = \sqrt{25} = 5$
   and $\cos\theta = \dfrac{6+0-8}{3\cdot 5} = -\dfrac{2}{15}$

$$\theta = \cos^{-1}\left(-\frac{2}{15}\right)$$

3. Find the three angles of the triangle with vertices $A = (2,-1,1),\ B = (1,-3,-5)$, and $C = (3,-4,-4)$. The sides are: $\vec{AB} = -\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$, $\vec{AC} = \mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$, and $\vec{BC} = 2\mathbf{i} - \mathbf{j} + bk$.
   Therefore $\cos\theta_1 = \dfrac{\vec{AB}\cdot\vec{AC}}{||\vec{AB}||||\vec{AC}||} = \dfrac{-1+6+30}{\sqrt{1+4+36}\sqrt{1+9+25}} = \dfrac{35}{\sqrt{41}\sqrt{35}} = \sqrt{\dfrac{35}{41}}$

$$\theta_1 = \arccos\left(\sqrt{\frac{35}{41}}\right)$$

   Since we need vectors emanating from the point B, we should use $-\vec{AB}$,
   $\cos\theta_2 = \dfrac{-\vec{AB}\cdot\vec{BC}}{||\vec{AB}||||\vec{BC}||} = \dfrac{-(-2+2-6)}{\sqrt{1+4+36}\sqrt{4+1+1}} = \dfrac{6}{\sqrt{41}\sqrt{6}} = \sqrt{\dfrac{6}{41}}$

$$\theta_2 = \arccos\left(\sqrt{\frac{6}{41}}\right)$$

$$\theta_3 = 180 - \theta_1 - \theta_2$$

7. Let the 4 sides be the vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$, so that

$$\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = 0.$$

Since the two opposing sides are equal and parallel, we have

$$\mathbf{A} = -\mathbf{C}$$

Therefore, when we substitute this in the previous equation, we get

$$\mathbf{B} + \mathbf{D} = 0$$

and thus

$$\mathbf{B} = -\mathbf{D}$$

and so the other two are parallel and equal.

9. Show that $AE = EC,\ BE = ED$, see figure.

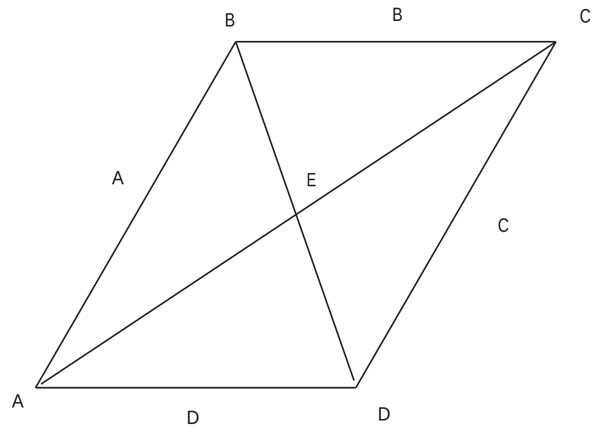

Figure 7: For Problem 9 of section 1.7

$$\vec{A}+\vec{B}+\vec{C}+\vec{D}=0$$

$$\vec{CA}=\vec{A}+\vec{B}$$

Let us look at $\vec{CE}$, and show that $\vec{CE}=\frac{1}{2}\vec{CA}=\frac{1}{2}\left(\vec{A}+\vec{B}\right)$ and similarly $\vec{BE}=\frac{1}{2}\vec{BD}$.

Now $\vec{CE}=s\left(\vec{A}+\vec{B}\right)$ since it is in the direction of $\vec{A}+\vec{B}$. The point E is on the line connecting the points B and D and so $-\vec{EC}=\vec{C}+t\left(\vec{B}+\vec{C}\right)$, but $\vec{C}=-\vec{A}$, so

$$-\vec{EC}=-\vec{A}+t\left(\vec{B}-\vec{A}\right)$$

Therefore

$$\vec{EC}=\vec{A}+t\left(\vec{A}-\vec{B}\right)=s\left(\vec{A}+\vec{B}\right)$$

so

$$(s-1-t)\vec{A}=(-s-t)\vec{B}$$

Since $\vec{A}$ and $\vec{B}$ are **not** parallel, the scalars are zero

$$s-1-t=0,\ \ s+t=0$$

Or

$$2s-1=0$$

so

$$s=\frac{1}{2}$$

and therefore substituting s above $\vec{CE}=\frac{1}{2}\left(\vec{A}+\vec{B}\right)$ as we wanted to show.

17. The equation of a sphere

$$||\mathbf{r}||^2=(x-2)^2+(y-3)^2+(z-4)^2$$

where **r** is the vector from the center (2,3,4) to the point (x,y,z) on the edge of the sphere, therefore $||\mathbf{r}||$ is the radius which is given as 3 and the equation

$$(x-2)^2 + (y-3)^2 + (z-4)^2 = 9$$

19.

$$x = y = z$$

is a line (two equations!) $x = y$ is a plane and $x = z$ is a plane and we are taking the intersection of these two.

1.8

Problems: 1, 4, 7, 9, 11, 15a

1. Line thru $(0,0,0)$ parallel to $3\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$

$$\frac{x-0}{3} = \frac{y-0}{-2} = \frac{z-0}{7}$$

or

$$\frac{x}{3} = -\frac{y}{2} = \frac{z}{7}$$

or in parametric form

$$\begin{aligned} x &= 3t \\ y &= -2t \\ z &= 7t \end{aligned}$$

4. Find the two unit vectors parallel to the line

$$\frac{x-1}{3} = \frac{y+2}{4}, \qquad z = 9$$

The vector $\mathbf{v}$ in the direction of the line is $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$, and $||\mathbf{v}|| = \sqrt{9+16} = 5$, therefore the unit vectors are $\dfrac{\mathbf{v}}{\pm||\mathbf{v}||} = \pm\dfrac{3}{5}\mathbf{i} \pm \dfrac{4}{5}\mathbf{j}$

7. Find equations of line thru $(0,0,0)$ parallel to $x - 3 = \dfrac{y+2}{4} = -z + 1$.

The vector $\mathbf{v}$ in the direction of the line is $\mathbf{v} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and so the equations are

$$\frac{x-0}{1} = \frac{y-0}{4} = \frac{z-0}{-1}$$

or

$$x = \frac{y}{4} = -z$$

9. Line thru $(1,4,-1)$ and $(2,2,7)$

$$\mathbf{v} = (2-1)\mathbf{i} + (2-4)\mathbf{j} + (7+1)\mathbf{k} = \mathbf{i} - 2\mathbf{j} + 8\mathbf{k}$$

and the line is

$$\frac{x-1}{1} = \frac{y-4}{-2} = \frac{z+1}{8}$$

or

$$x - 2 = \frac{2-y}{2} = \frac{z-7}{8}$$

11. Angle between lines

$$\frac{x-1}{3} = \frac{y-3}{4} = \frac{z}{5}$$

$$\frac{x-1}{2} = 3 - y = 2z$$

We need the angle between the vectors $\mathbf{v}_1 = 3\mathbf{i}+4\mathbf{j}+5\mathbf{k}$ and $\mathbf{v}_2 = 2\mathbf{i}-\mathbf{j}+\frac{1}{2}\mathbf{k}$ in the direction of these lines. $||\mathbf{v}_1|| = \sqrt{9+16+25} = \sqrt{50}$, $||\mathbf{v}_2|| = \sqrt{4+1+\dfrac{1}{4}} = \sqrt{\dfrac{21}{4}}$

Therefore

$$\cos\theta = \frac{3\cdot 2+4(-1)+5\cdot\frac{1}{2}}{\sqrt{50}\sqrt{\frac{21}{4}}} = \frac{6-4+\frac{5}{2}}{\sqrt{50}\frac{\sqrt{21}}{2}} = \frac{9}{\sqrt{50}\sqrt{21}} = \frac{9}{5\sqrt{21}\sqrt{2}}$$

15. a. Given the lines

$$\mathbf{R}_1 = (5\mathbf{i}+4\mathbf{j}+5\mathbf{k})t + 7\mathbf{i}+6\mathbf{j}+8\mathbf{k}$$

$$\mathbf{R}_2 = (6\mathbf{i}+4\mathbf{j}+6\mathbf{k})t + 8\mathbf{i}+6\mathbf{j}+9\mathbf{k}$$

We can also write them as follows

$$\frac{x-7}{5} = \frac{y-6}{4} = \frac{z-8}{5}$$

$$\frac{x-8}{6} = \frac{y-6}{4} = \frac{z-9}{6}$$

Find the point(s) of intersection. Since both have $\dfrac{y-4}{6}$, we get

$$\frac{x-7}{5} = \frac{x-8}{6}$$

or $6x-42 = 5x-40$ or

$$x = 2$$

Now substitute this into $\dfrac{x-7}{5} = \dfrac{y-6}{4}$ to get $\dfrac{-5}{5} = \dfrac{y-6}{4}$ or

$$y = 2$$

Now we take $x = y = 2$ and substitute in **both** equations

$$-1 = -1 = \frac{z-8}{5}$$

$$-1 = -1 = \frac{z-9}{6}$$

The first one yields $z-8 = -5$ or $z = 3$. This value of z satisfies the second equation. Therefore we have one point $(2, 2, 3)$

1.9

Problems: 1, 5, 7, 11, 12a, 21, 24

1. The scalar product of $(3\mathbf{i}+8\mathbf{j}-2\mathbf{k})\cdot(5\mathbf{i}+\mathbf{j}+2\mathbf{k}) = 3(5)+8(1)-2(2) = 15+8-4 = 19$

5. The angle between the two vectors $2\mathbf{i}$ and $3\mathbf{i}+4\mathbf{j}$ is

$$\cos\theta = \frac{2(3)+0(4)}{\sqrt{4}\sqrt{9+16}} = \frac{6}{2(5)} = 0.6$$

$$\theta = 53.13^\circ$$

7. Find the component of $\mathbf{B} = 8\mathbf{i}+\mathbf{j}$ in the direction of $\mathbf{A} = \mathbf{i}+2\mathbf{j}-2\mathbf{k}$

$$\frac{\mathbf{A}\cdot\mathbf{B}}{||\mathbf{A}||} = \frac{8(1)+1(2)+0(-2)}{\sqrt{1+4+4}} = \frac{10}{\sqrt{9}} = \frac{10}{3}$$

11. $\mathbf{A}\cdot\mathbf{A} = 0$ and $\mathbf{A}\cdot\mathbf{B} = 0$

From the first we have that $\mathbf{A}$ is the zero vector, substituting in the second we get identity no matter what the vector $\mathbf{B}$ is.

12. a. $\mathbf{B} = 6\mathbf{i}-3\mathbf{j}-6\mathbf{k}$, $\mathbf{A} = \mathbf{i}+\mathbf{j}+\mathbf{k}$, then

$$\mathbf{B}_{||} = \frac{\mathbf{B}\cdot\mathbf{A}}{\mathbf{A}\cdot\mathbf{A}}\mathbf{A} = \frac{6-3-6}{1+1+1}(\mathbf{i}+\mathbf{j}+\mathbf{k}) = -\mathbf{i}-\mathbf{j}-\mathbf{k}$$

$$\mathbf{B}_{\perp} = \mathbf{B}-\mathbf{B}_{||} = (6\mathbf{i}-3\mathbf{j}-6\mathbf{k})-(-\mathbf{i}-\mathbf{j}-\mathbf{k}) = 7\mathbf{i}-2\mathbf{j}-5\mathbf{k}$$

21. Show that $||\mathbf{A}+\mathbf{B}|| \le ||\mathbf{A}||+||\mathbf{B}||$

Let us square the left hand side

$$||\mathbf{A}+\mathbf{B}||^2 = (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}) = \mathbf{A}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{B}+\mathbf{B}\cdot\mathbf{A}+\mathbf{B}\cdot\mathbf{B} = ||\mathbf{A}||^2+2\mathbf{A}\cdot\mathbf{B}+||\mathbf{B}||^2$$

Now square the right hand side

$$(||\mathbf{A}||+||\mathbf{B}||)^2 = ||\mathbf{A}||^2+2||\mathbf{A}||||\mathbf{B}||+||\mathbf{B}||^2$$

So we need to show that

$$2\mathbf{A}\cdot\mathbf{B} \le 2||\mathbf{A}||||\mathbf{B}||$$

Now $\mathbf{A}\cdot\mathbf{B} = ||\mathbf{A}||||\mathbf{B}||\cos\theta$ and since $|\cos\theta| \le 1$, we have the triangle inequality.

24. Given two lines, line ℓ_1 thru $(5,1,-2)$ and $(2,-3,1)$ and line ℓ_2 thru $(3,8,1)$ and $(-3,0,7)$, then

$$\mathbf{v}_1 = 3\mathbf{i}+4\mathbf{j}-3\mathbf{k}$$

$$\mathbf{v}_2 = 6\mathbf{i}+8\mathbf{j}-6\mathbf{k}$$

$$\mathbf{v}_1\cdot\mathbf{v}_1 = 3(6)+4(8)+(-3)(-6) \ne 0$$

Therefore the lines are **not** perpendicular. In fact $\mathbf{v}_2 = 2\mathbf{v}_1$ so the lines are parallel

1.10

Problems: 2, 4, 6, 9, 12, 23

2. Plane thru the origin and perpendicular to $2\mathbf{i} - 8\mathbf{j} + 2\mathbf{k}$ is

$$2x - 8y + 2z = 0$$

We can divide the equation by 2 to get $x - 4y + z = 0$.

4. Plane parallel to $3x + y - z = 8$ thru $(1, 3, 3)$ is

$$3(x - 1) + y - 3 - (z - 3) = 0$$

or

$$3x + y - z = 3$$

6. The distance from the point $(3, 4, 7)$ to the plane $2x - y - 2z = 4$

Let $\mathbf{R}_1 = 3\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}$ the position vector to the point and $\mathbf{n} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $d = 4$ the constant on the right hand side in the equation of the plane, then the distance is

$$\frac{|\mathbf{R}_1 \cdot \mathbf{n} - d|}{||\mathbf{n}||} = \frac{|6 - 4 - 14 - 4|}{\sqrt{4 + 1 + 4}} = \frac{16}{3}$$

9. Show that the normal to the plane $2x - 8y + 2z = 5$ is parallel to the line $x = y = \dfrac{z + 2}{3}$

$$\mathbf{n} = 2\mathbf{i} - 8\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{v} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$$

$$\mathbf{n} \cdot \mathbf{v} = 2 - 8 + 6 = 0$$

Therefore the normal to the plane is perpendicular to the line.

12. Given the three point $O(0, 0, 0)$, $A(1, 2, 3)$, $B(0, -1, 1)$ and $C(2, 0, 2)$

a. Find a vector perpendicular to the plane OAB

Since $\vec{OA}$ and $\vec{OB}$ are in the plane, we take the cross product. Since we don't know yet how to find the cross product, we just get the equation of the plane and this will give us the normal.

$$a(x - 0) + b(y - 0) + c(z - 0) = 0, \qquad \text{since } O \text{ is on the plane}$$

Using A and B, we get

$$a + 2b + 3c = 0$$

$$-b + c = 0$$

From the second $b = c$, so $a + 2b + 3b = 0$ implies $a = -5b$. Clearly we have a choice, we take $b = 1$ so that $c = 1$ and $a = -5$. The equation of the plane is

$$-5x + y + z = 0$$

and the normal is

$$-5\mathbf{i} + \mathbf{j} + \mathbf{k}$$

Note that a different choice of b will give a multiple of this vector.
b. Find the distance from C to the plane

$$\frac{|-5(2) + 1(2) - 0|}{\sqrt{25 + 1 + 1}} = \frac{|-8|}{\sqrt{27}} = \frac{8}{\sqrt{27}}$$

23. Plane thru the points $(4, 0, 0)$, $(0, 6, 0)$, and $(0, 0, 12)$, find equation of another plane thru $(6, -2, 4)$ parallel to this one.

$$a(x - 4) + by + cz = 0, \qquad \text{since the point } (4, 0, 0) \text{ is on the plane}$$

$$a(-4) + 6b = 0$$

$$a(-4) + 12c = 0$$

So $a = 3c$ and $b = 2a/3$, choose $c = 1$ gives $a = 3$ and $b = 2$, so the normal is $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and the equations of the plane required is

$$3(x - 6) + 2(y + 2) + (z - 4) = 0$$

or

$$3x + 2y + z = 18 - 4 + 4 = 18$$

1.12

Problems: 1, 3, 5, 8, 10, 13, 19, 22, 23

1. a. $\mathbf{A} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{B} = \mathbf{i} + \mathbf{j} - 4\mathbf{k}$

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ 1 & 1 & -4 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -1 & 2 \\ 1 & -4 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 3 & 2 \\ 1 & -4 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} = \mathbf{i}(4-2) - \mathbf{j}(-12-2) + \mathbf{k}(3+1)$$

$$\mathbf{A}\times\mathbf{B} = 2\mathbf{i} + 14\mathbf{j} + 4\mathbf{k}$$

b.

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 7 \\ 3 & 1 & -1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 1 & 7 \\ 1 & -1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & 7 \\ 3 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = \mathbf{i}(-8) - \mathbf{j}(-2-21) + \mathbf{k}(2-3)$$

$$\mathbf{A}\times\mathbf{B} = -8\mathbf{i} + 23\mathbf{j} - \mathbf{k}$$

c.

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 6 \\ -1 & 2 & 1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 1 & 6 \\ 2 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 0 & 6 \\ -1 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} = -11\mathbf{i} - 6\mathbf{j} + \mathbf{k}$$

d.

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \mathbf{k}$$

e. Given $\mathbf{B}\times\mathbf{A} = \mathbf{i} - \mathbf{j}$, find $\mathbf{A}\times\mathbf{B} = -(\mathbf{B}\times\mathbf{A}) = -\mathbf{i} + \mathbf{j}$

3. Find the area of the triangle with vertices $(1, 1, 2)$, $(1, 5, 5)$ and $(2, 3, 5)$.
Let $\mathbf{A} = (1-1)\mathbf{i} + (5-1)\mathbf{j} + (5-2)\mathbf{k} = 4\mathbf{j} + 3\mathbf{k}$
and $\mathbf{B} = (2-1)\mathbf{i} + (3-1)\mathbf{j} + (5-2)\mathbf{k} = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. The area of the triangle is half the area of the parallelogram

$$\frac{1}{2}||\mathbf{A}\times\mathbf{B}||$$

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 4 & 3 \\ 1 & 2 & 3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 4 & 3 \\ 2 & 3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 0 & 3 \\ 1 & 3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = \mathbf{i}(12-6) - \mathbf{j}(-3) + \mathbf{k}(-4) = 6\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$$

and the area is

$$\frac{1}{2}\sqrt{6^2 + 3^2 + 4^2} = \frac{1}{2}\sqrt{36 + 9 + 16} = \frac{1}{2}\sqrt{61}$$

5. Given the vectors $3\mathbf{i}+\mathbf{j}$ and $2\mathbf{i}-\mathbf{j}-5\mathbf{k}$, the vector perpendicular to both is the cross product

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 0 \\ 2 & -1 & -5 \end{vmatrix} = \mathbf{i}(-5) - \mathbf{j}(-15) + \mathbf{k}(-5) = -5\mathbf{i} + 15\mathbf{j} - 5\mathbf{k}$$

To get a unit vector we divide by the magnitude, which is $\pm\sqrt{(-5)^2+15^2+(-5)^2} = \pm\sqrt{25+225+25} = \pm\sqrt{275} = \pm\sqrt{25(11)} = \pm 5\sqrt{11}$

Therefore the unit vector is

$$\frac{-5\mathbf{i}+15\mathbf{j}-5\mathbf{k}}{\pm 5\sqrt{11}} = \pm\frac{-\mathbf{i}+3\mathbf{j}-\mathbf{k}}{\sqrt{11}}$$

8. Given $\mathbf{A} = 2\mathbf{i}+2\mathbf{j}$, $\mathbf{B} = 3\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $\mathbf{C} = 8\mathbf{i}$, show that

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} \neq \mathbf{A}\times(\mathbf{B}\times\mathbf{C})$$

$$(\mathbf{A}\times\mathbf{B}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 3 & -1 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} - 8\mathbf{k}$$

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & -8 \\ 8 & 0 & 0 \end{vmatrix} = -64\mathbf{j} + 16\mathbf{k}$$

$$(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ 8 & 0 & 0 \end{vmatrix} = 8\mathbf{j} + 8\mathbf{k}$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 0 & 8 & 8 \end{vmatrix} = 16\mathbf{i} - 16\mathbf{j} + 16\mathbf{k}$$

Not equal.

10. Find a unit vector in the plane of $\mathbf{i}+2\mathbf{j}$ and $\mathbf{j}+2\mathbf{k}$ perpendicular to $2\mathbf{i}+\mathbf{j}+2\mathbf{k}$.

We cross the first two vectors to find a vector perpendicular to the plane

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

When we cross this with the third vector we get a vector in the plane

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -2 & 1 \\ 2 & 1 & 2 \end{vmatrix} = -5\mathbf{i} - 6\mathbf{j} + 8\mathbf{k}$$

The unit vector is then

$$\frac{-5\mathbf{i}-6\mathbf{j}+8\mathbf{k}}{\pm\sqrt{(-5)^2+(-6)^2+8^2}} = \frac{-5\mathbf{i}-6\mathbf{j}+8\mathbf{k}}{\pm\sqrt{25+36+64}} = \frac{-5\mathbf{i}-6\mathbf{j}+8\mathbf{k}}{\pm 5\sqrt{5}}$$

13. Find the distance from the point $(5, 7, 14)$ to the line thru $(2, 3, 8)$ and $(3, 6, 12)$, see figure.

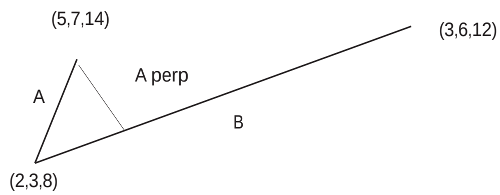

Figure 8: For Problem 13 of section 1.12

$$\mathbf{A} = 3\mathbf{i}+4\mathbf{j}+6\mathbf{k}$$

$$\mathbf{B} = \mathbf{i}+3\mathbf{j}+4\mathbf{k}$$

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 6 \\ 1 & 3 & 4 \end{vmatrix} = -2\mathbf{i}-6\mathbf{j}+5\mathbf{k}$$

The distance is

$$\frac{||\mathbf{A}\times\mathbf{B}||}{||\mathbf{B}||} = \frac{\sqrt{4+36+25}}{\sqrt{1+9+16}} = \sqrt{\frac{65}{26}} = \sqrt{\frac{5}{2}}$$

19. $\mathbf{A}\cdot\mathbf{B} = 0$ and $\mathbf{A}\times\mathbf{B} = 0$ imply

$$||\mathbf{A}||||\mathbf{B}||\cos\theta = 0$$

and

$$||\mathbf{A}||||\mathbf{B}||\sin\theta = 0$$

The only way this is possible is that either vector is the zero vector (since the sine and cosine functions can vanish at the same point).
22. Given that $\mathbf{A}$ and $\mathbf{B}$ are parallel to yz plane, i.e.

$$\mathbf{A} = a_2\mathbf{j}+a_3\mathbf{k}, \qquad \mathbf{B} = b_2\mathbf{j}+b_3\mathbf{k}$$

The norms are given

$$||\mathbf{A}|| = 2, \qquad ||\mathbf{B}|| = 4$$

and

$$\mathbf{A}\cdot\mathbf{B} = 0$$

then the angle θ between them satisfies $\cos\theta = 0$ since they are **not** zero vectors. This implies that $\sin\theta = \pm 1$ therefore $\mathbf{A} \times \mathbf{B} = 2(4)(\pm 1)\mathbf{n} = \pm 8\mathbf{n}$. Therefore $||\mathbf{A} \times \mathbf{B}|| = 8$ and since the vectors are in the yz plane, the normal $\mathbf{n} = \mathbf{i}$, so $\mathbf{A} \times \mathbf{B} = \pm 8\mathbf{i}$

23. a. Do the lines

$$\frac{x}{3} = \frac{y}{2} = \frac{z}{2}$$

$$\frac{x}{5} = \frac{y}{3} = \frac{z-4}{2}$$

intersect? Let's write the lines as

$$x = 3t_1,\ y = 2t_1,\ z = 2t_1$$

$$x = 5t_2,\ y = 3t_2,\ z = 4 + 2t_2$$

Matching the x and y we have $3t_1 = 5t_2$ and $2t_1 = 3t_2$ for which there is **no** solution. Therefore the line do not intersect.

b. The line perpendicular to both, will have a vector in the direction of the normal to the vectors in the direction of those lines, i.e.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 2 \\ 5 & 3 & 2 \end{vmatrix} = -2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$$

Now this line should cross the original two lines, so

$$(3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})t_1 - (4\mathbf{k} + (5\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})t_2) = (-2\mathbf{i} + 4\mathbf{j} - \mathbf{k})t_3$$

or component-wise

$$\begin{aligned} 3t_1 - 5t_2 &= -2t_3 \\ 2t_1 - 3t_2 &= 4t_3 \\ 2t_1 - 4 - 2t_2 &= -t_3 \end{aligned}$$

solve the last equation in the above system for t_3 gives $t_3 = -2t_1 + 2t_2 + 4$, substitute this in the first two equations

$$3t_1 - 5t_2 = -2(-2t_1 + 2t_2 + 4)$$

$$2t_1 - 3t_2 = 4(-2t_1 + 2t_2 + 4)$$

simplify

$$-t_1 - t_2 = -8$$

$$10t_1 - 11t_2 = 16$$

and solve for t_1 and t_2. So $t_1 = \dfrac{104}{21}$ and $t_2 = \dfrac{64}{21}$. We use the point $t_1 = \dfrac{104}{21}$) to get the equation

$$x = -2t + \frac{104}{21}3,\ y = 4t + \frac{104}{21}2,\ z = -t + \frac{104}{21}2$$

Note that when $t = 0$ we get the point on the first line and when $t = \frac{4}{21}$ we get a point on the other line. We can write the equations as

$$x = \frac{104}{7} - 2t, \ y = \frac{208}{21} + 4t, \ z = \frac{208}{21} - t$$

Another way: The normal line is

$$x = x_0 - 2t, \ y = y_0 + 4t, \ z = z_0 - t$$

Now for $t = t_3$ this line intersects the first line and for $t = t_4$ it intersects the second line, we write these 6 equations

$$\begin{aligned}
x_0 - 2t_3 &= 3t_1 \\
y_0 + 4t_3 &= 2t_1 \\
z_0 - t_3 &= 2t_1 \\
x_0 - 2t_4 &= 5t_2 \\
y_0 + 4t_4 &= 3t_2 \\
z_0 - t_4 &= 2t_2 + 4
\end{aligned}$$

Solving the system of 6 equations with 7 unknowns: $x_0, y_0, z_0, t_1, t_2, t_3, t_4$, we find a free parameter t_4 and all the other variables will be

$$t_1 = \frac{104}{21}, \ t_2 = \frac{64}{21}, \ t_3 = \frac{4}{21} + t_4$$

$$x_0 = \frac{320}{21} + 2t_4, \ y_0 = \frac{64}{7} - 4t_4, \ z_0 = \frac{212}{21} + t_4$$

c. To find the distance, we have already the normal to both $-2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and we just need two point, one on each line. We should take $(0, 0, 0)$ on the first and $(0, 0, 4)$ on the second (i.e., the parameters are zero) and create the vector connecting these two points which is $-4\mathbf{k}$. Now we take the dot product of these vectors and divide by the norm, i.e.

$$\text{distance} = \frac{-4\mathbf{k} \cdot (-2\mathbf{i} + 4\mathbf{j} - \mathbf{k})}{\sqrt{(-2)^2 + 4^2 + (-1)^2}} = \frac{4}{\sqrt{21}}$$

1.13

Problems: 1, 3, 5, 7

1. a. Given $\mathbf{A} = 2\mathbf{i}$, $\mathbf{B} = 3\mathbf{j}$, $\mathbf{C} = 5\mathbf{k}$ then

$$[\mathbf{A}, \mathbf{B}, \mathbf{C}] = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{vmatrix} = 2(3)(5)[\mathbf{i}, \mathbf{j}, \mathbf{k}] = 30$$

b.

Using the last column to expand.

$$[\mathbf{A}, \mathbf{B}, \mathbf{C}] = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & -1 & 5 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 0 & -1 \end{vmatrix} + 5 \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} = -3 + 5(1-3) = -3 - 10 = -13$$

c.

$$[2\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{i} + \mathbf{j} + \mathbf{k}, 2\mathbf{i} + 3\mathbf{k}] = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{vmatrix} = 2 \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} + 3 \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 2(-2) + 3(2+1) = -4 + 9 = 5$$

d.

$$[\mathbf{k}, \mathbf{i}, \mathbf{j}] = -[\mathbf{i}, \mathbf{k}, \mathbf{j}] = [\mathbf{i}, \mathbf{j}, \mathbf{k}] = 1$$

3. The volume is the triple $[\vec{AB}, \vec{AC}, \vec{AD}]$ which is

$$\begin{vmatrix} 1 & 0 & 0 \\ -3 & -1 & 3 \\ -3 & -2 & 6 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ -2 & 6 \end{vmatrix} = -6 + 6 = 0$$

5. The vecors $\mathbf{i} + \mathbf{j}$, $3\mathbf{i} + 4\mathbf{j}$, $\mathbf{k}$ give a volume of

$$\begin{vmatrix} 1 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} = 4 - 3 = 1$$

7. Plane parallel to $2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} - 3\mathbf{k}$ and thru $93, 4, -1)$. The normal is the cross product of the given vectors

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 1 & 0 & -3 \end{vmatrix} = \mathbf{i}(-3) - \mathbf{j}(-7) + \mathbf{k}(-1) = -3\mathbf{i} + 7\mathbf{j} - \mathbf{k}$$

and the equation is

$$-3(x-3) + 7(y-4) - (z+1) = 0$$

or

$$3x - 7y + z = -20$$

1.14

Problems: 5–7

5.

$$\mathbf{a} = \omega \times (\omega \times \mathbf{R}) = (\omega \cdot \mathbf{R})\omega - (\omega \cdot \omega)\mathbf{R}$$

6. a. $\mathbf{A} \times \mathbf{B} = \mathbf{B} \times \mathbf{A}$? No
6. b. $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$? No
6. c. $\mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{C}$ if and only if $\mathbf{B} = \mathbf{C}$. Not necessarily
6. d. $\mathbf{A} \times \mathbf{B} = 0$ if and only if one is zero. Not necessarily

7.

$$\underbrace{||\mathbf{A} \times \mathbf{B}||^2}_{(||\mathbf{A}||\,||\mathbf{B}||\,\sin\theta)^2} + \underbrace{(\mathbf{A} \cdot \mathbf{B})^2}_{(||\mathbf{A}||\,||\mathbf{B}||\,\cos\theta)^2} - ||\mathbf{A}||^2||\mathbf{B}||^2 = ||\mathbf{A}||^2||\mathbf{B}||^2 \sin^2\theta + ||\mathbf{A}||^2||\mathbf{B}||^2 \cos^2\theta - ||\mathbf{A}||^2||\mathbf{B}||^2$$

$$= ||\mathbf{A}||^2||\mathbf{B}||^2 \left(\sin^2\theta + \cos^2\theta - 1\right) = 0$$

2.1

Problems: 1, 3, 4, 5hi

1. $\mathbf{F}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$
   a. $\mathbf{F}'(t) = \cos t\mathbf{i} - \sin t\mathbf{j}$
   b. $\mathbf{F}'(t)$ parallel to xy plane since no component in z direction
   c. When is $\mathbf{F}'(t)$ parallel to xz plane? This means $-\sin t = 0$ or $t = \pm n\pi, \quad n = 0, 1, 2, \ldots$
   d. $||\mathbf{F}(t)|| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{1+1} = \sqrt{2}$, so $\mathbf{F}(t)$ has a constant magnitude.
   e. $||\mathbf{F}'(t)|| = \sqrt{\sin^2 t + \cos^2 t} = \sqrt{1}$, so $\mathbf{F}'$ has a constant magnitude.
   f. $\mathbf{F}''(t) = -\sin t\mathbf{i} - \cos t\mathbf{j}$.

3. a. $f(t) = (3t\mathbf{i} + 5t^2\mathbf{j}) \cdot (t\mathbf{i} - \sin t\mathbf{j})$

$$f'(t) = (3\mathbf{i} + 10t\mathbf{j}) \cdot (t\mathbf{i} - \sin t\mathbf{j}) + (3t\mathbf{i} + 5t^2\mathbf{j}) \cdot (\mathbf{i} - \cos t\mathbf{j})$$

$$f'(t) = 3t - 10t\sin t + 3t - 5t^2\cos t = 6t - 10t\sin t - 5t^2\cos t$$

   b. $f(t) = ||2t\mathbf{i} + 2t\mathbf{j} - \mathbf{k}||$, so $f^2(t) = (2t\mathbf{i} + 2t\mathbf{j} - \mathbf{k}) \cdot (2t\mathbf{i} + 2t\mathbf{j} - \mathbf{k})$ Now differentiate

$$2f(t)f'(t) = (2\mathbf{i} + 2\mathbf{j}) \cdot (2t\mathbf{i} + 2t\mathbf{j} - \mathbf{k}) + (2t\mathbf{i} + 2t\mathbf{j} - \mathbf{k}) \cdot (2\mathbf{i} + 2\mathbf{j}) = 2(4t + 4t + 0) = 16t$$

$$f'(t) = \frac{8t}{f(t)}$$

$$f'(t) = \frac{8t}{\sqrt{4t^2 + 4t^2 + 1}} = \frac{8t}{\sqrt{1 + 8t^2}}$$

   c. $f(t) = [(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \times (3t^4\mathbf{i} + t\mathbf{j})] \cdot \mathbf{k}$.

We can do this either by first computing the function $f(t)$ and then differentiating or use the rule to differentiate the product. Here we use the latter. Note that the only vector depending on t is the second one. First compute the cross product of the first vector with the derivative of the second vector

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 12t^3 & 1 & 0 \end{vmatrix} = 2\mathbf{i} - \mathbf{j}(-24t^3) + (1 - 12t^3)\mathbf{k}$$

Since we want to dot this with $\mathbf{k}$, the answer should be $f'(t) = 1 - 12t^3$

4. Prove $\dfrac{d}{dt}\left(\mathbf{R} \times \dfrac{d\mathbf{R}}{dt}\right) = \mathbf{R} \times \dfrac{d^2\mathbf{R}}{dt^2}$

   The left hand side is $\underbrace{\dfrac{d}{dt}\mathbf{R} \times \dfrac{d\mathbf{R}}{dt}}_{=0} + \left(\mathbf{R} \times \dfrac{d^2\mathbf{R}}{dt^2}\right)$ which is the right hand side.

5. h. $\mathbf{A} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$, $\mathbf{B} = 3\mathbf{i} + 4\mathbf{k}$, $\mathbf{C} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ then since $\mathbf{A}$ and $\mathbf{B}$ are constants

$$\frac{d}{dt}(\mathbf{A} + \mathbf{B}t) = \mathbf{B} = 3\mathbf{i} + 4\mathbf{k}$$

5. i.

$$\frac{d}{dt}(\mathbf{B}\times\mathbf{C}t) = \underbrace{\frac{d}{dt}\mathbf{B}}_{=0}\times\mathbf{C}t + \mathbf{B}\times\underbrace{\frac{d}{dt}(\mathbf{C}t)}_{\mathbf{C}} = \mathbf{B}\times\mathbf{C}$$

$$\mathbf{B}\times\mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 4 \\ 2 & -2 & 1 \end{vmatrix} = 8\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$$

2.2

Problems: 1–3, 5, 8, 13

1. Find unit tangent vector to: $x = a\cos t,\ y = b\sin t,\ z = 0$

$$\mathbf{R} = a\cos t\mathbf{i} + b\sin t\mathbf{j}$$

$$\frac{d\mathbf{R}}{dt} = -a\sin t\mathbf{i} + b\cos t\mathbf{j}$$

$$||\frac{d\mathbf{R}}{dt}|| = \sqrt{(-a\sin t)^2 + (b\cos t)^2}$$

$$\mathbf{T} = \frac{\frac{d\mathbf{R}}{dt}}{||\frac{d\mathbf{R}}{dt}||} = \frac{-a\sin t\mathbf{i} + b\cos t\mathbf{j}}{\sqrt{(-a\sin t)^2 + (b\cos t)^2}}$$

Now at $t = 3\pi/2$ we have $\sin(3\pi/2) = -1,\ \cos(3\pi/2) = 0$ and

$$\mathbf{T} = \frac{-a(-1)\mathbf{i} + b(0)\mathbf{j}}{\sqrt{(a)^2}} = \mathbf{i}$$

2. For the curve $x = \sin t - t\cos t,\ y = \cos t + t\sin t,\ z = t^2$,
   a. Find the arc length between $(0, 1, 0)$ and $(-2\pi, 1, 4\pi^2)$. First note that $0 \le t \le 2\pi$.

$$\frac{dx}{dt} = \cos t - (\cos t - t\sin t) = t\sin t$$

$$\frac{dy}{dt} = -\sin t + (\sin t + t\cos t) = t\cos t$$

$$\frac{dz}{dt} = 2t$$

Now

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \underbrace{t^2\sin^2 t + t^2\cos^2 t}_{=t^2} + 4t^2 = 5t^2$$

and the arc length is

$$s = \int_0^{2\pi} \sqrt{5t^2}dt = \sqrt{5}\frac{t^2}{2}\bigg|_0^{2\pi} = \sqrt{5}\frac{4\pi^2}{2} = 2\sqrt{5}\pi^2$$

   b. Find $\mathbf{T}(t)$

$$\mathbf{T}(t) = \frac{\frac{d\mathbf{R}}{dt}}{||\frac{d\mathbf{R}}{dt}||}$$

$$\frac{d\mathbf{R}}{dt} = t\sin t\mathbf{i} + t\cos t\mathbf{j} + 2t\mathbf{k} \qquad \text{from part a.}$$

$$||\frac{d\mathbf{R}}{dt}|| = \sqrt{5}t \qquad \text{from part a.}$$

$$\mathbf{T}(t) = \frac{t\sin t\mathbf{i} + t\cos t\mathbf{j} + 2t\mathbf{k}}{\sqrt{5}t} = \frac{\sin t\mathbf{i} + \cos t\mathbf{j} + 2\mathbf{k}}{\sqrt{5}}$$

b. Find $\mathbf{T}(\pi)$

$$\mathbf{T}(\pi) = \frac{\sin \pi\mathbf{i} + \cos \pi\mathbf{j} + 2\mathbf{k}}{\sqrt{5}} = \frac{\mathbf{j} + 2\mathbf{k}}{\sqrt{5}}$$

3. For the curve $x = \frac{t}{2\pi}$, $y = \sin t$, $z = \cos t$, find the arc length between the points $(0, 0, 1)$ and $(1, 0, 1)$. Clearly $0 \leq t \leq 2\pi$.

$$\frac{d\mathbf{R}}{dt} = \frac{1}{2\pi}\mathbf{i} + \cos t\mathbf{j} - \sin t\mathbf{k}$$

$$||\frac{d\mathbf{R}}{dt}|| = \sqrt{\frac{1}{4\pi^2} + \cos^2 t + \sin^2 t} = \sqrt{\frac{1}{4\pi^2} + 1} = \frac{\sqrt{1+4\pi^2}}{2\pi}$$

$$s = \int_0^{2\pi} \frac{\sqrt{1+4\pi^2}}{2\pi} dt = 2\pi\frac{\sqrt{1+4\pi^2}}{2\pi} = \sqrt{1+4\pi^2}$$

Now to the unit tangent vector

$$\mathbf{T}(t) = \frac{\frac{d\mathbf{R}}{dt}}{||\frac{d\mathbf{R}}{dt}||}$$

$$\mathbf{T}(t) = \frac{\frac{1}{2\pi}\mathbf{i} + \cos t\mathbf{j} - \sin t\mathbf{k}}{\frac{\sqrt{1+4\pi^2}}{2\pi}}$$

At the point $(0, 0, 1)$ we have $t = 0$, so

$$\mathbf{T}(0) = \frac{\frac{1}{2\pi}\mathbf{i} + \mathbf{j}}{\frac{\sqrt{1+4\pi^2}}{2\pi}} = \frac{\mathbf{i} + 2\pi\mathbf{j}}{\sqrt{1+4\pi^2}}$$

5. a. For the curve $x = e^t\cos t$, $y = e^t\sin t$, $z = 0$, find the arc length between $t = 1$ and $t = 1$

$$\frac{dx}{dt} = e^t\cos t - e^t\sin t$$

$$\frac{dy}{dt} = e^t\sin t + e^t\cos t$$

$$\frac{dz}{dt} = 0$$

Now

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \left(e^t\cos t - e^t\sin t\right)^2 + \left(e^t\sin t + e^t\cos t\right)^2$$

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2 = e^{2t}\left(\cos^2 t - 2\sin t\cos t + \sin^2 t\right)+e^{2t}\left(\sin^2 t + 2\sin t\cos t + \cos^2 t\right) = 2e^{2t}$$

and the arc length is

$$s = \int_0^1 e^t\sqrt{2}dt = \sqrt{2}e^t\Big|_0^1 = \sqrt{2}(e-1)$$

b. To reparametrize, we need $s(t) == \int_0^t e^t\sqrt{2}dt = \sqrt{2}e^t\Big|_0^t = \sqrt{2}(e^t-1)$, so $e^t = \frac{s(t)}{\sqrt{2}}+1$ and $t = \ln\left(\frac{s}{\sqrt{2}}+1\right)$ Therefore

$$x(s) = \left(\frac{s}{\sqrt{2}}+1\right)\cos\ln\left(\frac{s}{\sqrt{2}}+1\right)$$

$$y(s) = \left(\frac{s}{\sqrt{2}}+1\right)\sin\ln\left(\frac{s}{\sqrt{2}}+1\right)$$

$$z = 0$$

c. Sketch. The curve starts at $(0,1)$ and moves counter clockwise to $(0,e^{\pi/2})$ and down to $(-e^{\pi},0)$ continue to $(0,-e^{3\pi/2})$ and hits the x axis again at $(e^{2\pi},0)$. Spiraling outward.

8. Show that the curve $x(t) = t\ y(t) = 2t^2,\ z(t) = t^3$ intersects the plane $x+8y+12z = 162$ at right angle. For some t we need $\frac{d\mathbf{R}}{dt}$ to be perpendicular to the plane or parallel to $\mathbf{i}+8\mathbf{j}+12\mathbf{k}$ which is normal to the plane. Now

$$\frac{d\mathbf{R}}{dt} = \mathbf{i}+4t\mathbf{j}+3t^2\mathbf{k}$$

For $t=2$ we get

$$\frac{d\mathbf{R}}{dt} = \mathbf{i}+8\mathbf{j}+12\mathbf{k}$$

The question is if the curve and the plane intersect at that point $x(t=2) = 2$, $y(t=2) = 8$, $z(t=2) = 8$ and for these values the equation of the plane: $2+8(8)+12(8) = 2+64+96 = 162$ Therefore $(2,8,8)$ is the point of intersection and at that point the vectors are parallel.

13. No. For example $x^2+y^2 = C,\ \ x \geq 0,\ \ C > 0$ which is the right half circle. In this case $2x+2y\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{x}{y}$ which does NOT exist for $y = 0$.

2.3

Problems: 2, 5–7, 9, 15, 17

2. Given $x = 3t\cos t,\ y = 3t\sin t,\ z = 4t$

   a. To find the speed, we need the first time derivatives of x, y, z,

$$\dot{x} = 3\cos t - 3t\sin t,\ \dot{y} = 3\sin t + 3t\cos t,\ \dot{z} = 4$$

$$||\mathbf{v}|| = \sqrt{(3\cos t - 3t\sin t)^2 + (3\sin t + 3t\cos t)^2 + 4^2}$$

$$||\mathbf{v}|| = \sqrt{9\cos^2 t - 18t\sin t\cos t + 9t^2\sin^2 t + 9\sin^2 t + 18t\cos t\sin t + 9t^2\cos^2 t + 16}$$

$$||\mathbf{v}|| = \sqrt{9(\cos^2 t + \sin^2 t) + 9t^2(\sin^2 t + \cos^2 t) + 16}$$

$$||\mathbf{v}|| = \sqrt{9 + 9t^2 + 16} = \sqrt{25 + 9t^2}$$

$$\frac{ds}{dt} = \sqrt{25 + 9t^2}$$

   b. To find the acceleration we need the second derivative

$$\ddot{x} = -3\sin t - 3\sin t - 3t\cos t,\ \ddot{y} = 3\cos t + 3\cos t - 3t\sin t,\ \ddot{z} = 0$$

$$\mathbf{a} = (-6\sin t - 3t\cos t)\mathbf{i} + (6\cos t - 3t\sin t)\mathbf{j}$$

$$||\mathbf{a}|| = \sqrt{(-6\sin t - 3t\cos t)^2 + (6\cos t - 3t\sin t)^2}$$

$$||\mathbf{a}|| = \sqrt{36\sin^2 t + 36t\sin t\cos t + 9t^2\cos^2 t + 36\cos^2 t - 36t\sin t\cos t + 9t^2\sin^2 t}$$

$$||\mathbf{a}|| = \sqrt{36 + 9t^2}$$

$$\mathbf{a}_T = \frac{d^2 s}{dt^2} = \frac{d}{dt}\frac{ds}{dt} = \frac{d}{dt}\sqrt{25 + 9t^2} = \frac{18t}{2\sqrt{25 + 9t^2}} = \frac{9t}{\sqrt{25 + 9t^2}}$$

$$\mathbf{a}_T^2 + \mathbf{a}_N^2 = ||\mathbf{a}||^2 = 36 + 9t^2$$

Therefore

$$\mathbf{a}_N^2 = 36 + 9t^2 - \frac{81t^2}{25 + 9t^2}$$

$$\mathbf{a}_N = \sqrt{36 + 9t^2 - \frac{81t^2}{25 + 9t^2}}$$

   c. $\mathbf{T} = \dfrac{\mathbf{v}}{||\mathbf{v}||} = \dfrac{(3\cos t - 3t\sin t)\mathbf{i} + (3\sin t + 3t\cos t)\mathbf{j} + 4\mathbf{k}}{\sqrt{25 + 9t^2}}$

   d. Find the curvature $\kappa ||\mathbf{v}||^2 = \mathbf{a}_N$,

$$\kappa = \frac{\mathbf{a}_N}{||\mathbf{v}||^2} = \frac{\sqrt{36 + 9t^2 - \frac{81t^2}{25+9t^2}}}{9t^2 + 25}$$

5. $\mathbf{R} = (\cos t + \sin t)\mathbf{i} + (\sin t - \cos t)\mathbf{j} + \frac{1}{2}t\mathbf{k}$

a. $\dot{\mathbf{R}} = \mathbf{v} = (-\sin t + \cos t)\mathbf{i} + (\cos t + \sin t)\mathbf{j} + \dfrac{1}{2}\mathbf{k}$

$$||\mathbf{v}||^2 = (-\sin t + \cos t)^2 + (\cos t + \sin t)^2 + \frac{1}{4} = 1 - 2\sin t\cos t + 1 + 2\sin t\cos t + \frac{1}{4} = \frac{9}{4}$$

$$||\mathbf{v}|| = \frac{3}{2} = \frac{ds}{dt}$$

b.

$$\mathbf{a} = -(\cos t + \sin t)\mathbf{i} + (-\sin t + \cos t)\mathbf{j} +$$

c.

$$\mathbf{T} = \frac{2}{3}(-\sin t + \cos t)\mathbf{i} + \frac{2}{3}(\cos t + \sin t)\mathbf{j} + \frac{1}{3}\mathbf{k}$$

d.

$$\mathbf{a}_T = 0, \qquad \text{since } \tfrac{ds}{dt} \text{ is constant}$$

$$||\mathbf{a}||^2 = 1 + 2\sin t\cos t + 1 - 2\sin t\cos t = 2$$

Therefore

$$\mathbf{a}_N^2 = 2 - 0 = 2$$

$$\kappa = \frac{\mathbf{a}_N}{||\mathbf{v}||^2} = \frac{\sqrt{2}}{\frac{9}{4}} = \frac{4\sqrt{2}}{9}$$

Note that the curvature is constant.

e. Compare to (2.15)

$$\mathbf{e}_1 = \mathbf{i} - \mathbf{j}$$

$$\mathbf{e}_2 = \mathbf{i} + \mathbf{j}$$

$$\rho = 1$$

$$a = \frac{1}{2}$$

6. $x(t) = 3t^2 - t^3,\ y(t) = 3t^2,\ z(t) = 3t + t^3$

Then

$$\dot{x}(t) = 6t - 3t^2,\ \dot{y}(t) = 6t,\ \dot{z}(t) = 3 + 3t^2$$

$$\ddot{x}(t) = 6 - 6t,\ \ddot{y}(t) = 6,\ \ddot{z}(t) = 6t$$

The curvature is

$$\kappa = \frac{||\mathbf{R}' \times \mathbf{R}''||}{||\mathbf{R}'||^3}$$

Now

$$\mathbf{R}' \times \mathbf{R}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6t - 3t^2 & 6t & 3 + 3t^2 \\ 6 - 6t & 6 & 6t \end{vmatrix}$$

$$\mathbf{R}'\times\mathbf{R}'' = (36t^2-18-18t^2)\mathbf{i}-(36t^2-18t^3-18+18t-18t^2+18t^3)\mathbf{j}+(36t-18t^2-36t+36t^2)\mathbf{k}$$

$$\mathbf{R}'\times\mathbf{R}'' = 18(t^2-1)\mathbf{i}-18(t^2+t-1)\mathbf{j}+18t^2\mathbf{k}$$

$$||\mathbf{R}'\times\mathbf{R}''|| = \sqrt{18^2(t^2-1)^2+18^2(t^2+t-1)^2+18^2t^4}$$

$$||\mathbf{R}'\times\mathbf{R}''|| = 18\sqrt{t^4-2t^2+1+t^4+2t^3-2t^2+t^2-2t+1+t^4}$$

$$||\mathbf{R}'\times\mathbf{R}''|| = 18\sqrt{3t^4+2t^3-3t^2-2t+2}$$

$$||\mathbf{R}'||^2 = 36t^2-36t^3+9t^4+36t^2+9+18t^2+9t^4 = 18t^4-36t^3+90t^2+9$$

$$||\mathbf{R}'||^2 = 9(2t^4-4t^3+10t^2+1)$$

Therefore

$$\kappa = \frac{18\sqrt{3t^4+2t^3-3t^2-2t+2}}{(9(2t^4-4t^3+10t^2+1))^{3/2}} = \frac{2\sqrt{3t^4+2t^3-3t^2-2t+2}}{3(2t^4-4t^3+10t^2+1)^{3/2}}$$

7. $\mathbf{R}(t) = \sin t\mathbf{i}+\cos t\mathbf{j}+\log(\sec t)\mathbf{k}$

   a. Find ds

$$ds^2 = dx^2+dy^2+dz^2 = (\cos t\,dt)^2+(-\sin t\,dt)^2+\left(\frac{1}{\sec t}\tan t\sec t\,dt\right)^2$$

$$ds = \sqrt{\underbrace{\underbrace{\cos^2 t+\sin^2 t}_{=1}+\tan^2 t}_{\sec^2 t}}dt = \sec t\,dt$$

   b. $\mathbf{T} = \dfrac{\mathbf{v}}{||\mathbf{v}||} = \dfrac{\cos t\mathbf{i}-\sin t\mathbf{j}+\tan t\mathbf{k}}{\sec t}$

   c. $\mathbf{N} = \dfrac{\frac{d\mathbf{T}}{dt}}{||\frac{d\mathbf{T}}{dt}||}$

   Now

$$\frac{d\mathbf{T}}{dt} = \left(\frac{\cos t}{\sec t}\right)'\mathbf{i}-\left(\frac{-\sin t}{\sec t}\right)'\mathbf{j}+\left(\frac{\tan t}{\sec t}\right)'\mathbf{k}$$

   Note that $\dfrac{\tan t}{\sec t} = \sin t$, so

$$\frac{d\mathbf{T}}{dt} = -\underbrace{2\cos t\sin t}_{=\sin(2t)}\mathbf{i}-\underbrace{\left(\cos^2 t-\sin^2 t\right)}_{=\cos(2t)}\mathbf{j}+\cos t\mathbf{k}$$

$$\mathbf{N} = \frac{\frac{d\mathbf{T}}{dt}}{||\frac{d\mathbf{T}}{dt}||} = \frac{-\sin(2t)\mathbf{i}-\cos(2t)\mathbf{j}+\cos t\mathbf{k}}{\sqrt{\sin^2(2t)+\cos^2(2t)+\cos^2 t}}$$

$$\mathbf{N} = -\frac{\sin(2t)\mathbf{i}+\cos(2t)\mathbf{j}-\cos t\mathbf{k}}{\sqrt{1+\cos^2 t}}$$

d. For the curvature use $\kappa = \dfrac{||\mathbf{R}' \times \mathbf{R}''||}{||\mathbf{R}'||^3}$ or use the acceleration $\kappa = \dfrac{\mathbf{a}_N}{||\mathbf{v}||^2}$

$$\mathbf{R}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + \log(\sec t)\mathbf{k}$$

$$\mathbf{R}'(t) = \mathbf{v} = \cos t\mathbf{i} - \sin t\mathbf{j} + \tan t\mathbf{k}$$

$$\mathbf{R}''(t) = \mathbf{a} = -\sin t\mathbf{i} - \cos t\mathbf{j} + \sec^2 t\mathbf{k}$$

$$||\mathbf{v}|| = \frac{ds}{dt} = \sec t, \qquad \text{see part a}$$

$$\mathbf{a}_T = \frac{d^2s}{dt^2} = \tan t \sec t$$

$$\mathbf{a}_N^2 = ||\mathbf{a}||^2 - \mathbf{a}_T^2 = (-\sin t)^2 + (-\cos t)^2 + (\sec^2 t)^2 - (\tan t \sec t)^2$$

$$\mathbf{a}_N^2 = \underbrace{\sin^2 t + \cos^2 t}_{=1} + \underbrace{\sec^4 t - \tan^2 t \sec^2 t}_{=\sec^2 t(\sec^2 t - \tan^2 t)}$$

$$\mathbf{a}_N^2 = 1 + \sec^2 t \underbrace{(\sec^2 t - \tan^2 t)}_{=1} = 1 + \sec^2 t$$

$$\kappa = \frac{\sqrt{1 + \sec^2 t}}{\sec^2 t}$$

9. $\mathbf{R}(t) = \log(t^2+1)\mathbf{i} + (t - 2\tan^{-1} t)\mathbf{j} + 2\sqrt{2}t\mathbf{k}$

a.

$$\mathbf{R}'(t) = \mathbf{v} = \frac{1}{t^2+1} 2t\mathbf{i} + \underbrace{\left(1 - \frac{2}{1+t^2}\right)}_{=\frac{1+t^2-2}{1+t^2}} \mathbf{j} + 2\sqrt{2}\mathbf{k}$$

$$||\mathbf{v}||^2 = \frac{4t^2}{(1+t^2)^2} + \frac{(t^2-1)^2}{(1+t^2)^2} + (2\sqrt{2})^2$$

$$||\mathbf{v}||^2 = \frac{4t^2 + t^4 - 2t^2 + 1}{(1+t^2)^2} + 8 = \frac{t^4 + 2t^2 + 1}{(1+t^2)^2} + 8 = \frac{(1+t^2)^2}{(1+t^2)^2} + 8 = 1 + 8 = 9$$

$$||\mathbf{v}|| = \frac{ds}{dt} = 3, \qquad \text{constant}$$

$$\mathbf{a}_T = \frac{d^2s}{dt^2} = 0, \qquad \text{since } \frac{ds}{dt} \text{ is constant}$$

b.

$$\kappa = \frac{\mathbf{a}_N}{||\mathbf{v}||^2} = \frac{||\mathbf{a}||}{||\mathbf{v}||^2}$$

Now

$$\mathbf{a} = \left(\frac{2t}{1+t^2}\right)' \mathbf{i} + \left(\frac{t^2-1}{1+t^2}\right)' \mathbf{j}$$

$$\mathbf{a} = \frac{2(t^2+1) - 2t(2t)}{(1+t^2)^2}\mathbf{i} + \frac{2t(t^2+1) - (t^2-1)2t}{(1+t^2)^2}\mathbf{j}$$

$$\mathbf{a} = \frac{(2t^2 + 2 - 4t^2)\mathbf{i} + (2t^3 + 2t - 2t^3 + 2t)\mathbf{j}}{(1+t^2)^2}$$

$$\mathbf{a} = \frac{2(1-t^2)\mathbf{i} + 4t\mathbf{j}}{(1+t^2)^2}$$

$$||\mathbf{a}||^2 = \frac{4(1-t^2)^2 + (4t)^2}{(1+t^2)^4} = \frac{4 + 8t^2 + 4t^4}{(1+t^2)^4} = \frac{4(1+t^2)^2}{(1+t^2)^4} = \frac{4}{(1+t^2)^2}$$

or

$$||\mathbf{a}|| = \frac{2}{(1+t^2)}$$

Therefore

$$\kappa = \frac{2}{(1+t^2)}\,\frac{1}{3^2} = \frac{2}{9(1+t^2)}$$

15. a. $\underbrace{\dfrac{d\mathbf{R}}{ds}}_{=\mathbf{T}} \cdot \mathbf{T} = \mathbf{T} \cdot \mathbf{T} = ||\mathbf{T}||^2 = 1,$ unit vector

b. $\dfrac{d}{ds}(\mathbf{T} \cdot \mathbf{T}) = 0,$ since $\dfrac{d\mathbf{T}}{ds}$ is parallel to $\mathbf{N}$ which is perpendicular to $\mathbf{T}$

c. $\dfrac{d^2\mathbf{R}}{dt^2} \cdot \mathbf{T} = \mathbf{a} \cdot \mathbf{T} = \mathbf{a}_T,$

d. $\mathbf{T} \cdot \mathbf{N} = 0$

e. $\dfrac{d\mathbf{R}}{dt} \cdot \mathbf{T} = \left(\underbrace{\dfrac{d\mathbf{R}}{ds}}_{=\mathbf{T}} \underbrace{\dfrac{ds}{dt}}_{=||\mathbf{v}||} \right) \cdot \mathbf{T} = ||\mathbf{v}||\, \underbrace{||\mathbf{T}||^2}_{=1} = ||\mathbf{v}||,$

f. $\dfrac{d\mathbf{N}}{ds} \cdot \mathbf{B} = \tau$

g. $[\mathbf{T}, \mathbf{N}, \mathbf{B}] = 1$ right handed system

h. $\left|\left|\dfrac{d^2\mathbf{R}}{ds^2}\right|\right| = \left|\left|\dfrac{d}{ds} \underbrace{\dfrac{d\mathbf{R}}{ds}}_{=\mathbf{T}}\right|\right| = \left|\left|\dfrac{d\mathbf{T}}{ds}\right|\right| = \kappa||\mathbf{N}|| = \kappa,$

i. $\dfrac{d\mathbf{B}}{ds} = -\tau\mathbf{N},$ Frenet formula

17. $x(t) = \cos^3 t,\ y(t) = \sin^3 t,\ z(t) = 2\sin^2 t, \quad 0 \le t \le 2\pi$

$$\frac{d\mathbf{R}}{dt} = -3\cos^2 t \sin t\mathbf{i} + 3\sin^2 t \cos t\mathbf{j} + 4\sin t \cos t\mathbf{k}$$

$$||\frac{d\mathbf{R}}{dt}|| = \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2 + (4\sin t \cos t)^2}$$

$$||\frac{d\mathbf{R}}{dt}|| = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t + 16\sin^2 t \cos^2 t}$$

$$||\frac{d\mathbf{R}}{dt}|| = \sqrt{\cos^2 t \sin^2 t(9\cos^2 t + 9\sin^2 t + 16)} = \sqrt{\cos^2 t \sin^2 t(9 + 16)} = 5\sin t \cos t$$

$$\mathbf{T} = \frac{\frac{d\mathbf{R}}{dt}}{\frac{ds}{dt}} = -\frac{3}{5}\cos t\mathbf{i} + \frac{3}{5}\sin t\mathbf{j} + \frac{4}{5}\mathbf{k}$$

$$\mathbf{N} = \frac{\frac{d\mathbf{T}}{dt}}{||\frac{d\mathbf{T}}{dt}||} = \frac{\frac{3}{5}\sin t\mathbf{i} + \frac{3}{5}\cos t\mathbf{j}}{\sqrt{\frac{9}{25}\sin^2 t + \frac{9}{25}\cos^2 t}} = \frac{\frac{3}{5}\sin t\mathbf{i} + \frac{3}{5}\cos t\mathbf{j}}{\sqrt{\frac{9}{25}}} = \frac{5}{3}\left(\frac{3}{5}\sin t\mathbf{i} + \frac{3}{5}\cos t\mathbf{j}\right)$$

$$\mathbf{N} = \sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{3}{5}\cos t & \frac{3}{5}\sin t\mathbf{j} & \frac{4}{5} \\ \\ \sin t & \cos t & 0 \end{vmatrix}$$

$$\mathbf{B} = \mathbf{i}\left(-\frac{4}{5}\cos t\right) - \mathbf{j}\left(-\frac{4}{5}\sin t\right) + \mathbf{k}\left(-\frac{3}{5}\cos^2 t - \frac{3}{5}\sin^2 t\right)$$

$$\mathbf{B} = -\frac{4}{5}\cos t\mathbf{i} + \frac{4}{5}\sin t\mathbf{j} - \frac{3}{5}\mathbf{k}$$

$$\kappa = ||\frac{d\mathbf{T}}{ds}|| = ||\frac{d\mathbf{T}}{dt}||\,\frac{dt}{ds} = \frac{\frac{3}{5}\ \text{ from computing } \mathbf{N}}{5\sin t\cos t\ \text{ from computing } \mathbf{T}} = \frac{3}{25\sin t\cos t}$$

$$\tau = \pm||\frac{d\mathbf{B}}{ds}|| = \pm\frac{||\frac{d\mathbf{B}}{dt}||}{\frac{ds}{dt}}$$

$$\frac{d\mathbf{B}}{dt} = \frac{4}{5}\sin t\mathbf{i} + \frac{4}{5}\cos t\mathbf{j}$$

Substituting

$$\tau = \pm\frac{\sqrt{(\frac{4}{5}\sin t)^2 + (\frac{4}{5}\cos t)^2}}{5\sin t\cos t} = \pm\frac{\frac{4}{5}}{5\sin t\cos t} = \pm\frac{4}{25\sin t\cos t}$$

Compare with (2.45) to find that the sign is negative.

2.4

Problems: 1, 3, 4, 6, 10

1. $r = b(1-\cos\theta)$ implies $\dfrac{dr}{d\theta} = b\sin\theta$

   $\dfrac{d\theta}{dt} = 4$ implies $\dfrac{d^2\theta}{dt^2} = 0$

   Now $\mathbf{v} = \underbrace{\dfrac{dr}{dt}}_{\frac{dr}{d\theta}\frac{d\theta}{dt}}\mathbf{u}_r + r\underbrace{\dfrac{d\theta}{dt}}_{=4}\mathbf{u}_\theta = 4b\sin\theta\mathbf{u}_r + 4b(1-\cos\theta)\mathbf{u}_\theta$

   $\mathbf{a} = \left[\dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2\right]\mathbf{u}_r + \left[r\underbrace{\dfrac{d^2\theta}{dt^2}}_{=0} + 2\dfrac{dr}{dt}\dfrac{d\theta}{dt}\right]\mathbf{u}_\theta$

   $\dfrac{d^2r}{dt^2} = \dfrac{d}{dt}(4b\sin\theta) = 4b\cos\theta\underbrace{\dfrac{d\theta}{dt}}_{=4} = 16b\cos\theta$

   Substituting

$$\mathbf{a} = [16b\cos\theta - 16b(1-\cos\theta)]\,\mathbf{u}_r + 32b\sin\theta\mathbf{u}_\theta$$

or

$$\mathbf{a} = 16b(2\cos\theta - 1)\mathbf{u}_r + 32b\sin\theta\mathbf{u}_\theta$$

3. $r = 2(1+\sin\theta)$ implies $\dfrac{dr}{d\theta} = 2\cos\theta$ and $\dfrac{dr}{dt} = 2\cos\theta\dfrac{d\theta}{dt} = -2e^{-t}\cos\theta$

   $\mathbf{v} = \dfrac{dr}{dt}\mathbf{u}_r + r\dfrac{d\theta}{dt}\mathbf{u}_\theta = -2e^{-t}\cos\theta\mathbf{u}_r - 2(1+\sin\theta)e^{-t}\mathbf{u}_\theta$

4. $r = 2+\sin t$ implies $\dfrac{dr}{dt} = \cos t$

   $\mathbf{v} = \dfrac{dr}{dt}\mathbf{u}_r + r\dfrac{d\theta}{dt}\mathbf{u}_\theta = \cos t\mathbf{u}_r + (2+\sin t)\dfrac{d\theta}{dt}\mathbf{u}_\theta$

   To find $\dfrac{d\theta}{dt}$ we use the given speed $\nu = \sqrt{2}\cos t$

$$||\mathbf{v}||^2 = \cos^2 t + (2+\sin t)^2\left(\frac{d\theta}{dt}\right)^2 = 2\cos^2 t$$

Therefore

$$(2+\sin t)^2\left(\frac{d\theta}{dt}\right)^2 = \cos^2 t$$

$$\frac{d\theta}{dt} = \frac{\cos t}{2+\sin t}$$

$$\int d\theta = \int \frac{\cos t}{2+\sin t}\,dt$$

For this integral we use a substitution $\tau = 2+\sin t$, then $d\tau = \cos t\, dt$ and we have

$$\theta = \int \frac{1}{\tau}\, d\tau = \ln(\tau) + C = \ln|2+\sin t| + C$$

But $\theta = 0$ at $t = 0$, so $0 = \ln|2+\sin 0| + C$ and $C = -\ln 2$

Therefore $\theta = \ln|2+\sin t| - \ln 2 = \ln\dfrac{2+\sin t}{2} = \ln(1+\dfrac{1}{2}\sin t)$

At $t = \dfrac{\pi}{2}$ then $\theta = \ln\left(1 + \frac{1}{2}\underbrace{\sin(\frac{\pi}{2})}_{=1}\right) = \ln(\frac{3}{2})$ and $r = 2 + \underbrace{\sin(\frac{\pi}{2})}_{=1} = 3$, so the position at $t = \pi/2$ is $(r,\theta) = (3, \ln(3/2))$

6.

$$\mathbf{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_r + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\mathbf{u}_\theta$$

a. Moving around circle at $(0,0)$ with constant nonzero angular velocity $\dfrac{d\theta}{dt} = c$ so $\dfrac{d^2\theta}{dt^2} = 0$. Since r is constant, we have $\dfrac{dr}{dt} = 0$ and therefore

$$\mathbf{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_r + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\mathbf{u}_\theta = -r\left(\frac{d\theta}{dt}\right)^2\mathbf{u}_r$$

b. Moving around circle at $(0,0)$ with constant nonzero angular acceleration $\dfrac{d^2\theta}{dt^2} = c$ and as in part a $\dfrac{dr}{dt} = 0$ so

$$\mathbf{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_r + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\mathbf{u}_\theta = -r\left(\frac{d\theta}{dt}\right)^2\mathbf{u}_r + r\frac{d^2\theta}{dt^2}\mathbf{u}_\theta$$

c. Moving along a straight line with constant speed, therefore $\dfrac{dr}{dt} \neq 0$ and $\dfrac{d\theta}{dt} \neq 0$. Therefore all terms are non zero.

d. Walking from the center of a merry-go-round toward its outer edge. The acceleration depends on the details (see example 2.19)

10. Constant radial speed $\dfrac{dr}{dt} = 2\dfrac{cm}{sec}$

Platform rotates with uniform angular velocity of $30\dfrac{rev}{min} = 30\dfrac{rev}{60\, sec} = \dfrac{1}{2}\dfrac{rev}{sec} = \pi\dfrac{rad}{sec}$

a. radial acceleration $\mathbf{a}_r = \dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2 = -\pi^2 r$ (the sign means toward the center)

b. Coriolis acceleration $= 2\underbrace{\dfrac{dr}{dt}}_{=2}\underbrace{\dfrac{d\theta}{dt}}_{=\pi} = 4\pi\frac{cm}{sec^2}$

3.1

Problems: 1, 6, 7, 8, 16, 17, 21, 24, 31

1. a. $f = \sin x + e^{xy} + z$

$$\frac{\partial f}{\partial x} = \cos x + ye^{xy} + 0$$
$$\frac{\partial f}{\partial y} = 0 + xe^{xy} + 0$$
$$\frac{\partial f}{\partial z} = 0 + 0 + 1$$
$$\nabla f = (\cos x + ye^{xy})\mathbf{i} + xe^{xy}\mathbf{j} + \mathbf{k}$$

b. $f = \frac{1}{||\mathbf{R}||} = \frac{1}{\sqrt{x^2+y^2+z^2}}$

$$\frac{\partial f}{\partial x} = \frac{-\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\,(2x)}{x^2+y^2+z^2} = -\frac{x}{(x^2+y^2+z^2)^{3/2}}$$

By symmetry

$$\frac{\partial f}{\partial y} = -\frac{y}{(x^2+y^2+z^2)^{3/2}}$$
$$\frac{\partial f}{\partial z} = -\frac{z}{(x^2+y^2+z^2)^{3/2}}$$
$$\nabla f = -\frac{x}{(x^2+y^2+z^2)^{3/2}}\mathbf{i} - \frac{y}{(x^2+y^2+z^2)^{3/2}}\mathbf{j} - \frac{z}{(x^2+y^2+z^2)^{3/2}}\mathbf{k} = -\frac{\mathbf{R}}{||\mathbf{R}||^3}$$

c. $f = \mathbf{R} \cdot \underbrace{\mathbf{i} \times \mathbf{j}}_{=\mathbf{k}}$ therefore $f = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{k} = z$

The gradient is $\nabla f = \mathbf{k}$

6. Given $f(x, y, z)$ is the distance from the z axis then the gradient of f is a unit vector directed from the z axis except at points **on** the z axis where it is undefined.

Hint: Look at $f(x, y, z) = \sqrt{x^2 + y^2}$

7. a. $f = x^2 + y^2 + z^2$ then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ and at the point $(3, 0, 4)$ we have

$$\nabla f = 6\mathbf{i} + 8\mathbf{k}$$

The maximum value is $||\nabla f|| = \sqrt{36 + 64} = 10$

8. $z = h\,e^{-(x^2+2y^2)}$, so $f = z - h\,e^{-(x^2+2y^2)}$

a. $\nabla f = 2xh\,e^{-(x^2+2y^2)}\mathbf{i} + 4yh\,e^{-(x^2+2y^2)}\mathbf{j} + \mathbf{k}$ at $(1, 2, he^{-9})$ lava flows steepest descent is in the direction of the gradient and $\nabla f|_{(1,2,he^{-9})} = 2he^{-9}\mathbf{i} + 8he^{-9}\mathbf{j} + \mathbf{k}$ and the direction is

$$\frac{\nabla f}{||\nabla f||} = \frac{2he^{-9}\mathbf{i} + 8he^{-9}\mathbf{j} + \mathbf{k}}{\sqrt{4h^2e^{-18} + 64h^2e^{-18} + 1}}$$

The projection on the xy plane is $\nabla f = 2he^{-9}\mathbf{i} + 8he^{-9}\mathbf{j}$

b. The curve $\dfrac{x-1}{2h\,e^{-9}} = \dfrac{y-2}{8h\,e^{-9}} = \dfrac{z-he^{-9}}{1}$ in the direction of ∇f.

The projection on the xy plane is $\dfrac{x-1}{2h\,e^{-9}} = \dfrac{y-2}{8h\,e^{-9}}$ or $4(x-1) = y-2$ or $y = 4x-2$

16. $x^2 + z^2 = 8$ at $(2, 0, 2)$

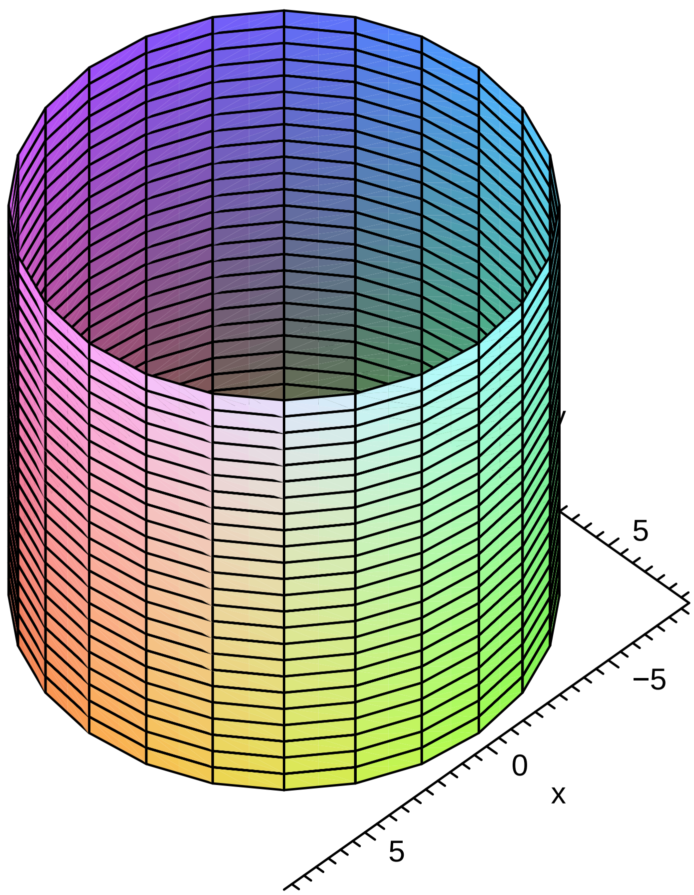

Figure 9: For Problem 16 of section 3.1. y is the vertical axis

a. This is a cylinder whose cross section is a circle in the xz plane. A vector normal to the cylinder is always perpendicular to the y axis.

b. $f = x^2 + z^2$ therefore $\nabla f = 2x\mathbf{i} + 2z\mathbf{k}$ at the given point we get $\nabla f = 4\mathbf{i} + 4\mathbf{k}$. This has no component in the y direction and therefore it is always perpendicular to y.

17. Plane tangent to $f = z^2 - xy - 14$ at $(2, 1, 4)$

$$\nabla f|_{(2,1,4)} = (-y\mathbf{i} - x\mathbf{j} + 2z\mathbf{k})|_{(2,1,4)} = -\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}$$

This vector is normal to the surface and also normal to the tangent plane. Therefore the equation of the plane is

$$-(x-2) - 2(y-1) + 8(z-4) = 0$$

or

$$-x - 2y + 8z = 28$$

21. Let $T(x, y, z) = x^2 + 2y^2 + 3z^2$ and let S be the isotimic surface $T = 1$, i.e. $S: \, x^2 + 2y^2 + 3z^2 = 1$

The normal to the surface is $2x\mathbf{i} + 4y\mathbf{j} + 6z\mathbf{k}$

At the point (x, y, z) we want the normal $\mathbf{i} + \mathbf{j} + \mathbf{k}$, therefore $4y = 2x, \; 6z = 2x$ and so $x = 2y = 3z$.

Now plug this in S: $x^2 + 2\left(\frac{x}{2}\right)^2 + 3\left(\frac{x}{3}\right)^2 = 1$ and we have $x^2 = \dfrac{6}{11}$ or $x = \pm\sqrt{\dfrac{6}{11}}$. Now we compute y and z, $y = \dfrac{1}{2}x = \pm\dfrac{1}{2}\sqrt{\dfrac{6}{11}}$ and $z = \dfrac{x}{3} = \pm\dfrac{1}{3}\sqrt{\dfrac{6}{11}}$

24.

$$\nabla f|_{(1/2,\,3/2,\,3\sqrt{6}/2)} = (2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})|_{(1/2,\,3/2,\,3\sqrt{6}/2)} = \mathbf{i} + 3\mathbf{j} + 3\sqrt{6}\mathbf{k}$$

$$\nabla g|_{(1/2,\,3/2,\,3\sqrt{6}/2)} = (2(x-1)\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})|_{(1/2,\,3/2,\,3\sqrt{6}/2)} = -\mathbf{i} + 3\mathbf{j} + 3\sqrt{6}\mathbf{k}$$

$$\cos\theta = \frac{\nabla f \cdot \nabla g}{||\nabla f||\,||\nabla g||} = \frac{-1 + 9 + 9(6)}{\sqrt{1 + 9 + 9(6)}\sqrt{1 + 9 + 9(6)}} = \frac{62}{\sqrt{64}\sqrt{64}} = \frac{31}{32}$$

$$\theta = \cos^{-1}\frac{31}{32}$$

31. $x^2 + y^2 + z^2 = 84$ closest to $x + 2y + 4z = 77$

First find the tangent plane and a point (x_0, y_0, z_0) so that the normal is parallel to the given plane normal, i.e. to $\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$.

The gradient is $\nabla f|_{(x_0,y_0,z_0)} = 2x_0 + 2y_0 + 2z_0 = \alpha(\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$

Now we find $x_0 = \dfrac{1}{2}\alpha, \; y_0 = \alpha, \; z_0 = 2\alpha$ and since this point is on the sphere we have

$$\left(\frac{1}{2}\alpha\right)^2 + \alpha^2 + (2\alpha)^2 = 84$$

or

$$(\frac{1}{4} + 1 + 4)\alpha^2 = 84$$

$$\alpha^2 = \frac{84}{\frac{21}{4}} = 16$$

$$\alpha = \pm 4$$

and the points are $(2, 4, 8)$ and $(-2, -4, -8)$. Note that the second point is the farthest away from the plane on the other side of the sphere. If you need the distance from $(2, 4, 8)$ to the plane we use

$$\text{distance} = \frac{|2 + 2(4) + 4(8) - 77|}{\sqrt{1 + 4 + 16}} = \frac{|2 + 8 + 32 - 77|}{\sqrt{21}} = \frac{35}{\sqrt{21}}$$

$$\text{the farthest point} = \frac{|-2+2(-4)+4(-8)-77|}{\sqrt{1+4+16}} = \frac{|-2-8-32-77|}{\sqrt{21}} = \frac{119}{\sqrt{21}}$$

$$\text{The difference between the two} = \frac{119-35}{\sqrt{21}} = \frac{84}{\sqrt{21}} = \text{the diameter}$$

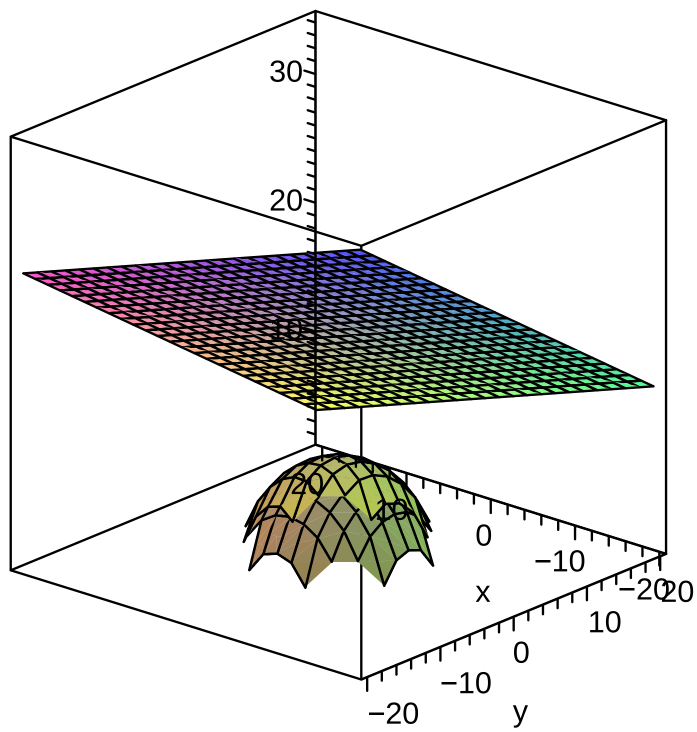

Figure 10: For Problem 31 of section 3.1

3.2

Problems: 1–4

1. $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$

   The equations of the curves are: $\dfrac{dx}{-y} = \dfrac{dy}{x}$

   The solution is $x\,dx = -y\,dy$, integration yields: $\dfrac{1}{2}x^2 = -\dfrac{1}{2}y^2 + c$ or

$$x^2 + y^2 = k, \qquad \text{these are circles centered at the origin for } k > 0$$

   We can create the vectors

$$\begin{array}{lcr}
\mathbf{F}(1,0) & = & \mathbf{j} \\
\mathbf{F}(0,1) & = & -\mathbf{i} \\
\mathbf{F}(-1,0) & = & -\mathbf{j} \\
\mathbf{F}(0,-1) & = & \mathbf{i} \\
\mathbf{F}(1,1) & = & -\mathbf{i}+\mathbf{j} \\
\mathbf{F}(-1,1) & = & -\mathbf{i}-\mathbf{j} \\
\mathbf{F}(-1,-1) & = & \mathbf{i}-\mathbf{j} \\
\mathbf{F}(1,-1) & = & \mathbf{i}-\mathbf{j}
\end{array}$$

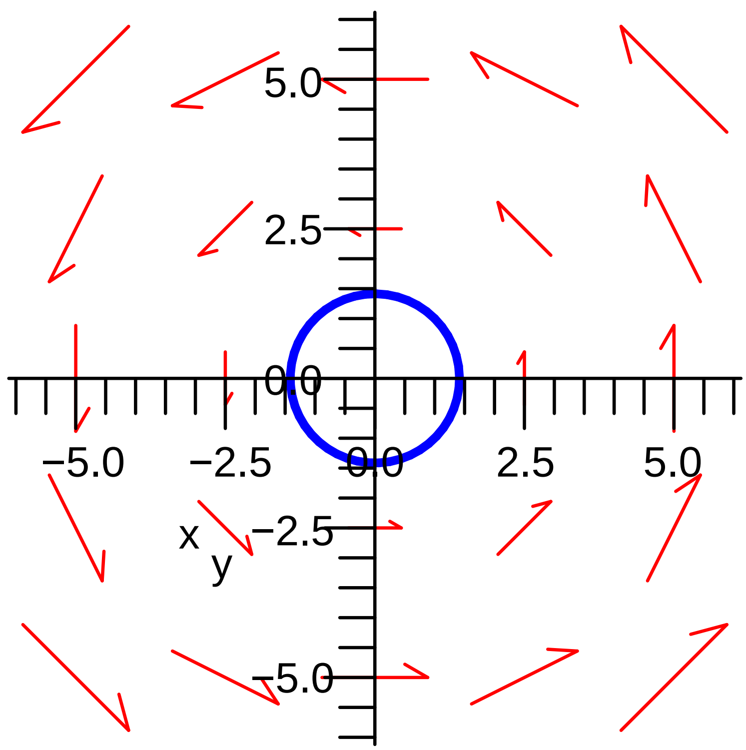

Figure 11: For Problem 1 of section 3.2

2. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + \mathbf{k}$

a. $\dfrac{dx}{x^2} = \dfrac{dy}{y^2} = \dfrac{dz}{1}$ Solve the right equality $-\dfrac{1}{y} = z + c$ or

$$y(z + c) = -1$$

Now solve $\dfrac{dx}{x^2} = dz$ to get $-\dfrac{1}{x} = z + k$ or

$$x(z + k) = -1$$

b. At the point $(1, 1, 2)$ we have $1(2 + k) = -1$ and $1(2 + c) = -1$, therefore $k = -3,\ c = -3$ and the equations are

$$x(z - 3) = -1,\ \ y(z - 3) = -1$$

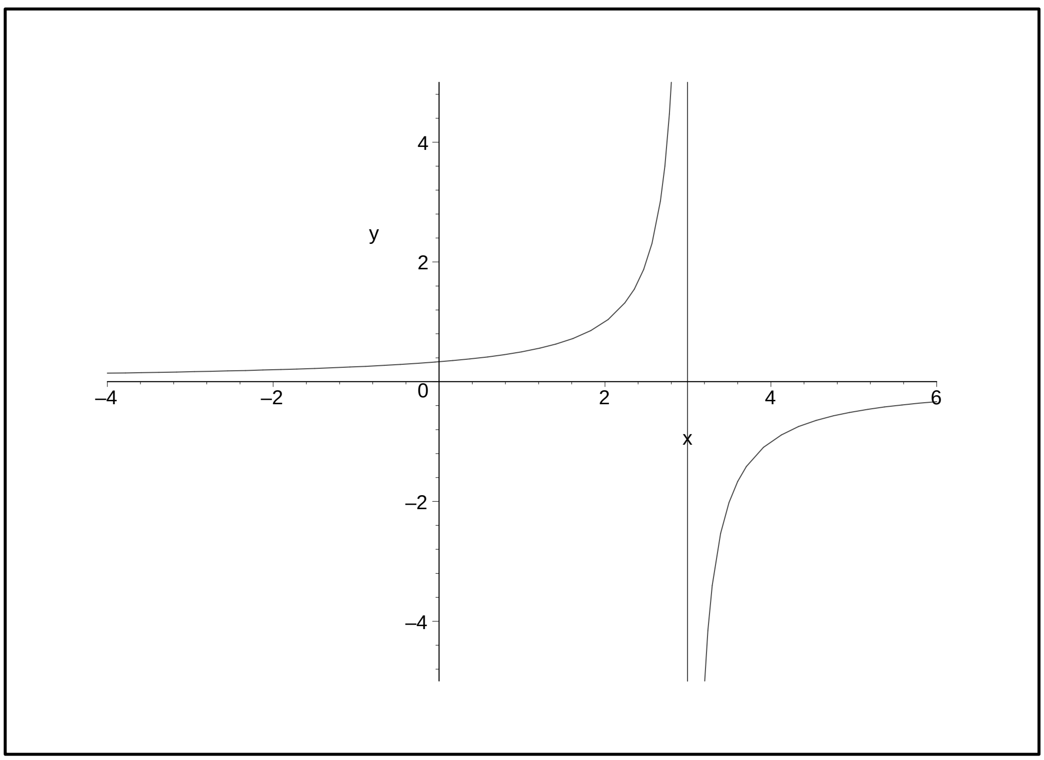

Figure 12: For Problem 2b of section 3.2 z axis is horizontal

3. $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ describe the flow lines

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Integrate

$$\ln x = \ln y + c$$

$$x = cy$$

and similarly

$$y = kz$$

which are lines aways from the origin.

4. The flow lines of the gradient field cross isotimic surfaces ortogonally. The distance between isotimic surfaces is constant and it is the normal to surfaces which are perpendicular to the gradients.

3.3

Problems: 1, 3, 5, 7, 10

1. $\mathbf{F} = e^{xy}\mathbf{i} + \sin(xy)\mathbf{j} + \cos^2(xz)\mathbf{k}$

$$\nabla\cdot\mathbf{F} = div\mathbf{F} = ye^{xy} + x\cos(xy) + x(2\cos(xz)\,(-\sin(xz)) = ye^{xy} + x\cos(xy) - 2x\cos(xz)\,\sin(xz)$$

3. $\mathbf{F} = \nabla(3x^2y^3z)$

$$\mathbf{F} = 6xy^3z\mathbf{i} + 9x^2y^2z\mathbf{j} + 3x^2y^3\mathbf{k}$$
$$div\mathbf{F} = 6y^3z + 18x^2yz + 0 = 6yz(y^2 + 3x^2)$$

5. $\nabla\cdot(\phi\mathbf{F}) = \phi\nabla\cdot\mathbf{F} + \mathbf{F}\cdot\nabla\phi$
   The left hand side is

$$\frac{\partial}{\partial x}(\phi\mathbf{F}_1) + \frac{\partial}{\partial y}(\phi\mathbf{F}_2) + \frac{\partial}{\partial x}(\phi\mathbf{F}_3)$$

$$= \frac{\partial\phi}{\partial x}\mathbf{F}_1 + \frac{\partial\mathbf{F}_1}{\partial x}\phi + \frac{\partial\phi}{\partial y}\mathbf{F}_2 + \frac{\partial\mathbf{F}_2}{\partial y}\phi + \frac{\partial\phi}{\partial z}\mathbf{F}_3 + \frac{\partial\mathbf{F}_3}{\partial z}\phi$$

$$= \underbrace{\frac{\partial\phi}{\partial x}\mathbf{F}_1 + \frac{\partial\phi}{\partial y}\mathbf{F}_2 + \frac{\partial\phi}{\partial z}\mathbf{F}_3}_{=(\nabla\phi)\cdot\mathbf{F}} + \phi\left(\frac{\partial\mathbf{F}_1}{\partial x} + \frac{\partial\mathbf{F}_2}{\partial y} + \frac{\partial\mathbf{F}_3}{\partial z}\right)$$

$$= (\nabla\phi)\cdot\mathbf{F} + \phi\cdot(\nabla\cdot\mathbf{F})$$

7. Nonconstant field with zero divergence, for example $\mathbf{F} = f(y,z)\mathbf{i} + g(x,z)\mathbf{j} + h(x,y)\mathbf{k}$ will have a zero divergence.

10. The divergence is zero since the arrows have the same length.

3.4

Problems: 1, 2, 4, 7, 10, 12

1. $\mathbf{F} = xy^2\mathbf{i} + xy\mathbf{j} + xy\mathbf{k}$
   The curl is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & xy & xy \end{vmatrix} = \mathbf{i}(x-0) - \mathbf{j}(y-0) + \mathbf{k}(y-2xy) = x\mathbf{i} - y\mathbf{j} + y(1-2x)\mathbf{k}$$

2. $\mathbf{F} = e^{xy}\mathbf{i} + \sin(xy)\mathbf{j} + \cos(yz^2)\mathbf{k}$
   The curl is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xy} & \sin(xy) & \cos(yz^2) \end{vmatrix} = \mathbf{i}(-z^2\sin(yz^2) - 0) - \mathbf{j}(0-0) + \mathbf{k}(y\cos(xy) - xe^{xy})$$

$$\nabla \times \mathbf{F} = -z^2\sin(yz^2)\mathbf{i} + (y\cos(xy) - xe^{xy})\mathbf{k}$$

4. $\mathbf{F} = (x + xz^2)\mathbf{i} + xy\mathbf{j} + yz\mathbf{k}$
   a. The divergence is:

$$\nabla \cdot \mathbf{F} = (1 + z^2) + x + y = 1 + x + y + z^2$$

   b. The curl is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + xz^2 & xy & yz \end{vmatrix} = \mathbf{i}(z-0) - \mathbf{j}(0-2xz) + \mathbf{k}(y-0) = z\mathbf{i} + 2xz\mathbf{j} + y\mathbf{k}$$

7. The flow lines of a velocity field $\mathbf{F}$ are straight lines, this does **not** mean the curl is zero.

10. $\mathbf{F} = y^2\mathbf{i} + z^2\mathbf{j} + x\mathbf{k}$
   a. The curl is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x \end{vmatrix} = \mathbf{i}(0-2z) - \mathbf{j}(1-0) + \mathbf{k}(0-2y) = -2z\mathbf{i} - \mathbf{j} - 2y\mathbf{k}$$

   b. Find the tangent to the curve $x = \cos(\pi t),\ y = \sin(\pi t),\ z = t^2$ when $t = 1$

$$\frac{\partial x}{\partial t} = -\pi\sin(\pi t), \quad \frac{\partial y}{\partial t} = \pi\cos(\pi t), \quad \frac{\partial z}{\partial t} = 2t$$

$$\mathbf{T} = \frac{-\pi\sin(\pi t)\mathbf{i} + \pi\cos(\pi t)\mathbf{j} + 2t\mathbf{k}}{\sqrt{(-\pi\sin(\pi t))^2 + (\pi\cos(\pi t))^2 + (2t)^2}}$$

$$\mathbf{T} = \frac{-\pi\sin(\pi t)\mathbf{i} + \pi\cos(\pi t)\mathbf{j} + 2t\mathbf{k}}{\sqrt{\pi^2 + 4t^2}}$$

At $t = 1$

$$\mathbf{T}(1) = \frac{-\pi\sin(\pi)\mathbf{i} + \pi\cos(\pi)\mathbf{j} + 2\mathbf{k}}{\sqrt{\pi^2 + 4}} = \frac{-\pi\mathbf{j} + 2\mathbf{k}}{\sqrt{\pi^2 + 4}}$$

The component (scalar) of $(\nabla \times \mathbf{F})$ parallel to $\mathbf{T}(1)$ is given by

$$||(\nabla \times \mathbf{F})_{||}|| = ||(\nabla \times \mathbf{F}) \cdot \mathbf{T}(1)||$$

Therefore (since $y = 0$, $z = 1$ at $t = 1$)

$$||(\nabla \times \mathbf{F})_{||}|| = ||\pi(-\pi\mathbf{j} + 2\mathbf{k})|| = ||-\pi^2\mathbf{j} + 2\pi\mathbf{k}|| = \sqrt{\pi^4 + 4\pi^2} = \pi\sqrt{\pi^2 + 1}$$

12. Find the curl of $f(||\mathbf{R}||)\mathbf{R}$

a.

$$\nabla \times (f(||\mathbf{R}||)\mathbf{R}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(||\mathbf{R}||)x & f(||\mathbf{R}||)y & f(||\mathbf{R}||)z \end{vmatrix}$$

Recalling that $||\mathbf{R}|| = \sqrt{x^2 + y^2 + z^2}$ we have

$$\nabla \times (f(||\mathbf{R}||)\mathbf{R}) = \mathbf{i}(z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}) - \mathbf{j}(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}) + \mathbf{k}(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y})$$

Let's compute these partials

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial ||\mathbf{R}||}\frac{\partial ||\mathbf{R}||}{\partial x} = \frac{\partial f}{\partial ||\mathbf{R}||}\frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2x)$$

Similarly

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial ||\mathbf{R}||}\frac{\partial ||\mathbf{R}||}{\partial y} = \frac{\partial f}{\partial ||\mathbf{R}||}\frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2y)$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial ||\mathbf{R}||}\frac{\partial ||\mathbf{R}||}{\partial z} = \frac{\partial f}{\partial ||\mathbf{R}||}\frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2z)$$

Now we evaluate the components of the curl,

$$z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z} = \frac{\partial f}{\partial ||\mathbf{R}||}\left(zy(x^2 + y^2 + z^2)^{-1/2} - yz(x^2 + y^2 + z^2)^{-1/2}\right) = 0$$

The same for the other components, therefore $\nabla \times (f(||\mathbf{R}||)\mathbf{R}) = 0$.

b. Use geometrical interpretation.

3.5

Problems: 1, 2, 4–8

1. $f = x^2 + y$ then $f(2,3,4) = 2^2(3) + 4 = 16$

2. $f = x^2 y + z$

$$\nabla f = 2xy\mathbf{i} + x^2\mathbf{j} + \mathbf{k}$$
$$\nabla f|_{(2,3,4)} = 2(2)(3)\mathbf{i} + 2^2\mathbf{j} + \mathbf{k} = 12\mathbf{i} + 4\mathbf{j} + \mathbf{k}$$

4. $\mathbf{F} = x^2 y\mathbf{i} + z\mathbf{j} - (x + y - z)\mathbf{k}$
   a. $\nabla \cdot \mathbf{F} = 2xy + 0 + 1 = 1 + 2xy$
   b.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & z & -(x + y - z) \end{vmatrix}$$

$$\nabla \times \mathbf{F} = \mathbf{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -(x + y - z) \end{vmatrix} - \mathbf{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x^2 y & -(x + y - z) \end{vmatrix} + \mathbf{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x^2 y & z \end{vmatrix}$$

$$\nabla \times \mathbf{F} = (-1 - 1)\mathbf{i} - (-1 + 0)\mathbf{j} + (0 - x^2)\mathbf{k} = -2\mathbf{i} + \mathbf{j} - x^2\mathbf{k}$$

   c. $\nabla(\nabla \cdot \mathbf{F}) = \nabla(1 + 2xy)$, using part a, so

$$\nabla(\nabla \cdot \mathbf{F}) = 2y\mathbf{i} + 2x\mathbf{j}$$

5. $\underbrace{\nabla \cdot \underbrace{(\nabla \times \mathbf{F})}_{\text{vector}}}_{\text{scalar}}$

6. $\underbrace{\nabla \times \underbrace{(\nabla \times \mathbf{F})}_{\text{vector}}}_{\text{vector}}$

7. $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\nabla \cdot \mathbf{R} = 1 + 1 + 1 = 3$ and

$$\nabla \times \mathbf{R} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{i}\underbrace{\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z \end{vmatrix}}_{=0} - \mathbf{j}\underbrace{\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x & z \end{vmatrix}}_{=0} + \mathbf{k}\underbrace{\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x & y \end{vmatrix}}_{=0} = \mathbf{0}$$

8. Find $\nabla \cdot (\nabla f)$ where $f = xyz + e^{xz}$

$$\nabla f = \left(yz + ze^{xz}\right)\mathbf{i} + xz\mathbf{j} + \left(xy + xe^{xz}\right)\mathbf{k}$$

and

$$\nabla \cdot (\nabla f) = z^2 e^{xz} + 0 + x^2 e^{xz} = \left(x^2 + z^2\right)e^{xz}$$

3.6

Problems: 1, 3–5, 7

1. $f = x^5yz^3$

$$\frac{\partial^2 f}{\partial x^2} = 5(4)(x^3yz^3) = 20x^3yz^3$$

$$\frac{\partial^2 f}{\partial y^2} = 0$$

$$\frac{\partial^2 f}{\partial z^2} = 3(2)(x^5yz) = 6x^5yz$$

$$\nabla^2 f = 20x^3yz^3 + 6x^5yz = 2x^3yz(10z^2 + 3x^2)$$

3. $\mathbf{F} = 3\mathbf{i} + \mathbf{j} - x^2y^3z^4\mathbf{k}$

$$\frac{\partial^2 \mathbf{F}}{\partial x^2} = -2(1)y^3z^4\mathbf{k} = -2y^3z^4\mathbf{k}$$

$$\frac{\partial^2 \mathbf{F}}{\partial y^2} = -3(2)x^2yz^4\mathbf{k} = -6x^2yz^4\mathbf{k}$$

$$\frac{\partial^2 \mathbf{F}}{\partial x^2} = -4(3)x^2y^3z^2\mathbf{k} = -12x^2y^3z^2\mathbf{k}$$

$$\nabla^2 \mathbf{F} = \left[(-2y^3z^4) + (-6x^2yz^4) + (-12x^2y^3z^2)\right]\mathbf{k} = -2yz^2(y^2z^2 + 3x^2z^2 + 6x^2y^2)\mathbf{k}$$

4. a. $f = e^z \sin y$

$$\nabla^2 f = \underbrace{\frac{\partial^2}{\partial x^2}(e^z \sin y)}_{=0} + \frac{\partial^2}{\partial y^2}(e^z \sin y) + \frac{\partial^2}{\partial z^2}(e^z \sin y) = -e^z \sin y + e^z \sin y = 0$$

Yes

b. $f = \sin x \sinh y + \cos x \cosh z$

$$\nabla^2 f = -\sin x \sinh y - \cos x \cosh z + \sin x \sinh y + 0 + 0 + \cos x \cosh z = 0$$

Yes

c. $f = \sin(px)\sinh(qy)$

$$\nabla^2 f = -p^2 \sin(px)\sinh(qy) + q^2 \sin(px)\sinh(qy) = (-p^2 + q^2)\sin(px)\sinh(qy)$$

In order for this to vanish we have to have $p^2 = q^2$.

5. a. ∇f is a vector
 b. $\nabla \cdot \mathbf{F}$ is a scalar
 c. $\nabla \times \mathbf{F}$ is a vector
 d. $\nabla \cdot (\nabla f)$ is a scalar $(\nabla^2 f)$

e. $\nabla \times (\nabla f)$ is a vector
f. $\nabla \times f$ is meaningless
g. $\nabla^2 \mathbf{F}$ is a vector
h. $\nabla \times \underbrace{\nabla^2 \mathbf{F}}_{\text{vector}}$ is a vector
i. $\nabla \times \underbrace{\nabla^2 f}_{\text{scalar}}$ is meaningless
j. $\nabla(\underbrace{\nabla^2 f}_{\text{scalar}})$ is a vector

7. $f = 2x^2 + y$ and $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
a. $\nabla f = 4x\mathbf{i} + \mathbf{j}$
b. $\nabla \cdot \mathbf{R} = 1 + 1 + 1 = 3$
c. $\nabla^2 f = 4$
d. $\nabla \times (f\mathbf{R}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x(2x^2+y) & y(2x^2+y) & z(2x^2+y) \end{vmatrix}$

$$\nabla \times (f\mathbf{R}) = \mathbf{i}(z - 0) - \mathbf{j}(4xz - 0) + \mathbf{k}(4xy - x) = z\mathbf{i} - 4xz\mathbf{j} + x(4y - 1)\mathbf{k}$$

3.8

Problems: 1, 5, 6, 10, 12

1. Prove that $\nabla(\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1$
   Starting on the left hand side we have

$$\nabla(\phi_1\phi_2) = \left[\left(\frac{\partial\phi_1}{\partial x}\right)\phi_2 + \left(\frac{\partial\phi_2}{\partial x}\right)\phi_1\right]\mathbf{i} + \dots$$

Similarly for the other components. Now collect the second terms for each component to get the first term on the right and similarly for the first terms giving the second term on the right.

Prove $\nabla\cdot(\phi\mathbf{F}) = \phi\nabla\cdot\mathbf{F} + \mathbf{F}\cdot(\nabla\phi)$
Again start with the left hand side

$$\nabla\cdot(\phi\mathbf{F}) = \left(\frac{\partial\phi}{\partial x}\right)\mathbf{F}_1 + \phi\frac{\partial\mathbf{F}_1}{\partial x} + \left(\frac{\partial\phi}{\partial y}\right)\mathbf{F}_2 + \phi\frac{\partial\mathbf{F}_2}{\partial y} + \left(\frac{\partial\phi}{\partial z}\right)\mathbf{F}_3 + \phi\frac{\partial\mathbf{F}_3}{\partial z}$$

$$\nabla\cdot(\phi\mathbf{F}) = \underbrace{\left(\frac{\partial\phi}{\partial x}\right)\mathbf{F}_1 + \left(\frac{\partial\phi}{\partial y}\right)\mathbf{F}_2 + \left(\frac{\partial\phi}{\partial z}\right)\mathbf{F}_3}_{\mathbf{F}\cdot\nabla\phi} + \phi\left[\underbrace{\frac{\partial\mathbf{F}_1}{\partial x} + \frac{\partial\mathbf{F}_2}{\partial y} + \frac{\partial\mathbf{F}_3}{\partial z}}_{\nabla\cdot\mathbf{F}}\right]$$

So

$$\nabla\cdot(\phi\mathbf{F}) = \mathbf{F}\cdot(\nabla\phi) + \phi(\nabla\cdot\mathbf{F})$$

5. Why $\nabla\cdot(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot(\nabla\times\mathbf{F}) + \mathbf{F}\cdot(\nabla\times\mathbf{G})$ is not valid?
   The right hand side is symmetric relative to $\mathbf{F}$ and $\mathbf{G}$ but the left hand side is not since $\mathbf{F}\times\mathbf{G} \neq \mathbf{G}\times\mathbf{F}$.

6. Show that $\nabla\cdot\dfrac{\mathbf{A}\times\mathbf{R}}{||\mathbf{R}||} = 0$

$$\mathbf{A}\times\mathbf{R} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix} = (A_2 z - yA_3)\mathbf{i} - (A_1 z - A_3 x)\mathbf{j} + (A_1 y - A_2 x)\mathbf{k}$$

Therefore

$$\frac{\mathbf{A}\times\mathbf{R}}{||\mathbf{R}||} = \frac{(A_2 z - yA_3)\mathbf{i} - (A_1 z - A_3 x)\mathbf{j} + (A_1 y - A_2 x)\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$$

Now compute the divergence of the above

$$\begin{aligned}\nabla\cdot\frac{\mathbf{A}\times\mathbf{R}}{||\mathbf{R}||} &= \frac{-\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x)(A_2 z - A_3 y)}{x^2+y^2+z^2} + \frac{\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2y)(A_1 z - A_3 x)}{x^2+y^2+z^2} \\ &+ \frac{-\frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2z)(A_1 y - A_2 x)}{x^2+y^2+z^2}\end{aligned}$$

$$\nabla \cdot \frac{\mathbf{A} \times \mathbf{R}}{||\mathbf{R}||} = \frac{-x(A_2 z - A_3 y) + y(A_1 z - A_3 x) - z(A_1 y - A_2 x)}{(x^2 + y^2 + z^2)^{3/2}}$$

You can easily check the all terms cancel and we are left with zero.

10. a. Evaluate $\nabla \cdot \left(||\mathbf{R}||^2 \mathbf{A} \right)$

$$\nabla \cdot \left(||\mathbf{R}||^2 \mathbf{A} \right) = 2xA_1 + 2yA_2 + 2zA_3 = 2\mathbf{R} \cdot \mathbf{A}$$

b. Evaluate $\nabla \times \left(||\mathbf{R}||^2 \mathbf{A} \right)$

$$\nabla \cdot \left(||\mathbf{R}||^2 \mathbf{A} \right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ||\mathbf{R}||^2 A_1 & ||\mathbf{R}||^2 A_2 & ||\mathbf{R}||^2 A_3 \end{vmatrix} = (2yA_3 - 2zA_2)\mathbf{i} - (2xA_3 - 2zA_1)\mathbf{j} + (2xA_2 - 2yA_1)\mathbf{k}$$

Therefore

$$\nabla \cdot \left(||\mathbf{R}||^2 \mathbf{A} \right) = 2(\mathbf{R} \times \mathbf{A})$$

c. $\mathbf{R} \cdot \nabla \left(||\mathbf{R}||^2 \mathbf{A} \right) = \mathbf{R} \cdot (2\mathbf{R} \cdot \mathbf{A}) = 2(\mathbf{R} \cdot \mathbf{R})\mathbf{A} = 2||\mathbf{R}||^2 \mathbf{A}$

d. $\nabla (\mathbf{A} \cdot \mathbf{R})^4 = 4(\mathbf{A} \cdot \mathbf{R})^3 \mathbf{A}$

e. $\nabla \cdot (||\mathbf{R}||\mathbf{A}) = \dfrac{\mathbf{A} \cdot \mathbf{R}}{||\mathbf{R}||}$

f. $\mathbf{R} \cdot \nabla (\mathbf{A} \cdot \mathbf{R}\mathbf{A}) = \mathbf{R} \cdot \left[\underbrace{\nabla (\mathbf{A} \cdot \mathbf{R})}_{\mathbf{A}} \right] \mathbf{A} = (\mathbf{R} \cdot \mathbf{A})\mathbf{A}$

g. $\nabla \cdot (\mathbf{A} \times \mathbf{R}) = \mathbf{R} \cdot \underbrace{\nabla \times \mathbf{A}}_{=0} - \mathbf{A} \cdot \underbrace{\nabla \times \mathbf{R}}_{=0 \text{ by } (3.32)}$

h. $\nabla \times (\mathbf{A} \times \mathbf{R}) = \underbrace{(\mathbf{R} \cdot \nabla)\mathbf{A}}_{=0\ \mathbf{A} \text{ is constant}} - (\mathbf{A} \cdot \nabla)\mathbf{R} + \underbrace{(\nabla \cdot \mathbf{R})}_{=3} \mathbf{A} - \underbrace{(\nabla \cdot \mathbf{A})}_{=0} \mathbf{R}$

$$\nabla \times (\mathbf{A} \times \mathbf{R}) = -(\mathbf{A} \cdot \nabla)\mathbf{R} + 3\mathbf{A} = -A_1\mathbf{i} - A_2\mathbf{j} - A_3\mathbf{k} + 3\mathbf{A} = 2\mathbf{A}$$

i. $\nabla^2 (\mathbf{R} \cdot \mathbf{R}) = \nabla^2 (x^2 + y^2 + z^2) = 6$

12. $\mathbf{V} = (x + 4y)\mathbf{i} + (y - 3z)\mathbf{j} + Cz\mathbf{k} = \nabla \times \mathbf{F}$

Since $\nabla \cdot (\nabla \times \mathbf{F}) = 0$, we get

$$\frac{\partial}{\partial x}(x + 4y) + \frac{\partial}{\partial y}(y - 3z) + \frac{\partial}{\partial z}(Cz) = 0$$

The left hand side is $1 + 1 + C$, therefore $C = -2$

3.10

Problems: 2, 4, 6, 8, 11

2. Use (3.50)-(3.51) to derive $\mathbf{e}_z = \mathbf{k}$, $\mathbf{e}_\rho = \dfrac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}}$, $\mathbf{e}_\theta = \dfrac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2+y^2}}$

$$\mathbf{e}_z = \frac{grad\, z}{||grad\, z||} = \frac{\mathbf{k}}{1} = \mathbf{k}, \qquad \text{since } z = z$$

$$\mathbf{e}_\rho = \frac{grad\,\rho}{||grad\,\rho||} = \frac{\frac{\partial}{\partial x}\sqrt{x^2+y^2}\mathbf{i} + \frac{\partial}{\partial y}\sqrt{x^2+y^2}\mathbf{j}}{||grad\,\rho||} = \frac{\frac{x}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2+y^2}}\mathbf{j}}{1} = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}}$$

$$\mathbf{e}_\theta = \frac{grad\,\theta}{||grad\,\theta||} = \frac{\frac{\partial}{\partial x}\sin^{-1}\frac{y}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{\partial}{\partial y}\sin^{-1}\frac{y}{\sqrt{x^2+y^2}}\mathbf{j}}{||grad\,\theta||}$$

Now differentiate the arc sine

$$\frac{\partial}{\partial x}\sin^{-1}\frac{y}{\sqrt{x^2+y^2}} = \frac{1}{\sqrt{1-\frac{y^2}{x^2+y^2}}}\,\frac{-\frac{y}{\sqrt{x^2+y^2}}x}{x^2+y^2} = \frac{\sqrt{x^2+y^2}}{x}\,\frac{-xy\left(x^2+y^2\right)^{-1/2}}{x^2+y^2} = -\frac{y}{x^2+y^2}$$

$$\frac{\partial}{\partial y}\sin^{-1}\frac{y}{\sqrt{x^2+y^2}} = \frac{1}{\sqrt{1-\frac{y^2}{x^2+y^2}}}\,\frac{\sqrt{x^2+y^2} - y^2\left(x^2+y^2\right)^{-1/2}}{x^2+y^2} = \frac{x^2+y^2-y^2}{x\left(x^2+y^2\right)^2} = \frac{x}{x^2+y^2}$$

$$||grad\,\theta||^2 = \frac{x^2}{\left(x^2+y^2\right)^2} + \frac{y^2}{\left(x^2+y^2\right)^2} = \frac{1}{x^2+y^2}$$

So

$$\mathbf{e}_\theta = \frac{-\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}}{\frac{1}{\sqrt{x^2+y^2}}} = \frac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2+y^2}}$$

4. Use (3.61)-(3.62) to derive $\mathbf{e}_r = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$

Since $r = \sqrt{x^2+y^2+z^2}$ we find $\nabla r = \dfrac{x}{\sqrt{x^2+y^2+z^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2+z^2}}\mathbf{j} + \dfrac{z}{\sqrt{x^2+y^2+z^2}}\mathbf{k}$ and

$$||\nabla r||^2 = \frac{x^2+y^2+z^2}{x^2+y^2+z^2} = 1$$

So $\mathbf{e}_r = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$

Now to $\mathbf{e}_\theta$. Recall that $\theta = \arccos \dfrac{x}{\sqrt{x^2+y^2}}$.

$$\frac{\partial\theta}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2}} \frac{\sqrt{x^2+y^2} - x\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$= \frac{1}{\sqrt{1-\frac{x^2}{x^2+y^2}}} \frac{x^2+y^2-x^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$= \frac{1}{\sqrt{\frac{y^2}{x^2+y^2}}} \frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$= \frac{y}{x^2+y^2}$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2}} \frac{-x\frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$= \frac{1}{\sqrt{1-\frac{x^2}{x^2+y^2}}} \frac{-xy}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$= -\frac{x}{x^2+y^2}$$

$$\frac{\partial\theta}{\partial z} = 0$$

Now to the norm of $\nabla\theta$,

$$||\nabla\theta|| = \sqrt{\frac{y^2}{(x^2+y^2)^2} + \frac{(-x)^2}{(x^2+y^2)^2}} = \frac{1}{\sqrt{x^2+y^2}}$$

But $x^2+y^2 = r^2\sin^2\phi\cos^2\theta + r^2\sin^2\phi\sin^2\theta = r^2\sin^2\phi$ and so

$$||\nabla\theta|| = \frac{1}{r\sin\phi}$$

Therefore

$$e_\theta = r\sin\phi\nabla\theta.$$

Now to $\mathbf{e}_\phi$. Recall $\phi = \arccos \dfrac{z}{\sqrt{x^2+y^2+z^2}}$. Therefore

$$\frac{\partial\phi}{\partial x} = \frac{1}{\sqrt{1-\frac{z^2}{x^2+y^2+z^2}}}\frac{-z\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}$$

$$= \frac{-xz}{\sqrt{x^2+y^2}(x^2+y^2+z^2)}$$

$$\frac{\partial\phi}{\partial y} = \frac{1}{\sqrt{1-\frac{z^2}{x^2+y^2+z^2}}}\frac{-z\frac{1}{2}\frac{2y}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}$$

$$= \frac{-yz}{\sqrt{x^2+y^2}(x^2+y^2+z^2)}$$

$$\frac{\partial\phi}{\partial z} = \frac{1}{\sqrt{1-\frac{z^2}{x^2+y^2+z^2}}}\frac{\sqrt{x^2+y^2+z^2}-z\frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}$$

$$= \frac{1}{\sqrt{\frac{x^2+y^2}{x^2+y^2+z^2}}}\frac{x^2+y^2+z^2-z^2}{\sqrt{x^2+y^2+z^2}(x^2+y^2+z^2)}$$

$$= \frac{1}{\sqrt{x^2+y^2}}\frac{x^2+y^2}{(x^2+y^2+z^2)}$$

$$= \frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2}$$

We now compute the magnitude of $\nabla\phi$.

$$||\nabla\phi|| = \sqrt{\frac{x^2z^2}{(x^2+y^2)(x^+y^2+z^2)^2}+\frac{y^2z^2}{(x^2+y^2)(x^+y^2+z^2)^2}+\frac{x^2+y^2}{(x^+y^2+z^2)^2}}$$

$$= \frac{1}{x^2+y^2+z^2}\sqrt{\frac{x^2z^2+y^2z^2}{x^2+y^2}+x^2+y^2} = \frac{1}{x^2+y^2+z^2}\sqrt{\frac{(x^2+y^2)z^2}{x^2+y^2}+x^2+y^2}$$

$$= \frac{1}{x^2+y^2+z^2}\sqrt{x^2+y^2+z^2}$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{1}{r}$$

Therefore

$$e_\phi = r\nabla\phi.$$

6. $\nabla^2 f$ in cylindrical coordinates

Let $\mathbf{F} = \nabla f = \underbrace{\frac{\partial f}{\partial \rho}}_{=\mathbf{F}_\rho} \mathbf{e}_\rho + \underbrace{\frac{1}{\rho}\frac{\partial f}{\partial \theta}}_{=\mathbf{F}_\theta} \mathbf{e}_\theta + \underbrace{\frac{\partial f}{\partial z}}_{=\mathbf{F}_z} \mathbf{e}_z$

Now

$$\nabla^2 f = \nabla \cdot \mathbf{F} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho \mathbf{F}_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \theta}\mathbf{F}_\theta + \frac{\partial}{\partial z}\mathbf{F}_z = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

In spherical coordinates:

$\mathbf{F} = \nabla f = \underbrace{\frac{\partial f}{\partial r}}_{=\mathbf{F}_r} \mathbf{e}_r + \underbrace{\frac{1}{r}\frac{\partial f}{\partial \phi}}_{=\mathbf{F}_\phi} \mathbf{e}_\phi + \underbrace{\frac{1}{r\sin\phi}\frac{\partial f}{\partial \theta}}_{=\mathbf{F}_\theta} \mathbf{e}_\theta$

Now

$$\begin{aligned}
\nabla^2 f &= \nabla \cdot \mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \mathbf{F}_r) + \frac{1}{r\sin\phi}\frac{\partial}{\partial \theta}\mathbf{F}_\theta + \frac{1}{r\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi \mathbf{F}_\phi) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r\sin\phi}\frac{\partial}{\partial \theta}\left(\frac{1}{r\sin\phi}\frac{\partial f}{\partial \theta}\right) + \frac{1}{r\sin\phi}\frac{\partial}{\partial \phi}\left(\frac{\sin\phi}{r}\frac{\partial f}{\partial \phi}\right) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial f}{\partial \phi}\right)
\end{aligned}$$

8. a. Use Problem 2 to write $\mathbf{F} = \dfrac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2}$ in cylindrical coordinates.

From problem 2, we have: $\mathbf{e}_\rho = \dfrac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}}$, $\mathbf{e}_\theta = \dfrac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2+y^2}}$, $\mathbf{e}_z = \mathbf{k}$. Therefore the vector field $\mathbf{F}$ becomes: $\frac{\mathbf{e}_\rho}{\rho}$ and we have $\mathbf{F}(\rho, \theta, z) = \dfrac{\mathbf{e}_\rho}{\rho} = \dfrac{1}{\rho}\mathbf{e}_\rho$ therefore $\mathbf{F}_\rho = \dfrac{1}{\rho}$ and $\mathbf{F}_\theta = \mathbf{F}_z = 0$ and the divergence is

$\nabla \cdot \mathbf{F} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\underbrace{(\rho \mathbf{F}_\rho)}_{=1} = 0$

and the curl is

$$\nabla \times \mathbf{F} = \frac{1}{\rho}\begin{vmatrix} \mathbf{e}_\rho & \frac{1}{\rho}\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ \frac{1}{\rho} & 0 & 0 \end{vmatrix} = 0$$

b. Again use Problem 2 to get: $\mathbf{e}_\rho = \dfrac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}}$, $\mathbf{e}_\theta = \dfrac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2+y^2}}$, $\mathbf{e}_z = \mathbf{k}$. Therefore the vector field $\mathbf{F}$ becomes: $\mathbf{F} = \dfrac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2} = \dfrac{\mathbf{e}_\theta}{\rho}$ therefore $\mathbf{F}_\rho = 0$, $\mathbf{F}_\theta = \dfrac{1}{\rho}$, $\mathbf{F}_z = 0$ and the divergence is (note that the only non zero component is $\mathbf{F}_\theta$)

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho}\frac{\partial \mathbf{F}_\theta}{\partial \theta} = 0$$

and the curl is

$$\nabla \times \mathbf{F} = \frac{1}{\rho}\begin{vmatrix} \mathbf{e}_\rho & \frac{1}{\rho}\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & 1 & 0 \end{vmatrix} = 0$$

11. $f(r,\theta,\phi) = \dfrac{\cos\phi}{r^2}$

$$\nabla f = \underbrace{\frac{\partial f}{\partial r}}_{=\frac{-2\cos\phi}{r^3}} \mathbf{e}_r + \frac{1}{r\sin\phi}\underbrace{\frac{\partial f}{\partial \theta}}_{=0}\mathbf{e}_\theta + \frac{1}{r}\underbrace{\frac{\partial f}{\partial \phi}}_{=-\frac{\sin\phi}{r^2}}\mathbf{e}_\phi = -\frac{2\cos\phi}{r^3}\mathbf{e}_r - \frac{\sin\phi}{r^3}\mathbf{e}_\phi$$

3.11

Problems: 3–5, 8, 9, 12, 13

3. Since x, y, z are orthogonal then any order of x, y, z will do the same for ρ, θ, z

4. One way to do this is to find the differentials for the transformation:

$$u_1 = e^x, \; u_2 = y, \; u_3 = z$$

then $du_1\, du_2\, du_3 = e^x dx\, dy\, dz$ and since $dV = dx\, dy\, dz$ we have $dV = e^{-x} du_1\, du_2\, du_3 = \frac{1}{u_1} du_1\, du_2\, du_3$.

Two other ways are possible. First to use $h_i = \left|\left|\frac{1}{\nabla u_i}\right|\right|$. Now $u_1 = e^x, \; u_2 = y, \; u_3 = z$ then $\nabla u_1 =< e^x, 0, 0 >$, $\nabla u_2 =< 0, 1, 0 >$, and $\nabla u_3 =< 0, 0, 1 >$. The magnitudes of these vectors are $||\nabla u_1|| = e^x, \; ||\nabla u_2|| = 1, \; ||\nabla u_3|| = 1$. The reciprocals are the scale factors.

Second way is to first find the inverse transformation:

$$x = \ln u_1, \; y = u_2, \; z = u_3$$

and find the scale factors using

$$h_i = \left|\left|\frac{\partial \mathbf{R}}{\partial u_i}\right|\right|.$$

Therefore

$$h_1 = \left|\left|\frac{\partial}{\partial u_1}\left(\ln u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}\right)\right|\right| = \frac{1}{|u_1|} = e^{-x}$$

$$h_2 = \left|\left|\frac{\partial}{\partial u_2}\left(\ln u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}\right)\right|\right| = 1$$

$$h_3 = \left|\left|\frac{\partial}{\partial u_3}\left(\ln u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}\right)\right|\right| = 1$$

and therefore

$$dV = h_1 h_2 h_3 du_1\, du_2\, du_3 = \frac{1}{u_1} du_1\, du_2\, du_3 = e^{-x} du_1\, du_2\, du_3$$

5. Find the Laplacian of $g = u_1^3 + u_2^3 + u_3^3$ where

$$x = u_1 - u_2, \; y = u_1 + u_2, \; z = u_3^2$$

First we can find u_i in terms of x, y, z compute g and its Laplacian in Cartesian coordinates and transform back. This will give us $u_1 = \frac{1}{2}(x + y), \; u_2 = \frac{1}{2}(y - x), \; u_3 = \sqrt{z}$ and

$$g = \frac{1}{8}(x + y)^3 + \frac{1}{8}(y - x)^3 + z^{3/2}$$

$$\nabla^2 g = \underbrace{\frac{3}{4}(x+y) + \frac{3}{4}(y-x)}_{=\frac{\partial^2 g}{\partial x^2}} + \underbrace{\frac{3}{4}(x+y) + \frac{3}{4}(y-x)}_{=\frac{\partial^2 g}{\partial y^2}} + \frac{3}{4}z^{-1/2}$$

$$\nabla^2 g = \frac{3}{2}(x+y) + \frac{3}{2}(y-x) + \frac{3}{4}z^{-1/2}$$

and in terms of u_i

$$\nabla^2 g = 3u_1 + 3u_2 + \frac{3}{4u_3}$$

Another way is to use the scale factors $h_1 = \sqrt{2},\ h_2 = \sqrt{2},\ h_3 = 2u_3$ and substitute in the equation for the Laplacian

$$\nabla^2 g = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right]$$

$$\nabla^2 g = \frac{1}{4u_3}\left[\frac{\partial}{\partial u_1}\left(\frac{2\sqrt{2}u_3}{\sqrt{2}}(3u_1^2)\right) + \frac{\partial}{\partial u_2}\left(2u_3(3u_2^2)\right) + \frac{\partial}{\partial u_3}\left(\frac{2}{2u_3}(3u_3^2)\right)\right]$$

$$\nabla^2 g = \frac{1}{4u_3}\left[12u_1u_3 + 12u_2u_3 + 3\right] = 3u_1 + 3u_2 + \frac{3}{4u_3}$$

8. $x = u_1^2 - u_2^2,\ y = 2u_1u_2,\ z = u_3$

$$\mathbf{R} = (u_1^2 - u_2^2)\mathbf{i} + 2u_1u_2\mathbf{j} + u_3\mathbf{k}$$

a.

$$\mathbf{e}_1 = \frac{\frac{\partial \mathbf{R}}{\partial u_1}}{||\frac{\partial \mathbf{R}}{\partial u_1}||} = \frac{2u_1\mathbf{i} + 2u_2\mathbf{j}}{\sqrt{4u_1^2 + 4u_2^2}} = \frac{u_1\mathbf{i} + u_2\mathbf{j}}{\sqrt{u_1^2 + u_2^2}}$$

$$\mathbf{e}_2 = \frac{\frac{\partial \mathbf{R}}{\partial u_2}}{||\frac{\partial \mathbf{R}}{\partial u_2}||} = \frac{-2u_2\mathbf{i} + 2u_1\mathbf{j}}{\sqrt{4u_2^2 + 4u_1^2}} = \frac{-u_2\mathbf{i} + u_1\mathbf{j}}{\sqrt{u_1^2 + u_2^2}}$$

$$\mathbf{e}_3 = \frac{\frac{\partial \mathbf{R}}{\partial u_3}}{||\frac{\partial \mathbf{R}}{\partial u_3}||} = \frac{\mathbf{k}}{1} = \mathbf{k}$$

Orthogonality: $\mathbf{e}_1 \cdot \mathbf{e}_2 = \dfrac{-u_1u_2 + u_1u_2}{u_1^2 + u_2^2} = 0$ and similarly for the rest $\mathbf{e}_1 \cdot \mathbf{e}_3 = 0$ and $\mathbf{e}_2 \cdot \mathbf{e}_3 = 0$.

The system is right handed because $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$ and so on.

b. scale factors can be read from above $h_1 = h_2 = 2\sqrt{u_1^2 + u_2^2}$ and $h_3 = 1$

c. $\nabla^2 g = \dfrac{1}{4(u_1^2 + u_2^2)}\left[\dfrac{\partial^2 g}{\partial u_1^2} + \dfrac{\partial^2 g}{\partial u_2^2} + \dfrac{\partial}{\partial u_3}\left(4(u_1^2 + u_2^2)g\right)\right]$

$$\nabla^2 g = \frac{1}{4(u_1^2 + u_2^2)}\left[\frac{\partial^2 g}{\partial u_1^2} + \frac{\partial^2 g}{\partial u_2^2}\right] + \frac{\partial^2 g}{\partial u_3^2}$$

d. $\mathbf{F} = u_3\mathbf{e}_1 + u_1\mathbf{e}_2 + u_2\mathbf{e}_3$ then

$$\nabla\cdot\mathbf{F} = \frac{1}{4(u_1^2+u_2^2)}\left[\frac{\partial}{\partial u_1}\left(2\sqrt{u_1^2+u_2^2}\,u_3\right) + \frac{\partial}{\partial u_2}\left(2\sqrt{u_1^2+u_2^2}\,u_1\right) + \underbrace{\frac{\partial}{\partial u_3}\left(4(u_1^2+u_2^2)u_2\right)}_{=0}\right]$$

$$\nabla\cdot\mathbf{F} = \frac{1}{4(u_1^2+u_2^2)}\left[2u_3\,\frac{1}{2}(u_1^2+u_2^2)^{-1/2}(2u_1) + 2u_1\,\frac{1}{2}(u_1^2+u_2^2)^{-1/2}(2u_2)\right] = \frac{u_1(u_3+u_2)}{2(u_1^2+u_2^2)^{3/2}}$$

$$\nabla\times\mathbf{F} = \frac{1}{4(u_1^2+u_2^2)}\begin{vmatrix} 2\sqrt{u_1^2+u_2^2}\,\mathbf{e}_1 & 2\sqrt{u_1^2+u_2^2}\,\mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ 2u_3\sqrt{u_1^2+u_2^2} & 2u_1\sqrt{u_1^2+u_2^2} & u_2 \end{vmatrix}$$

$$\begin{aligned}\nabla\times\mathbf{F} &= \frac{1}{4(u_1^2+u_2^2)}\left[2\sqrt{u_1^2+u_2^2}\,\mathbf{e}_1(1-0) - 2\sqrt{u_1^2+u_2^2}\,\mathbf{e}_2(0-2\sqrt{u_1^2+u_2^2})\right.\\ &+ \left.\mathbf{e}_3\left(2\sqrt{u_1^2+u_2^2} + 2u_1\frac{2u_1}{2\sqrt{u_1^2+u_2^2}} - 2u_3\frac{2u_2}{2\sqrt{u_1^2+u_2^2}}\right)\right]\\ &= \frac{1}{2\sqrt{u_1^2+u_2^2}}\,\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\left(\frac{1}{2\sqrt{u_1^2+u_2^2}} + \frac{u_1^2-u_2u_3}{2(u_1+u_2^2)^{3/2}}\right)\end{aligned}$$

9. $x = u_3,\ y = e^{u_2}\cos u_1,\ z = e^{u_2}\sin u_1$

a. $\dfrac{\partial\mathbf{R}}{\partial u_1} = -\sin u_1\,(e^{u_2})\mathbf{j} + \cos u_1\,(e^{u_2})\mathbf{k}$

$\dfrac{\partial\mathbf{R}}{\partial u_2} = \cos u_1\,(e^{u_2})\mathbf{j} + \sin u_1\,(e^{u_2})\mathbf{k}$

$\dfrac{\partial\mathbf{R}}{\partial u_3} = \mathbf{i}$

Since $\dfrac{\partial\mathbf{R}}{\partial u_3} = \mathbf{i}$ and the other have no component in the direction of $\mathbf{i}$ we get the orthogonality to $\dfrac{\partial\mathbf{R}}{\partial u_3}$. Now to the orthogonality of $\dfrac{\partial\mathbf{R}}{\partial u_1}$ and $\dfrac{\partial\mathbf{R}}{\partial u_2}$

$$\frac{\partial\mathbf{R}}{\partial u_1}\cdot\frac{\partial\mathbf{R}}{\partial u_2} = -\sin u_1\,\cos u_1\,e^{u_2} + \sin u_1\,\cos u_1\,e^{u_2} = 0$$

b. $h_1 = ||\dfrac{\partial\mathbf{R}}{\partial u_1}|| = \sqrt{\sin^2 u_1\,e^{2u_2} + \cos^2 u_1\,e^{2u_2}} = e^{u_2}$, similarly $h_2 = e^{u_2}$ and $h_3 = 1$

c. $\nabla^2(u_1^2+u_2^2+u_3^2) = \dfrac{1}{e^{2u_2}}\left[\underbrace{\dfrac{\partial}{\partial u_1}\left(\dfrac{\partial g}{\partial u_1}\right)}_{=2} + \underbrace{\dfrac{\partial}{\partial u_2}\left(\dfrac{\partial g}{\partial u_2}\right)}_{=2} + \dfrac{\partial}{\partial u_3}\left(\dfrac{e^{2u_2}\partial g}{\partial u_3}\right)\right]$

$$\nabla^2(u_1^2+u_2^2+u_3^2) = e^{-2u_2}\left[2+2+2e^{2u_2}\right] = 4e^{-2u_2}+2$$

d. $$\nabla\cdot(-e^{u_2}\,\mathbf{e}_3+u_3\,\mathbf{e}_1) = \frac{1}{e^{2u_2}}\left[\underbrace{\frac{\partial}{\partial u_1}(u_3e^{u_2}))}_{=0}+0+\underbrace{\frac{\partial}{\partial u_3}(-e^{u_2}e^{2u_2}))}_{=0}\right] = 0$$

$$\nabla\times\mathbf{F} = \frac{1}{e^{2u_2}}\begin{vmatrix} e^{u_2}\,\mathbf{e}_1 & e^{u_2}\,\mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ e^{u_2}\,u_3 & 0 & -e^{u_2} \end{vmatrix}$$

$$\nabla\times\mathbf{F} = \frac{1}{e^{2u_2}}\left[e^{u_2}\,\mathbf{e}_1\,(-e^{u_2}) - e^{u_2}\,\mathbf{e}_2(-e^{u_2}) + \mathbf{e}_3\,(e^{u_2}u_3)\right]$$

$$\nabla\times\mathbf{F} = \frac{1}{e^{2u_2}}\left[-e^{2u_2}\,\mathbf{e}_1 + e^{2u_2}\,\mathbf{e}_2 + e^{u_2}u_3\,\mathbf{e}_3\right]$$

$$\nabla\times\mathbf{F} = \frac{1}{e^{u_2}}\left[-e^{u_2}\,\mathbf{e}_1 + e^{u_2}\,\mathbf{e}_2 + u_3\,\mathbf{e}_3\right]$$

12. $x = \frac{1}{2}(u^2-v^2),\ y = uv,\ z = z$

where $-\infty < u < \infty,\ v \geq 0,\ -\infty < z < \infty$ This is parabolic cylindrical coordinate system

$h_u = ||\frac{\partial \mathbf{R}}{\partial u}|| = ||u\mathbf{i}+v\mathbf{j}|| = \sqrt{u^2+v^2}$

$h_v = ||\frac{\partial \mathbf{R}}{\partial v}|| = ||-v\mathbf{i}+u\mathbf{j}|| = \sqrt{v^2+u^2}$

$h_z = ||\frac{\partial \mathbf{R}}{\partial z}|| = ||\mathbf{k}|| = 1$

13. For the previous problem: $dV = h_1h_2h_3du_1\,du_2\,du_3 = (u^2+v^2)du\,dv\,dz$

4.1

Problems: 1, 3, 6, 10, 14, 18

1. $\mathbf{F} = x^2\mathbf{i} + \mathbf{j} + yz\mathbf{k}$ and the curve C is

$$x = t,\ y = 2t^2,\ z = 3t,\ 0 \le t \le 1$$

Therefore $dx = dt,\ dy = 4tdt,\ dz = 3dt$ and the integral is

$$\int_C (x^2dx+dy+yzdz) = \int_0^1 [t^2+4t+2t^2(3t)(3)]dt = \int_0^1 (t^2+4t+18t^3)dt = \frac{1}{3}t^3+2t^2+18\left(\frac{1}{4}t^4\right)\Big|_0^1$$

$$\int_C (x^2dx + dy + yzdz) = \frac{1}{3} + 2 + \frac{9}{2} = \frac{41}{6}$$

3. Integrate $\int_C (x\mathbf{i} - y\mathbf{j} + z\mathbf{k}) \cdot d\mathbf{R}$ from $(1,0,0)$ to $(1,0,4)$
   a. along line joining points $x = 1,\ y = 0,\ z = 4t,\ 0 \le t \le 1$
   Therefore $dx = 0,\ dy = 0,\ dz = 4dt$ and the integral is

$$\int_C (x\mathbf{i} - y\mathbf{j} + z\mathbf{k}) \cdot d\mathbf{R} = \int_0^1 (4t)(4)dt = 16\left(\frac{1}{2}t^2\right)\Big|_0^1 = 8$$

   b. along $x = \cos(2\pi t),\ y = \sin(2\pi t),\ z = 4t,\ 0 \le t \le 1$
   Therefore $dx = -2\pi\sin(2\pi t)dt,\ dy = 2\pi\cos(2\pi t)dt,\ z = 4dt,$

$$\int_C (x\mathbf{i} - y\mathbf{j} + z\mathbf{k}) \cdot d\mathbf{R} = \int_0^1 \left[(\cos(2\pi t)(-2\pi\sin(2\pi t)) - \sin(2\pi t)(2\pi\cos(2\pi t)) + 4t(4)\right] dt$$

The first two terms add up and we have

$$\int_C (x\mathbf{i} - y\mathbf{j} + z\mathbf{k}) \cdot d\mathbf{R} = \int_0^1 \left[\underbrace{-4\pi\sin(2\pi t)\cos(2\pi t)}_{\text{substitute } u=\sin(2\pi t)} + 16t\right] dt = -\sin^2(2\pi t) + 8t^2\Big|_0^1 = 0 + 8 = 8$$

6. $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + xyz^2\mathbf{k}$ and the curve C is

$$x^2 - 2x + y^2 = 2,\ z = 1$$

Note that we have a circle centered at $(1,0)$ with radius $\sqrt{3}$ in a plane parallel to xy coordinate plane.

Let's parametrize the circle $x = \sqrt{3}\cos\theta + 1,\ y = \sqrt{3}\sin\theta,\ z = 1,\ 0 \le \theta \le 2\pi$ so $dx = -\sqrt{3}\sin\theta d\theta,\ dy = \sqrt{3}\cos\theta d\theta,\ dz = 0$

The integral is

$$\int_0^{2\pi} \left[\sqrt{3}\sin\theta(-\sqrt{3}\sin\theta) + (\sqrt{3}\cos\theta + 1)(\sqrt{3}\cos\theta) + 0\right] d\theta = \int_0^{2\pi} \left[\underbrace{-3\sin^2\theta + 3\cos^2\theta}_{=3\cos(2\theta)} + \sqrt{3}\cos\theta)\right] d\theta$$

$$= \frac{3}{2}\sin(2\theta) + \sqrt{3}\sin\theta\Big|_0^{2\pi} = 0$$

10. Evaluate $\int_C \Big[(y + yz\cos(xyz))dx + (x^2 + xz\cos(xyz))dy + (z + xy\cos(xyz))dz\Big]$ along the ellipse

$$x = 2\cos\theta,\ y = 3\sin\theta,\ z = 1,\ \ 0 \le \theta \le 2\pi$$

The last term vanishes since $z = 1$. Now $dx = -2\sin\theta d\theta$, $dy = 3\cos\theta d\theta$, $dz = 0$. Substituting in the integral we have

$$\begin{aligned}
&\int_0^{2\pi} \left(3\sin\theta + 3\sin\theta(1)(\cos(\underbrace{6\sin\theta\cos\theta}_{3\sin 2\theta})\right)(-2\sin\theta)d\theta \\
&+ \int_0^{2\pi} \left(4\cos^2\theta + 2\cos\theta(\cos(\underbrace{6\sin\theta\cos\theta}_{3\sin 2\theta})\right)(3\cos\theta)d\theta \\
&= \int_0^{2\pi} \Big(-6\sin^2\theta\cos(3\sin(2\theta))\Big)\,d\theta + \int_0^{2\pi} \Big(6\cos^2\theta\cos(3\sin(2\theta)\Big)\,d\theta \\
&+ \int_0^{2\pi} \Big(-6\sin^2\theta + 12\cos^3\theta\Big)\,d\theta \\
&= \underbrace{\int_0^{2\pi} (6\cos(2\theta)\cos(3\sin(2\theta)))\,d\theta}_{=0 \text{ Substitute } u=3\sin(2\theta)} + \int_0^{2\pi} \left(-6\ \underbrace{\sin^2\theta}_{(1-\cos(2\theta))/2}\right)d\theta + \int_0^{2\pi} 12\ \ \underbrace{(1-\sin^2\theta)}_{\text{Substitute } u=\sin\theta}\ \ \cos\theta d\theta \\
&= 0 + \left[-3\theta + 3\left(\frac{1}{2}\right)\sin(2\theta)\right]\Bigg|_0^{2\pi} + \Big[12\sin\theta - 4\sin^3\theta\Big]\Bigg|_0^{2\pi} \\
&= -6\pi
\end{aligned}$$

14. $\mathbf{F} = \omega \times \mathbf{R}$ where ω is constant. Therefore $\mathbf{F}$ is perpendicular to $\mathbf{R}$ and so $\mathbf{F} \cdot d\mathbf{R} = 0$ and $\int_C \mathbf{F} \cdot d\mathbf{R} = 0$

18. a. $\mathbf{F} = \frac{x^2}{y}\mathbf{i} + y\mathbf{j} + \mathbf{k}$ Find the flow lines through $(1, 1, 0)$

$$\frac{dx}{\mathbf{F}_1} = \frac{dy}{\mathbf{F}_2} = \frac{dz}{\mathbf{F}_3}$$

$$\underbrace{\frac{y\,dx}{x^2} = \frac{dy}{y}}_{\text{solve this first}} \quad = \frac{dz}{1}$$

$$-\frac{1}{x} = -\frac{1}{y} - C$$

at $(1, 1, 0)$ we have $C = 0$, so

$$y = x$$

Now solve the other one

$$\frac{dy}{y} = dz$$

$$\ln y = z + K$$

at $(1,1,0)$ we have $K = 0$, so

$$\ln y = z$$

Let's parametrize it

$$x = t,\ \ y = t,\ \ z = \ln t$$

b. At $(e, e, 1)$ then $t = e$ and $x = y = e,\ \ z = \ln e = 1$ which is on flow line.

c. $\displaystyle\int_{(1,1,0)}^{(e,e,1)} \left(\frac{x^2}{y}\mathbf{i} + y\mathbf{j} + \mathbf{k}\right)\cdot d\mathbf{R} = \int_{(1,1,0)}^{(e,e,1)} \left(\frac{x^2}{y}dx + ydy + dz\right)$

The flow line is $x = t,\ \ y = t,\ \ z = \ln t$ and so $dx = dt,\ \ dy = dt,\ \ dz = \frac{1}{t}dt$ and the integral becomes

$$= \int_1^e \left(\frac{t^2}{t} + t + \frac{1}{t}\right) dt = \left(\frac{1}{2}t^2 + \frac{1}{2}t^2 + \ln t\right)\Big|_1^e = \frac{1}{2}e^2 - \frac{1}{2} + \frac{1}{2}e^2 - \frac{1}{2} + 1 - 0 = e^2$$

4.3

Problems: 2, 4–7

2. a. $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= -y \\ \frac{\partial \phi}{\partial y} &= x \\ \frac{\partial \phi}{\partial z} &= 0\end{aligned}$$

Solving the first equation we have $\phi = -yx + p(y,z)$. Substitute this ϕ in the second equation we have $-x + \dfrac{\partial p(y,z)}{\partial y} = x$ which is **not** possible since $p(y,z)$ is not a function of x and therefore the field is not conservative.

b. $\mathbf{F} = y\mathbf{i} + y(x-1)\mathbf{j}$

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= y \\ \frac{\partial \phi}{\partial y} &= y(x-1) \\ \frac{\partial \phi}{\partial z} &= 0\end{aligned}$$

Solving the first equation we have $\phi = yx + p(y,z)$. Substitute this ϕ in the second equation we have $x + \dfrac{\partial p(y,z)}{\partial y} = y(x-1)$ which is **not** possible since $p(y,z)$ is not a function of x and therefore the field is not conservative.

c. $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + x^2\mathbf{k}$

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= y \\ \frac{\partial \phi}{\partial y} &= x \\ \frac{\partial \phi}{\partial z} &= x^2\end{aligned}$$

Solving the first equation we have $\phi = yx + p(y,z)$. Substitute this ϕ in the second equation we have $x + \dfrac{\partial p(y,z)}{\partial y} = x$ which yields $p(y,z) = g(z)$ and $\phi = yx + g(z)$. Now substitute in the third equation above $x^2 = g'(z)$ and therefore the field is not conservative.

d. $\mathbf{F} = z\mathbf{i} + z\mathbf{j} + (y-1)\mathbf{k}$

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= z \\ \frac{\partial \phi}{\partial y} &= z \\ \frac{\partial \phi}{\partial z} &= y - 1\end{aligned}$$

Solving the first equation we have $\phi = zx + p(y,z)$. Substitute this ϕ in the second equation we have $x + \dfrac{\partial p(y,z)}{\partial y} = z$ which yields $p(y,z) = yz + g(z)$ and $\phi = yz + zx + g(z)$. Now substitute in the third equation above $y - 1 = y + x + g'(z)$ and therefore the field is not conservative.

e. $\mathbf{F} = \dfrac{x}{x^2+y^2}\mathbf{i} + \dfrac{x}{x^2+y^2}\mathbf{j}$

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= \frac{x}{x^2+y^2} \\ \frac{\partial \phi}{\partial y} &= \frac{x}{x^2+y^2}\end{aligned}$$

Solving the second equation we have $\phi = \tan^{-1}\dfrac{y}{x} + p(x)$. Substitute this ϕ in the first equation we have $\dfrac{\frac{-y}{x^2}}{1+\frac{y^2}{x^2}} + p'(x) = \dfrac{x}{x^2+y^2}$ which yields $p'(x) = \frac{x+y}{x^2+y^2}$ and therefore the field is not conservative.

4. Evaluate $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{R}$ where $\mathbf{F} = \dfrac{-y\mathbf{i} + x\mathbf{j}}{x^2+y^2}$ and the curve C is given by $x^2 + y^2 = r^2$

Using polar coordinates $x = r\cos\theta,\ y = r\sin\theta$, so $dx = -r\sin\theta d\theta$ and $dy = r\cos\theta d\theta$ and the integral is

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \int_0^{2\pi} \left[\frac{-r\sin\theta}{r^2}(-r\sin\theta d\theta) + \frac{r\cos\theta}{r^2}(r\cos\theta d\theta)\right] = \int_0^{2\pi} d\theta = 2\pi$$

5. $\tan^{-1}\left(\dfrac{y}{x}\right)$ is a multiple valued function.

6. $\mathbf{F} = (y + z\cos(xz))\mathbf{i} + x\mathbf{j} + x\cos(xz)\mathbf{k}$

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= y + z\cos(xz)\\ \frac{\partial\phi}{\partial y} &= x\\ \frac{\partial\phi}{\partial z} &= x\cos(xz)\end{aligned}$$

Solving the second equation we have $\phi = xy + p(x,z)$. Substitute this ϕ in the first equation we have $y + z\cos(xz) = y + \dfrac{\partial p(x,z)}{\partial x}$ which yields $\dfrac{\partial p(x,z)}{\partial x} = z\cos(xz)$ and $p(x,z) = \sin(xz) + g(z)$ and $\phi = xy + \sin(xz) + g(z)$. Now substitute in the third equation above $x\cos(xz) = 0 + x\cos(xz) + g'(z)$ and therefore $g(z)$ is a constant and the field is conservative with a potential $\phi = xy + \sin(xz)$.

7. $\mathbf{F} = 2xy\mathbf{i} + (x^2 + z)\mathbf{j} + y\mathbf{k}$

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= 2xy\\ \frac{\partial\phi}{\partial y} &= x^2 + z\\ \frac{\partial\phi}{\partial z} &= y\end{aligned}$$

Solving the first equation we have $\phi = x^2y + p(y,z)$. Substitute this ϕ in the second equation we have $x^2 + z = x^2 + \dfrac{\partial p(y,z)}{\partial y}$ which yields $\dfrac{\partial p(y,z)}{\partial y} = z$ and $p(y,z) = yz + g(z)$ and $\phi = x^2y + yz + g(z)$. Now substitute in the third equation above $y = 0 + y + g'(z)$ and therefore $g(z)$ is a constant and the field is conservative with a potential $\phi = x^2y + yz$.

4.4

Problems: 1, 2, 6, 7, 9, 10

1. a. $\mathbf{F} = (12xy + yz)\mathbf{i} + (6x^2 + xz)\mathbf{j} + xy\mathbf{k}$.
Is $\nabla \times \mathbf{F} = 0$?

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 12xy + yz & 6x^2 + xz & xy \end{vmatrix} = \mathbf{i}(x - x) - \mathbf{j}(y - y) + \mathbf{k}(12x + z - (12x + z)) = 0$$

YES

b. $\mathbf{F} = ze^{xz}\mathbf{i} + xe^{xz}\mathbf{k}$.
Is $\nabla \times \mathbf{F} = 0$?

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ ze^{xz} & 0 & xe^{xz} \end{vmatrix} = \mathbf{i}(0 - 0) - \mathbf{j}(xze^{xz} + e^{xz} - (xze^{xz} + e^{xz})) + \mathbf{k}(0 - 0) = 0$$

YES

c. $\mathbf{F} = \sin x\mathbf{i} + y^2\mathbf{j} + e^z\mathbf{k}$.
Is $\nabla \times \mathbf{F} = 0$?

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sin x & y^2 & e^z \end{vmatrix} = \mathbf{i}(0 - 0) - \mathbf{j}(0 - 0) + \mathbf{k}(0 - 0) = 0$$

YES

d. $\mathbf{F} = 3x^2yz^2\mathbf{i} + x^3z^2\mathbf{j} + x^3yz\mathbf{k}$.
Is $\nabla \times \mathbf{F} = 0$?

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x^2yz^2 & x^3z^2 & x^3yz \end{vmatrix} = \mathbf{i}(x^3z - 2x^3z) - \mathbf{j}(3x^2yz - 6x^2yz) + \mathbf{k}(3x^2z^2 - 3x^2z^2) \neq 0$$

NO

e. $\mathbf{F} = \dfrac{2x}{x^2+y^2}\mathbf{i} + \dfrac{2y}{x^2+y^2}\mathbf{j} + 2z\mathbf{k}$.

Is $\nabla \times \mathbf{F} = 0$? This test is not applicable because of the singularity at $x = y = 0$ but we can try to find the potential.

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= \frac{2x}{x^2+y^2} \\ \frac{\partial \phi}{\partial y} &= \frac{2y}{x^2+y^2} \\ \frac{\partial \phi}{\partial z} &= 2z \end{aligned}$$

Solving the first equation we have

$$\phi = \ln\left(1 + \frac{x^2}{y^2}\right) + p(y,z).$$

Substitute this ϕ in the second equation we have

$$\frac{2y}{x^2+y^2} = \frac{1}{1+\frac{x^2}{y^2}}(-2x^2y^{-3}) + \frac{\partial p(y,z)}{\partial y}$$

which yields

$$\frac{\partial p(y,z)}{\partial y} = \frac{2y}{x^2+y^2} + \frac{2x^2}{y(x^2+y^2)}$$

or

$$\frac{\partial p(y,z)}{\partial y} = \frac{2y^2+2x^2}{y(x^2+y^2)} = \frac{2}{y}$$

and $p(y,z) = 2\ln|y| + g(z)$ and

$$\phi = \ln\left(1 + \frac{x^2}{y^2}\right) + 2\ln|y| + g(z).$$

Now substitute in the third equation above $2z = 0 + 0 + g'(z)$ and therefore $g(z) = z^2$ and the field is conservative with a potential $\phi = \ln\left(1 + \dfrac{x^2}{y^2}\right) + 2\ln|y| + z^2$.

There is a problem when $y = 0$ but not when $x = 0$

2. Given $\mathbf{F} = \nabla\phi$ and $\mathbf{G} = \nabla\psi$, is $\mathbf{F} + \mathbf{G}$ conservative?

The field $\mathbf{F} + \mathbf{G}$ is conservative with a potential $\phi + \psi$

6. $\mathbf{F} = \left(6x - 2e^{2x}y^2\right)\mathbf{i} - 2ye^{2x}\mathbf{j} + \cos z\mathbf{k}$ To check if the field is conservative,

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= 6x - 2e^{2x}y^2 \\
\frac{\partial \phi}{\partial y} &= -2ye^{2x} \\
\frac{\partial \phi}{\partial z} &= \cos z
\end{aligned}$$

Solving the first equation we have

$$\phi = 3x^2 - y^2e^{2x} + p(y, z).$$

Substitute this ϕ in the second equation we have

$$-2ye^{2x} = -2ye^{2x} + \frac{\partial p(y, z)}{\partial y}$$

which yields

$$\frac{\partial p(y, z)}{\partial y} = 0$$

and $p(y, z) = g(z)$ and

$$\phi = 3x^2 - y^2e^{2x} + g(z).$$

Now substitute in the third equation above $\cos z = 0 + 0 + g'(z)$ and therefore $g(z) = \sin z$ and the field is conservative with a potential $\phi = 3x^2 - y^2e^{2x} + \sin z$.

b. $\mathbf{R} = \underbrace{t}_{x}\,\mathbf{i} + \underbrace{(t-1)(t-2)}_{y}\,\mathbf{j} + \underbrace{\frac{\pi}{2}t^3}_{z}\,\mathbf{k}, \qquad 0 \le t \le 1$

Since $\mathbf{F}$ is conservative, the integral $\int_C \mathbf{F} \cdot d\mathbf{R}$ can be found by evaluating the potential at the end points. At $t = 1$, we have $x = 1$, $y = 0$, $z = \frac{\pi}{2}$. At $t = 0$, we have $x = 0$, $y = (-1)(-2) = 2$, $z = 0$. The values at $t = 1$ is $\phi(1, 0, \frac{\pi}{2}) = 4$ and at $t = 0$ $\phi(0, 2, 0) = -4$, therefore the integral is $4 - (-4) = 8$

c. $\mathbf{R} = \underbrace{\frac{1}{2}(t-1)}_{x}\,\mathbf{i} + \underbrace{(t)(3-t)}_{y}\,\mathbf{j} + \underbrace{\frac{\pi}{4}(t-1)}_{z}\,\mathbf{k}, \qquad 1 \le t \le 3$

Since $\mathbf{F}$ is conservative, the integral $\int_C \mathbf{F} \cdot d\mathbf{R}$ can be found by evaluating the potential at the end points. At $t = 1$, we have $x = 0$, $y = 2$, $z = 0$. At $t = 3$, we have $x = 1$, $y = 0$, $z = \frac{\pi}{2}$. The values at $t = 3$ is $\phi(1, 0, \frac{\pi}{2}) = 4$ and at $t = 1$ $\phi(0, 2, 0) = -4$, therefore the integral is $4 - (-4) = 8$

7. $\mathbf{F} = (1+x)e^{x+y}\mathbf{i} + \left(xe^{x+y} + 2y\right)\mathbf{j} - 2z\mathbf{k}$ To check if the field is conservative,

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= (1+x)e^{x+y}\\ \frac{\partial \phi}{\partial y} &= xe^{x+y} + 2y\\ \frac{\partial \phi}{\partial z} &= -2z\end{aligned}$$

Solving the first equation we have

$$\phi = e^{x+y} + e^y \underbrace{\int xe^x dx}_{(x-1)e^x} = e^{x+y} + (x-1)e^{x+y} + p(y,z) = xe^{x+y} + p(y,z).$$

Substitute this ϕ in the second equation we have

$$xe^{x+y} + 2y = xe^{x+y} + \frac{\partial p(y,z)}{\partial y}$$

which yields

$$\frac{\partial p(y,z)}{\partial y} = 2y$$

and $p(y,z) = y^2 + g(z)$ and

$$\phi = xe^{x+y} + y^2 + g(z).$$

Now substitute in the third equation above $-2z = 0 + 0 + g'(z)$ and therefore $g(z) = -z^2$ and the field is conservative with a potential $\phi = xe^{x+y} + y^2 - z^2$.

b. $\mathbf{R} = \underbrace{(1-t)e^t}_{x}\mathbf{i} + \underbrace{t}_{y}\mathbf{j} + \underbrace{2t}_{z}\mathbf{k}, \qquad 0 \le t \le 1$ and

$$\mathbf{G} = (1+x)e^{x+y}\mathbf{i} + \left(xe^{x+y} + 2z\right)\mathbf{j} - 2y\mathbf{k}$$

Note that $\mathbf{G}$ is **not** conservative, and it is messy to try and do it using brute force. But we can write $\mathbf{G} = \mathbf{F} + 2(z-y)\mathbf{j} + 2(z-y)\mathbf{k}$. Since $\mathbf{F}$ is conservative, the integral $\int_C \mathbf{F}\cdot d\mathbf{R}$ can be found by evaluating the potential at the end points. At $t=1$, we have $x=0,\ y=1,\ z=2$. At $t=0$, we have $x=1,\ y=0,\ z=0$. The values at $t=1$ is $\phi(0,1,2) = (1-4) = -3$ and at $t=0$ $\phi(1,0,0) = e$, therefore the integral is $-3-e$. Now to the integral over the rest

$$\int_C [2(z-y)dy + 2(z-y)dz] = \int_0^1 [2(2t-t)dt + 2(2t-t)(2dt)] = \int_0^1 6tdt = 3t^2\Big|_0^1 = 3$$

Therefore

$$\int_C \mathbf{G}\cdot d\mathbf{R} = -3 - e + 3 = -e$$

9. $\mathbf{F} = (2xyz + z^2 - 2y^2 + 1)\mathbf{i} + (x^2z - 4xy)\mathbf{j} + (x^2y + 2xz - 2)\mathbf{k}$
To check if the field is conservative,

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= (2xyz + z^2 - 2y^2 + 1) \\ \frac{\partial \phi}{\partial y} &= (x^2z - 4xy) \\ \frac{\partial \phi}{\partial z} &= (x^2y + 2xz - 2) \end{aligned}$$

Solving the first equation we have

$$\phi = x^2yz + (z^2 - 2y^2 + 1)x + p(y, z).$$

Substitute this ϕ in the second equation we have

$$x^2z - 4xy = x^2z - 4xy + \frac{\partial p(y, z)}{\partial y}$$

which yields

$$\frac{\partial p(y, z)}{\partial y} = 0$$

and $p(y, z) = g(z)$ and

$$\phi = x^2yz + (z^2 - 2y^2 + 1)x + g(z).$$

Now substitute in the third equation above $x^2y + 2xz - 2 = x^2y + 2zx + g'(z)$ and therefore $g(z) = -2z$ and the field is conservative with a potential $\phi = x^2yz + (z^2 - 2y^2 + 1)x - 2z$.

b. $\mathbf{G} = \dfrac{x}{(x^2 + z^2)^2}\mathbf{i} + \dfrac{z}{(x^2 + z^2)^2}\mathbf{k}$

To check if this field is conservative we should not use the fact that $\nabla \times \mathbf{G} = 0$. The reason is that this is true except on the y axis (since then $x^2 + z^2 = 0$) and that is not a star shaped domain.

10. $\mathbf{F} = (15x^4 - 3x^2y^2)\mathbf{i} - 2x^3y\mathbf{j}$ To check if the field is conservative,

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= (15x^4 - 3x^2y^2) \\ \frac{\partial \phi}{\partial y} &= -2x^3y \end{aligned}$$

Solving the first equation we have

$$\phi = 3x^5 - x^3y^2 + p(y).$$

Substitute this ϕ in the second equation we have

$$-2x^3y = -2x^3y + p'(y)$$

which yields

$$p'(y) = 0$$

and $p(y) = c$ and

$$\phi = 3x^5 - x^3y^2.$$

Therefore the field is conservative. To evaluate

$$\int_{(0,0)}^{(1,2)} \mathbf{F} \cdot d\mathbf{R} = \phi(1,2) - \phi(0,0) = (3-4) - 0 = -1$$

4.6

Problems: 1, 3–6

1. The surface S is given by $x = u^2,\ y = \sqrt{2}\,uv,\ z = v^2$

$$d\mathbf{S} = \frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v} dudv$$

$$\frac{\partial \mathbf{R}}{\partial u} = 2u\mathbf{i} + \sqrt{2}\,v\mathbf{j}$$

$$\frac{\partial \mathbf{R}}{\partial v} = \sqrt{2}\,u\mathbf{j} + 2v\mathbf{k}$$

$$\frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & \sqrt{2}\,v & 0 \\ 0 & \sqrt{2}\,u & 2v \end{vmatrix} = \mathbf{i}\begin{vmatrix} \sqrt{2}\,v & 0 \\ \sqrt{2}\,u & 2v \end{vmatrix} - 2u\begin{vmatrix} \mathbf{j} & \mathbf{k} \\ \sqrt{2}\,u & 2v \end{vmatrix} = 2\sqrt{2}\,v^2\mathbf{i} - 4uv\mathbf{j} + 2\sqrt{2}\,u^2\mathbf{k}$$

$$d\mathbf{S} = \left(2\sqrt{2}\,v^2\mathbf{i} - 4uv\mathbf{j} + 2\sqrt{2}\,u^2\mathbf{k}\right)\, dudv$$

$$||d\mathbf{S}|| = \sqrt{4(2)v^4 + 16u^2v^2 + 4(2)u^4}\, dudv = \sqrt{8v^4 + 16u^2v^2 + 8u^4}\, dudv = \sqrt{8(u^2+v^2)^2}\, dudv$$

$$||d\mathbf{S}|| = 2\sqrt{2}(u^2+v^2)\, dudv$$

3. $x = a\cos u,\ y = a\sin u,\ z = v$

$$d\mathbf{S} = \frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v} dudv$$

$$\frac{\partial \mathbf{R}}{\partial u} = -a\sin u\mathbf{i} + a\cos u\mathbf{j}$$

$$\frac{\partial \mathbf{R}}{\partial v} = \mathbf{k}$$

$$\frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin u & a\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = a\cos u\mathbf{i} + a\sin u\mathbf{j}$$

$$d\mathbf{S} = (a\cos u\mathbf{i} + a\sin u\mathbf{j})\, dudv$$

$$||d\mathbf{S}|| = \sqrt{a^2\cos^2 u + a^2\sin^2 u}\, dudv = a\, dudv$$

4. $z = x^2 + y^2$ so $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + (x^2+y^2)\mathbf{k}$

$$d\mathbf{S} = \frac{\partial \mathbf{R}}{\partial x} \times \frac{\partial \mathbf{R}}{\partial y} dxdy$$

$$\frac{\partial \mathbf{R}}{\partial x} = \mathbf{i} + 2x\mathbf{k}$$

$$\frac{\partial \mathbf{R}}{\partial y} = \mathbf{j} + 2y\mathbf{k}$$

$$\frac{\partial \mathbf{R}}{\partial x} \times \frac{\partial \mathbf{R}}{\partial y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} = -2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$$

$$d\mathbf{S} = (-2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k})\ dxdy$$

$$||d\mathbf{S}|| = \sqrt{4x^2 + 4y^2 + 1}\ dxdy$$

5. $x = u^2,\ y = uv,\ z = \frac{1}{2}v^2$

$$d\mathbf{S} = \frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v} dudv$$

$$\frac{\partial \mathbf{R}}{\partial u} = 2u\mathbf{i} + v\mathbf{j}$$

$$\frac{\partial \mathbf{R}}{\partial v} = u\mathbf{j} + v\mathbf{k}$$

$$\frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & v & 0 \\ 0 & u & v \end{vmatrix} = v^2\mathbf{i} - 2uv\mathbf{j} + 2u^2\mathbf{k}$$

$$d\mathbf{S} = \left(v^2\mathbf{i} - 2uv\mathbf{j} + 2u^2\mathbf{k}\right)\ dudv$$

$$||d\mathbf{S}|| = \sqrt{v^4 + 4u^2v^2 + 4u^4}\ dudv = \sqrt{(v^2 + 2v^2)^2}\ dudv = (v^2 + 2u^2)\ dudv$$

$$\begin{aligned} \int_0^3 \int_0^1 \left(v^2 + 2u^2\right)\ dudv &= \int_0^3 \left(v^2 u + \frac{2}{3}u^3\right)\Big|_{u=0}^{u=1} dv \\ &= \int_0^3 \left(\frac{2}{3} + v^2\right) dv = \left(\frac{2}{3}v + \frac{1}{3}v^3\right)\Big|_0^3 \\ &= 2 + 9 = 11 \end{aligned}$$

6. The triangle with vertices at $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. The normal to this plane is

$$\mathbf{n} = (\mathbf{j} - \mathbf{k}) \times (\mathbf{i} - \mathbf{j}) = -\mathbf{i} - \mathbf{j} - \mathbf{k}$$

Try writing the equation of the plane through those three points. The unit normal is then

$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}$$

b. $\cos\gamma$ for this vector is (γ is the angle with z axis),

$$\cos\gamma = \frac{\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}} \cdot \mathbf{k}}{1 \cdot 1} = \frac{1}{\sqrt{3}}$$

c. $\displaystyle \int\int \frac{dxdy}{|\cos\gamma|} = \int_0^1 \int_0^{1-y} \frac{dxdy}{\frac{1}{\sqrt{3}}}$

d. $\displaystyle \int_0^1 \sqrt{3}\, x \Big|_0^{1-y} dy = \int_0^1 \sqrt{3}\,(1-y)dy = \sqrt{3}\,\left(y - \frac{1}{2}y^2\right)\Big|_0^1 = \sqrt{3}\left(1 - \frac{1}{2}\right) = \frac{\sqrt{3}}{2}$

4.7

Problems: 1, 2d, 5, 6, 10, 15, 19 (Also worked 11 and 20)

1. $\mathbf{F} = z\mathbf{k}$ on a cylinder with base on xy plane (disk or radius 3 centered at origin) and height of 2.

On the lateral surface $\mathbf{F} \cdot \mathbf{n} = 0$
On the top surface $\mathbf{F} \cdot \mathbf{n} = z$
On the bottom surface $\mathbf{F} \cdot \mathbf{n} = -z$

$$\int\int_{top} \underbrace{z}_{=2} dS - \int\int_{bottom} \underbrace{z}_{=0} dS = 2 \underbrace{\int\int_{top} dS}_{=\pi r^2 = 9\pi} = 18\pi$$

2. d. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ on the surface of a cube bounded by the planes

$$x = \pm 1, \; y = \pm 1, \; z = \pm 1$$

For $x = 1$ we have $\mathbf{F} \cdot \mathbf{n} = (x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}) \cdot \mathbf{i} = x^2 = 1$
For $x = -1$ we have $\mathbf{F} \cdot \mathbf{n} = (x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}) \cdot (-\mathbf{i}) = (-x^2) = -1$
For $y = 1$ we have $\mathbf{F} \cdot \mathbf{n} = (x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}) \cdot \mathbf{j} = y^2 = 1$
For $y = -1$ we have $\mathbf{F} \cdot \mathbf{n} = (x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}) \cdot (-\mathbf{j}) = -y^2 = -1$
For $z = 1$ we have $\mathbf{F} \cdot \mathbf{n} = (x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}) \cdot \mathbf{k} = z^2 = 1$
For $z = -1$ we have $\mathbf{F} \cdot \mathbf{n} = (x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}) \cdot (-\mathbf{k}) = -z^2 = -1$
Now look at the integral over 2 opposing faces, eg for $x = 1$ and $x = -1$

$$\int\int_{x=1} dS = \underbrace{2(2)}_{area\ of\ square} = 4$$

$$\int\int_{x=-1} (-1) dS = - \underbrace{2(2)}_{area\ of\ square} = -4$$

and the sum is zero. Similarly for the other 2 pairs of faces.

5. $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (z^2 - 1)\mathbf{k}$ on the surface bounded by the planes $z = 0, \; z = 1$ and the cylinder $x^2 + y^2 = a^2$

On the bottom $\mathbf{n} = -\mathbf{k}$ and $\mathbf{F} \cdot \mathbf{n} = -(z^2 - 1)\Big|_{z=0} = 1$ and the integral is

$$\int\int_{bottom} \mathbf{F} \cdot \mathbf{n} dS = \pi a^2$$

On the top $\mathbf{n} = \mathbf{k}$ and $\mathbf{F} \cdot \mathbf{n} = (z^2 - 1)\Big|_{z=1} = 0$ and the integral is

$$\int\int_{top} \mathbf{F} \cdot \mathbf{n} dS = 0$$

On the lateral surface $\mathbf{n} = \dfrac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}} = \dfrac{x\mathbf{i} + y\mathbf{j}}{a}$ and

$\mathbf{F}\cdot\mathbf{n} = (x\mathbf{i} + y\mathbf{j} + (z^2-1)\mathbf{k}) \cdot \dfrac{x\mathbf{i} + y\mathbf{j}}{a} = \dfrac{x^2+y^2}{a}\Big|_{lateral} = a$

and the integral is

$$\int\int_{lateral} \mathbf{F}\cdot\mathbf{n}dS = a \underbrace{\int\int_{lateral} dS}_{=2\pi a} = 2\pi a^2$$

Therefore the total area is $2\pi a^2 + \pi a^2 = 3\pi a^2$

6. $\mathbf{F} = y\mathbf{i} + \mathbf{k}$ on the surface of a box without a top bounded by coordinate planes and the plane through the points $(2,0,0)$, $(2,1,0)$, $(1,1,1)$, $(1,0,1)$

On the bottom $\mathbf{n} = -\mathbf{k}$ and $\mathbf{F}\cdot\mathbf{n} = -1$ and the integral is

$$\int\int_{bottom} \mathbf{F}\cdot\mathbf{n}dS = -\underbrace{2(1)}_{=area} = -2$$

On the back $\mathbf{n} = -\mathbf{i}$ and $\mathbf{F}\cdot\mathbf{n} = -y$ and the integral is

$$\int\int_{back} \mathbf{F}\cdot\mathbf{n}dS = -\int\int_{back} ydS = -\int_0^1\int_0^1 ydydz = -\frac{1}{2}$$

On the right $\mathbf{n} = \mathbf{j}$ and $\mathbf{F}\cdot\mathbf{n} = 0$ and the integral is

$$\int\int_{right} \mathbf{F}\cdot\mathbf{n}dS = 0$$

On the left $\mathbf{n} = -\mathbf{j}$ and $\mathbf{F}\cdot\mathbf{n} = 0$ and the integral is

$$\int\int_{left} \mathbf{F}\cdot\mathbf{n}dS = 0$$

On the front (slanted) face $\mathbf{n} = \dfrac{\mathbf{j}\times(-\mathbf{i}+\mathbf{k})}{1(\sqrt{2})} = \dfrac{\mathbf{i}+\mathbf{k}}{\sqrt{2}}$

(Another way find the equation of the plane)

Now $\mathbf{F}\cdot\mathbf{n} = (y\mathbf{i}+\mathbf{k})\cdot\dfrac{\mathbf{i}+\mathbf{k}}{\sqrt{2}} = \dfrac{y+1}{\sqrt{2}}$ and the integral is

$$\begin{aligned}
\int\int_{front} \mathbf{F}\cdot\mathbf{n}dS &= \int_0^1\int_1^2 \frac{y+1}{\sqrt{2}} \underbrace{\sqrt{2}}_{see\ below} dxdy \\
&= \int_0^1 x(y+1)\Big|_{x=1}^{x=2} = \int_0^2 [2(y+1)-(y+1)]dy \\
&= \frac{1}{2}y^2 + y\Big|_0^1 = \frac{1}{2} + 1 = \frac{3}{2}
\end{aligned}$$

Therefore the total is $-2-\frac{1}{2}+\frac{3}{2}=-1$

Now to evaluating $d\mathbf{S}$. First we parametrize the plane $x=u,\ y=v,\ z=2-u$ therefore

$$\frac{\partial \mathbf{R}}{\partial u}=\mathbf{i}-\mathbf{k}$$

$$\frac{\partial \mathbf{R}}{\partial v}=\mathbf{j}$$

$$\frac{\partial \mathbf{R}}{\partial u}\times\frac{\partial \mathbf{R}}{\partial v}=\mathbf{i}+\mathbf{k}$$

$$dS=||\frac{\partial \mathbf{R}}{\partial u}\times\frac{\partial \mathbf{R}}{\partial v}||dudv=\sqrt{2}dxdy$$

Another way is to use the divergence theorem (remember that we need D to be closed)

$$\int\int_S \mathbf{F}\cdot d\mathbf{S}=\int\int\int_D \underbrace{\nabla\cdot\mathbf{F}}_{=0}dV-\underbrace{\int\int_{top}\mathbf{F}\cdot d\mathbf{S}}_{=1}=-1$$

Because on the top $\mathbf{n}=\mathbf{k}$ and $\mathbf{F}\cdot\mathbf{n}=1$

10. $\mathbf{F}=(x+1)\mathbf{i}-(2y+1)\mathbf{j}+z\mathbf{k}$ and the surface is the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ The normal is $(-\mathbf{i}+\mathbf{k})\times(-\mathbf{i}+\mathbf{j})=-\mathbf{i}-\mathbf{j}-\mathbf{k}$ and to get the unit outward normal

$$\mathbf{n}=\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$$

$$\mathbf{F}\cdot\mathbf{n}=\frac{(x+1)-(2y+1)+z}{\sqrt{3}}=\frac{x-2y+z}{\sqrt{3}}$$

$$\cos\gamma=\mathbf{n}\cdot\mathbf{k}=\frac{1}{\sqrt{3}}$$

$$\begin{aligned}
\int\int \mathbf{F}\cdot\mathbf{n}dS &= \int_0^1\int_0^{1-x}\frac{(x-2y+z)\frac{1}{\sqrt{3}}}{\frac{1}{\sqrt{3}}}dydx \\
&= \int_0^1\int_0^{1-x}(1-3y)dydx=\int_0^1\left(y-\frac{3}{2}y^2\right)\Big|_0^{1-x}dx \\
&= \int_0^1\left[1-x-\frac{3}{2}(1-x)^2\right]dx=\left[x-\frac{x^2}{2}+\frac{3}{2}\frac{(1-x)^3}{3}\right]\Big|_0^1 \\
&= 1-\frac{1}{2}-\frac{1}{2}=0
\end{aligned}$$

11. $\mathbf{F}=x^2\mathbf{i}+y^2\mathbf{j}+z^2\mathbf{k}$. The cone is given by

$$\begin{aligned}
x&=u\\
y&=v\\
z&=\sqrt{u^2+v^2}
\end{aligned}$$

Therefore

$$\frac{\partial \mathbf{R}}{\partial u} = \mathbf{i} + \frac{u}{\sqrt{u^2+v^2}}\mathbf{k}$$
$$\frac{\partial \mathbf{R}}{\partial v} = \mathbf{j} + \frac{v}{\sqrt{u^2+v^2}}\mathbf{k}$$

and

$$\frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{u}{\sqrt{u^2+v^2}} \\ 0 & 1 & \frac{v}{\sqrt{u^2+v^2}} \end{vmatrix} = -\frac{u}{\sqrt{u^2+v^2}}\mathbf{i} - \frac{v}{\sqrt{u^2+v^2}}\mathbf{j} + \mathbf{k}$$

and

$$\|\frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v}\| = \sqrt{\frac{u^2}{u^2+v^2} + \frac{v^2}{u^2+v^2} + 1} = \sqrt{2}$$

Therefore

$$\mathbf{F} \cdot d\mathbf{S} = -\frac{u^3}{\sqrt{u^2+v^2}} - \frac{v^3}{\sqrt{u^2+v^2}} + u^2 + v^2$$

The domain of integration in the xy plane is in between the two disks of radius 1 and radius 2 (projecting the piece of the cone), therefore we should use polar coordinates with

$$u = r\cos\theta$$
$$v = r\sin\theta$$

and the integral becomes

$$\int_0^{2\pi} \int_1^2 \left(\frac{-r^3\cos^3\theta}{r} + \frac{-r^3\sin^3\theta}{r} + r^2 \right) r\, dr\, d\theta$$

Integrating over r gives $\displaystyle\int_1^2 r^3\, dr = \frac{r^4}{4}\Big|_1^2 = 4 - \frac{1}{4} = \frac{15}{4}$.

The integral over θ is $\displaystyle\int_0^{2\pi} (-cos^3\theta - \sin^3\theta + 1)\, d\theta$.

The only non zero contribution comes from the last term which is 2π.

Therefore we have $\displaystyle\int\int \mathbf{F}\cdot d\mathbf{S} = \frac{15}{4}2\pi = \frac{15\pi}{2}$.

15. $\mathbf{E} = -\nabla \left(||\mathbf{R}||^{-1} \right)$

a. Show that $\mathbf{E} = \dfrac{\mathbf{R}}{||\mathbf{R}||^3}$

Since $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ then $||\mathbf{R}|| = \sqrt{x^2+y^2+z^2}$ and $\dfrac{1}{||\mathbf{R}||} = (x^2+y^2+z^2)^{-1/2}$

Now find the gradient (using the symmetry we only need one of the partials)

$$\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x) = -x(x^2+y^2+z^2)^{-3/2}$$

similarly

$$\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2y) = -y(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2z) = -z\underbrace{(x^2+y^2+z^2)^{-3/2}}_{\frac{1}{||\mathbf{R}||^3}}$$

Therefore

$$-\nabla\left(||\mathbf{R}||^{-1}\right) = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{||\mathbf{R}||^3} \qquad \text{as required}$$

b. Evaluate $\int_C \mathbf{E}\cdot d\mathbf{R}$ where C is the line segment from $(0,1,0)$ to $(0,0,1)$ which is $z=1-y$ and $x=0$

$$\begin{aligned}
\int_C \mathbf{E}\cdot d\mathbf{R} &= \int \frac{\mathbf{R}\cdot d\mathbf{R}}{||\mathbf{R}||^3}\bigg|_{x=0} = \int \frac{ydy+(1-y)(-dy)}{(y^2+(1-y)^2)^{3/2}} \\
&= \int_1^0 \frac{(-1+2y)dy}{(1-2y+2y^2)^{3/2}} = \frac{1}{2}\int \frac{du}{u^{3/2}} = -u^{-1/2} \\
&= -(1-2y+2y^2)^{-1/2}\bigg|_0^1 = 1-(1-2+2)^{-1/2} = 0
\end{aligned}$$

where we used the substitution $u = 1-2y+2y^2$ and so $du = (-2+4y)dy$

c. Evaluate $\int\int_S \mathbf{E}\cdot d\mathbf{S}$ over the sphere $x^2+y^2+z^2=9$

$$\int\int_S \mathbf{E}\cdot d\mathbf{S} = \int\int_S \underbrace{\mathbf{E}}_{\frac{\mathbf{R}}{||\mathbf{R}||^3}} \cdot \underbrace{\mathbf{n}}_{\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}} dS = \int\int \frac{1}{||\mathbf{R}||^2}dS = \frac{1}{9}(4\pi 3^2) = 4\pi$$

Note that $||\mathbf{R}||$ on the sphere is exactly 3.

19. Given the hollow sphere $a \le r \le b$ with temperatures T_a, T_b.

a. Find the steady state temperature as a function of r. As in the cylindrical example: $\mathbf{Q}\cdot\mathbf{n} = -k\nabla T\cdot\mathbf{n} = -k\frac{dT}{dr}$ because of the spherical symmetry.

$$H = \int\int_S \mathbf{Q}\cdot\mathbf{n} = -k\frac{dT}{dr}\underbrace{\int\int_S dS}_{4\pi r^2} = -4k\pi r^2\frac{dT}{dr}$$

$$\int_a^b H\frac{dr}{r^2} = -4\pi k\int_{T_a}^{T_b} dT$$

$$-H\frac{1}{r}\bigg|_a^b = -4\pi k\left(T_b - T_a\right)$$

$$H\left(\frac{1}{a}-\frac{1}{b}\right) = -4\pi k\left(T_b-T_a\right)$$

$$H = 4\pi k\frac{T_b-T_a}{\frac{1}{b}-\frac{1}{a}}$$

Now substitute H in

$$H\int_a^r \frac{dr}{r^2} = -4\pi k \int_{T_a}^{T} dT$$

$$4\pi k \frac{T_b - T_a}{\frac{1}{b}-\frac{1}{a}}\left(-\frac{1}{r}\right)\Big|_a^r = -4\pi k\,(T - T_a)$$

$$\frac{T_b - T_a}{\frac{1}{b}-\frac{1}{a}}\left(\frac{1}{a}-\frac{1}{r}\right) = T_a - T$$

Solving for T we have

$$T = T_a + \frac{T_b - T_a}{\frac{1}{b}-\frac{1}{a}}\left(\frac{1}{r}-\frac{1}{a}\right)$$

b. If r is halfway between a and b, i.e. $r = \dfrac{a+b}{2}$ then

$$T_{\frac{a+b}{2}} = T_a + \frac{T_b - T_a}{\underbrace{\frac{1}{b}-\frac{1}{a}}_{\frac{a-b}{ab}}}\left(\underbrace{\frac{2}{a+b}-\frac{1}{a}}_{\frac{a-b}{a(a+b)}}\right)$$

$$T_{\frac{a+b}{2}} = T_a + (T_b - T_a)\underbrace{\frac{ab}{a-b}\frac{a-b}{a(a+b)}}_{\frac{b}{a+b}}$$

$$T_{\frac{a+b}{2}} = T_a\underbrace{\left(1-\frac{b}{a+b}\right)}_{\frac{a}{a+b}} + T_b\frac{b}{a+b}$$

$$T_{\frac{a+b}{2}} = T_a\frac{a}{a+b} + T_b\frac{b}{a+b} \neq \frac{T_a + T_b}{2}$$

20. Given the spiral ramp in cylindrical coordinates

$$\begin{aligned}\rho &= u\\ \theta &= \tfrac{\pi}{2} - v\\ z &= v\end{aligned}$$

We can write this in Cartesian coordinates as

$$\begin{aligned}x &= u\cos(\frac{\pi}{2} - v)\\ y &= u\sin(\frac{\pi}{2} - v)\\ z &= v\end{aligned}$$

Therefore

$$\frac{\partial \mathbf{R}}{\partial u} = \cos(\frac{\pi}{2} - v)\mathbf{i} + \sin(\frac{\pi}{2} - v)\mathbf{j}$$

$$\frac{\partial \mathbf{R}}{\partial v} = +u\sin(\frac{\pi}{2} - v)\mathbf{i} - u\cos(\frac{\pi}{2} - v)\mathbf{j} + \mathbf{k}$$

$$\begin{aligned}
\frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\frac{\pi}{2} - v) & \sin(\frac{\pi}{2} - v) & 0 \\ u\sin(\frac{\pi}{2} - v) & -u\cos(\frac{\pi}{2} - v) & 1 \end{vmatrix} \\
&= -\sin(\frac{\pi}{2} - v)\mathbf{i} - \cos(\frac{\pi}{2} - v)\mathbf{j} + \Big(-u\cos^2(\frac{\pi}{2} - v) - u\sin^2(\frac{\pi}{2} - v)\Big)\mathbf{k} \\
&= -\sin(\frac{\pi}{2} - v)\mathbf{i} - \cos(\frac{\pi}{2} - v)\mathbf{j} - u\mathbf{k}
\end{aligned}$$

Therefore

$$\|\frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v}\|^2 = \sin^2(\frac{\pi}{2} - v) + \cos^2(\frac{\pi}{2} - v) + u^2 = 1 + u^2$$

The surface area is then

$$\int_0^1 \int_0^2 \sqrt{1+u^2}\, dv\, du$$

The integration over v is easy and we have

$$\int_0^1 \int_0^2 \sqrt{1+u^2}\, dv\, du = 2\int_0^1 \sqrt{1+u^2}\, du$$

This integral can be found in the tables:

$$\int \sqrt{u^2+a^2} du = \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln|u + \sqrt{u^2+a^2}|$$

In our case $a = 1$ and we have

$$\int_0^1 \sqrt{1+u^2}\, du = \left.\frac{u}{2}\sqrt{u^2+1}\right|_0^1 + \left.\frac{1}{2}\ln|u+\sqrt{u^2+1}|\right|_0^1 = \frac{\sqrt{2}}{2} + \frac{1}{2}\ln(1+\sqrt{2})$$

Therefore

$$\int_0^1 \int_0^2 \sqrt{1+u^2}\, dv\, du = 2(\frac{\sqrt{2}}{2} + \frac{1}{2}\ln(1+\sqrt{2})) = \sqrt{2} + \ln(1+\sqrt{2})$$

Remark: We can use cylindrical coordinates

$$\mathbf{R} = \rho\mathbf{e}_\rho + z\mathbf{e}_z$$

Then

$$\frac{\partial \mathbf{R}}{\partial u} = \underbrace{\frac{\partial \rho}{\partial u}}_{=1}\mathbf{e}_\rho + \rho\frac{\partial \mathbf{e}_\rho}{\partial \theta}\underbrace{\frac{\partial \theta}{\partial u}}_{=0} + \underbrace{\frac{\partial z}{\partial u}}_{=0}\mathbf{e}_z + z\underbrace{\frac{\partial \mathbf{e}_z}{\partial u}}_{=0}$$

$$\frac{\partial \mathbf{R}}{\partial v} = \underbrace{\frac{\partial \rho}{\partial v}}_{=0} \mathbf{e}_\rho + \rho \underbrace{\frac{\partial \mathbf{e}_\rho}{\partial \theta}}_{\mathbf{e}_\theta} \underbrace{\frac{\partial \theta}{\partial v}}_{=-1} + \underbrace{\frac{\partial z}{\partial v}}_{=1} \mathbf{e}_z + z \underbrace{\frac{\partial \mathbf{e}_z}{\partial v}}_{=0}$$

Computing the cross product and its magnitude yield the same integral to evaluate.

4.8

Problems: 2–4, 6. Also worked number 5.

2. The volume in example 4.28 is 3

$$\begin{aligned}
\int_0^2\int_0^1\int_0^{1+x} 1dzdxdy &= \int_0^2\int_0^1 z\Big|_0^{1+x} dxdy \\
&= \int_0^2\int_0^1 (1+x)dxdy \\
&= \int_0^2 \left(x+\frac{1}{2}x^2\right)\Big|_0^1 dy \\
&= \int_0^2 \left(1+\frac{1}{2}\right)dy = \frac{3}{2}y\Big|_0^2 = 3
\end{aligned}$$

Three other ways

$$\int_0^1\int_0^2\int_0^{1+x} 1dzdydx$$
$$\int_0^1\int_0^{1+x}\int_0^2 1dydzdx$$
$$\int_0^1\int_0^2\int_0^1 1dxdydz + \int_1^2\int_0^2\int_{z-1}^1 1dxdydz$$

3. $\displaystyle\int_0^2\int_0^3\int_0^{\sqrt{9-y^2}} dxdydz$ where $x^2+y^2\le 9,\ 0\le z\le 2$ and $0\le y\le 3$. This is a quarter of a cylinder, see figure 13.

4. a. $\mathbf{F}=x^2\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ Compute $\displaystyle\int\int_S \mathbf{F}\cdot d\mathbf{S}$ over the surface of the cube of side 1 in the first quadrant.

For $x=1$ (front) we have $\mathbf{F}\cdot\mathbf{n}=(x^2\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot\mathbf{i}=x^2=1$
For $x=0$ (back) we have $\mathbf{F}\cdot\mathbf{n}=(x^2\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(-\mathbf{i})=(-x^2)=0$
For $y=1$ (right) we have $\mathbf{F}\cdot\mathbf{n}=(x^2\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot\mathbf{j}=y=1$
For $y=0$ (left) we have $\mathbf{F}\cdot\mathbf{n}=(x^2\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(-\mathbf{j})=-y=0$
For $z=1$ (top) we have $\mathbf{F}\cdot\mathbf{n}=(x^2\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot\mathbf{k}=z=1$
For $z=0$ (bottom) we have $\mathbf{F}\cdot\mathbf{n}=(x^2\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(-\mathbf{k})=-z=0$

The integral over 3 of the 6 faces vanishes. The other 3 integrals are just the surface area of those faces. Each face has a surface area of unity and therefore

$$\int\int_S dS = 1+1+1=3$$

b. $f=\nabla\cdot\mathbf{F}=2x+1+1=2+2x$. Now the volume integral

$$\int_0^1\int_0^1\int_0^1 (2+2x)dxdydz = \int_0^1\int_0^1 (2x+x^2)\Big|_0^1 dydz = \int_0^1\int_0^1 3dydz = 3$$

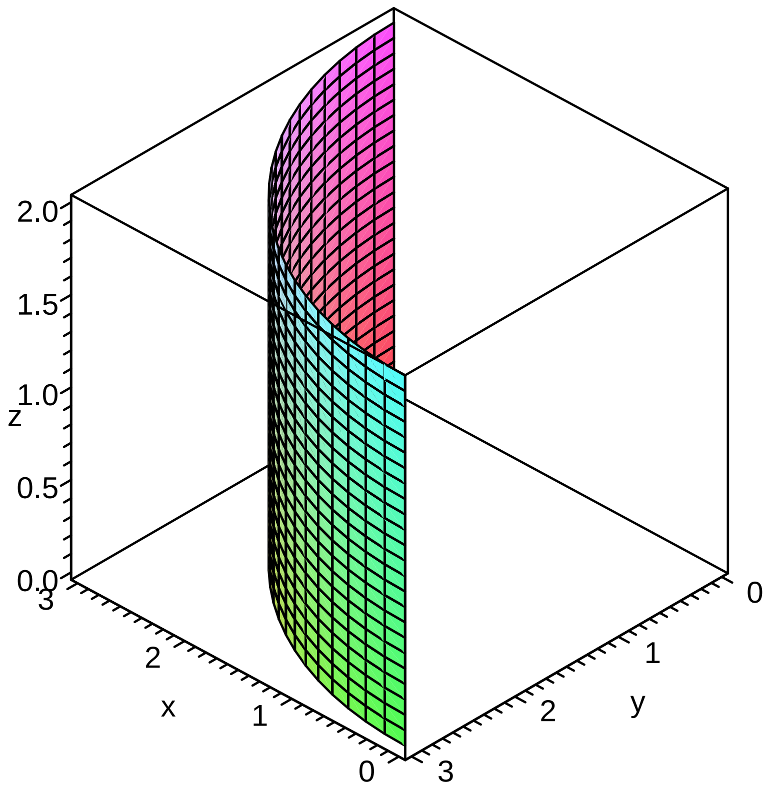

Figure 13: For Problem 3 of section 4.8

c. The answers are the same

d. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$
It is easy to see that we get the same answer both ways.

e. $\int\int\int_V \nabla \cdot \mathbf{F} dV = \int\int_S \mathbf{F} \cdot d\mathbf{S}$

5. The volume of V is v and $\mathbf{F} =< x, y, z >$.

a. $\int\int\int_V \nabla \cdot \mathbf{F} dV = \int\int\int_V 3dV = 3v$

b. $\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int\int_V \nabla \cdot \mathbf{F} dV$ based on problem 4. Therefore the surface integral is also $3v$.

6. The volume of the surface bounded by $z = e^{-(x^2+y^2)}$, the cylinder $x^2 + y^2 = 1$ and the plane $z = 0$

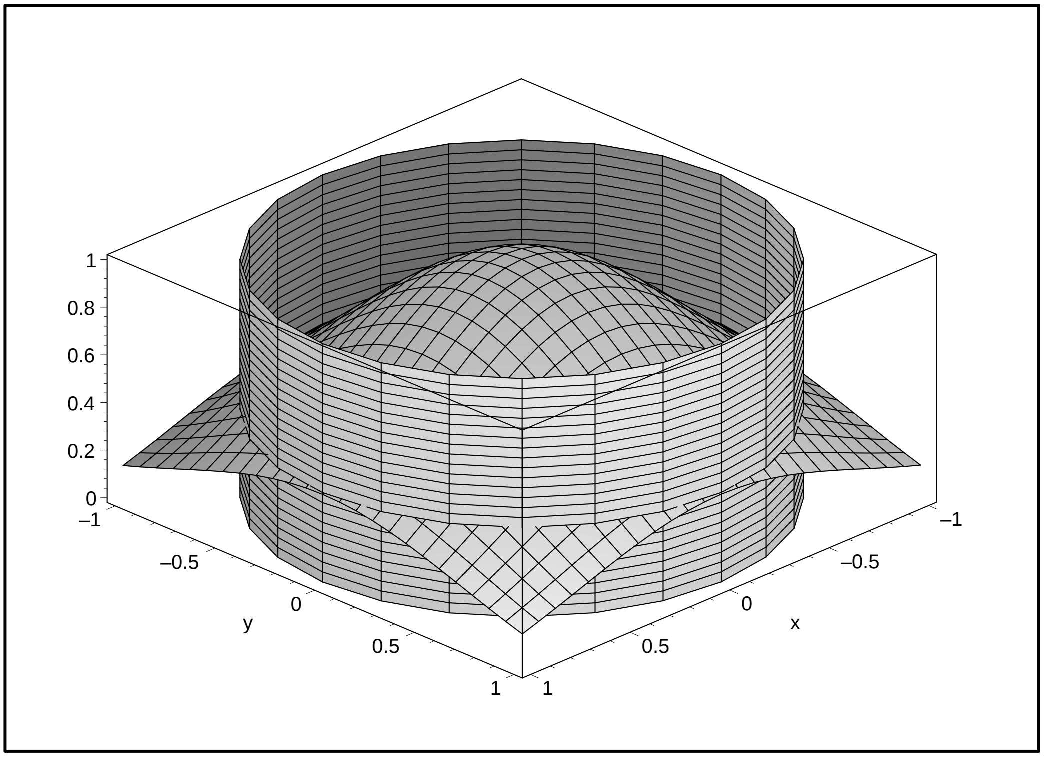

Figure 14: For Problem 6 of section 4.8

$$
\begin{aligned}
\textstyle\int\int\int_D \, dV &= \int_0^1 \int_0^{2\pi} \int_0^{e^{-r^2}} r dz d\theta dr = \int_0^1 \int_0^{2\pi} z\Big|_0^{e^{-r^2}} r d\theta dr \\
&= \int_0^1 \int_0^{2\pi} r e^{-r^2} d\theta dr = 2\pi \int_0^1 r e^{-r^2} dr \\
&= 2\pi \left(-\frac{1}{2} e^{-r^2}\right)\Big|_0^1 = -\pi \left(e^{-1} - 1\right) \\
&= \pi \left(1 - \tfrac{1}{e}\right)
\end{aligned}
$$

4.9

Problems: 3, 4, 6, 9, 14, 15, 17

3. a. $\mathbf{F} = x\mathbf{i} - y\mathbf{j}$

$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS = \int\int\int_D \underbrace{\nabla\cdot\mathbf{F}}_{=1-1=0}\,dV = \int\int\int_D 0dV = 0$$

b. $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (z^2-1)\mathbf{k}$ then $\nabla\cdot\mathbf{F} = 1+1+2z = 2+2z$
The integral over the cylinder of radius a and height 1 is

$$\int_0^{2\pi}\int_0^a\int_0^1 (2+2z)dz\,rdr\,d\theta = 2\int_0^{2\pi}\int_0^a \underbrace{\left(z+\frac{z^2}{2}\right)\Big|_0^1}_{=3/2} rdrd\theta = 3(2\pi)\frac{1}{2}r^2\Big|_0^a = 3\pi a^2$$

4. $\mathbf{F} = (x^2+xy)\mathbf{i} + (y^2+yz)\mathbf{j} + (z^2+zx)\mathbf{k}$ then $\nabla\cdot\mathbf{F} = 2x+y+2y+z+2z+x = 3(x+y+z)$

$$\int_{-1}^1\int_{-1}^1 \underbrace{\int_{-1}^1 3(x+y+z)dx}_{=3(\frac{x^2}{2}+xy+xz)\Big|_{x=-1}^{x=1}}\; dydz = \int_{-1}^1\int_{-1}^1 3(\frac{1}{2}+y+z-\frac{1}{2}+y+z)dydz$$

$$= \int_{-1}^1 \underbrace{\int_{-1}^1 6(y+z)dy}_{=6(\frac{y^2}{2}+yz)\Big|_{y=-1}^{y=1}}\; dz$$

$$= \int_{-1}^1 6(\frac{1}{2}+z-\frac{1}{2}+z)dz = \int_{-1}^1 12zdz$$

$$= 6z^2\Big|_{-1}^1 = 0$$

6. $\mathbf{F} = F_r(r)\mathbf{e}_r$ and $\nabla\cdot\mathbf{F} = r^m,\;\; m\geq 0$ Recall that

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2F_r(r)\right) = r^m$$

Therefore

$$\frac{\partial}{\partial r}\left(r^2F_r(r)\right) = r^{m+2}$$

$$r^2F_r(r) = \frac{r^{m+3}}{m+3}, \qquad \text{never mind the constant of integration}$$

Therefore

$$F_r(r) = \frac{r^{m+1}}{m+3}$$

b. $\displaystyle \int\int\int_D \underbrace{r^m}_{\nabla\cdot\mathbf{F}} dV = \int\int\int_D \nabla\cdot\mathbf{F}dV = \int\int_S \mathbf{F}\cdot\mathbf{n}dS$

Substituting $\displaystyle \int\int_S \frac{r^{m+1}}{m+3}\mathbf{e}_r\cdot\mathbf{n}dS = \frac{1}{m+3}\int\int_S r^{m+1}\mathbf{e}_r\cdot d\mathbf{S}$

9. a. $\mathbf{F} = x\mathbf{i}$

Use Stokes' theorem. $\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & 0 & 0 \end{vmatrix} = 0$ therefore $\displaystyle \int\int_S \nabla\times\mathbf{F}\cdot\mathbf{n}dS = 0$

Now to the line integral connectinting the points $(1,0)$ to $(0,1)$ to $(-1,0)$ to $(0,-1)$ and back to $(1,0)$. The segments are denoted by $C_1,\ldots,C_4$

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{R} &= \int_{C_1} xdx + \int_{C_2} xdx + \int_{C_3} xdx + \int_{C_4} xdx \\
&= \int_1^0 xdx + \int_0^{-1} xdx + \int_{-1}^0 xdx + \int_0^1 xdx \\
&= \frac{1}{2}x^2\Big|_1^0 + \frac{1}{2}x^2\Big|_0^{-1} + \frac{1}{2}x^2\Big|_{-1}^0 + \frac{1}{2}x^2\Big|_0^1 \\
&= 0 - \frac{1}{2} + \frac{1}{2} - 0 + 0 - \frac{1}{2} + \frac{1}{2} - 0 = 0
\end{aligned}$$

c. $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$

Use Stokes' theorem. $\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = 2\mathbf{k}$

and $\mathbf{n} = \mathbf{k}$ since the square is in the xy plane,

therefore $\displaystyle \int\int_S \nabla\times\mathbf{F}\cdot\mathbf{n}dS = 2\int\int_S dS = 2(2) = 4$ area of rombus is 2

Now to the line integral connectinting the points $(1,0)$ to $(0,1)$ to $(-1,0)$ to $(0,-1)$ and back to $(1,0)$. The segments are denoted by $C_1, \ldots, C_4$

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{R} &= \int_{C_1+C_2+C_3+C_4} (-ydx + xdy) \\
&= \int_{C_1} \underbrace{[-(1-x)dx + x(-dx)]}_{y=1-x,\ dy=-dx} + \int_{C_2} \underbrace{[(-x-1)dx + xdy]}_{y=x+1} + \int_{C_3} \underbrace{[(1+x)dx - xdx]}_{y=-1-x} \\
&\qquad + \textstyle\int_{C_4} \underbrace{[(1-x)dx + xdx]}_{y=x-1} \\
&= \int_1^0 (-1)dx + \int_0^{-1} (-1)dx + \int_{-1}^0 1dx + \int_0^1 1dx \\
&= -x\Big|_1^0 - x\Big|_0^{-1} + x\Big|_{-1}^0 + x\Big|_0^1 \\
&= -0 - (-1) - (-1) + 0 + 0 - (-1) + 1 - 0 = 4
\end{aligned}$$

Both integrals give the same answer.

14. Use Stokes' theorem to evaluate $\displaystyle\int_C \underbrace{(x\sin y\mathbf{i} - y\sin x\mathbf{j} + (x+y)z^2\mathbf{k})}_{=\mathbf{F}} \cdot d\mathbf{R}$

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x\sin y & -y\sin x & (x+y)z^2 \end{vmatrix} = \mathbf{i}(z^2-0) - \mathbf{j}(z^2-0) + \mathbf{k}(-y\cos x - x\cos y) \text{ so}$$

$\nabla\times\mathbf{F} = z^2\mathbf{i} - z^2\mathbf{j} - (x\cos y + y\cos x)\mathbf{k}$

The surface S is made up of two surfaces S_1 is a rectangle in the xz plane whose vertices are $(0,0,0), (\pi/2,0,0), (\pi/2,0,1), (0,0,1)$ and S_2 is a rectangle in the yz plane whose vertices are $(0,0,1), (0,\pi/2,1), (0,\pi/2,0), (0,0,0)$. Based on the orientation of the boundary curve (starting at $(0,0,0)$ going on the x axis and so on) we have the outward normal to S_1 is $\mathbf{n}_1 = -\mathbf{j}$ and the outward normal to S_2 is $\mathbf{n}_2 = -\mathbf{i}$. Therefore $\nabla\times\mathbf{F}\cdot\mathbf{n}_1 = z^2$ and $\nabla\times\mathbf{F}\cdot\mathbf{n}_2 = -z^2$ and

$$\int\!\!\int_S \nabla\times\mathbf{F}\cdot\mathbf{n}dS = \int\!\!\int_{S_1} \nabla\times\mathbf{F}\cdot\mathbf{n}_1 dS + \int\!\!\int_{S_2} \nabla\times\mathbf{F}\cdot\mathbf{n}_2 dS = \int\!\!\int_{S_1} z^2 dS + \int\!\!\int_{S_2} (-z^2) dS$$

On S_1 the parameters are x, z which $0 \le x \le \pi/2,\ \ 0 \le z \le 1$

$$\int\!\!\int_{S_1} z^2 dS = \int_0^{\pi/2} \underbrace{\int_0^1 z^2\,dz}_{=z^3/3\Big|_0^1=1/3}\ dx = \frac{1}{3}\left(\frac{\pi}{2}\right) = \frac{\pi}{6}$$

On S_2 the parameters are y, z which $0 \le y \le \pi/2, \;\; 0 \le z \le 1$

$$\int\int_{S_2} (-z^2) dS = \int_0^{\pi/2} \underbrace{\int_0^1 (-z^2)\, dz}_{= -z^3/3 \Big|_0^1 = -1/3} \, dy = -\frac{1}{3}\left(\frac{\pi}{2}\right) = \frac{\pi}{6}$$

and the sum is zero.

15. Use Stokes' theorem to evaluate $\int_C \underbrace{(ye^x \mathbf{i} + (x + e^x)\,\mathbf{j} + z^2\mathbf{k})}_{=\mathbf{F}} \cdot d\mathbf{R}$ along the curve C given by $x = 1 + \cos t, \; y = 1 + \sin t, \; z = 1 - \sin t - \cos t, \qquad 0 \le t \le 2\pi$

Note that C encloses an ellipse in three dimension (slanted ellipse on the plane $z = 3 - x - y$).

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ ye^x & x + e^x & z^2 \end{vmatrix} = \mathbf{i}(0-0) - \mathbf{j}(0-0) + \mathbf{k}(1 + e^x - e^x) = \mathbf{k}$$

so $\nabla \times \mathbf{F} = \mathbf{k}$ and we only need the z component of $d\mathbf{S}$

We parametrize S by $x = u, \; y = v, \; z = 3 - u - v$ where we have taken $u = 1 + \cos t$ and $v = 1 + \sin t$ and $\dfrac{\partial \mathbf{R}}{\partial u} = \mathbf{i} - \mathbf{k}$ and $\dfrac{\partial \mathbf{R}}{\partial v} = \mathbf{j} - \mathbf{k}$ so

$$\frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\int\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int\int_S \mathbf{k} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) du dv = \underbrace{\int_0^1 \int_0^{2\pi} r dr d\theta}_{\text{circle centered at } (1,1) \text{ with radius } 1} = \pi$$

Note that $(u-1)^2 + (v-1)^2 = 1$

Another way is to see if the field is conservative

$$\frac{\partial \phi}{\partial x} = ye^x$$

$$\frac{\partial \phi}{\partial y} = x + e^x$$

$$\frac{\partial \phi}{\partial z} = z^2$$

From the first $\phi = ye^x + g(y, z)$ differentiate with respect to y and compare to the second, we have $\dfrac{\partial g(y,z)}{\partial y} = 0$ (if we drop the term x) so $g(y, z) = h(z)$ and $\phi = ye^x + h(z)$. Differentiate this with respect to z and compare to the third we get $h'(z) = z^2$ or $h(z) = \dfrac{1}{3}z^3$ and

$$\phi = ye^x + \frac{1}{3}z^3$$

The field is not conservative but can be written as

$$\mathbf{F} = \nabla\phi + x\mathbf{j}$$

The line integral on the conservative part is zero (closed curve), so

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \underbrace{\int_C (\nabla\phi)\cdot d\mathbf{R}}_{=0} + \int_C (x\mathbf{j})\cdot d\mathbf{R} = \int_0^{2\pi} \underbrace{(1+\cos t)}_{=x}\underbrace{\cos t dt}_{=dy} = \underbrace{\sin t\Big|_0^{2\pi}}_{=0} + \int_0^{2\pi} \cos^2 t dt$$

Now

$$\int_0^{2\pi} \cos^2 t dt = \int_0^{2\pi} \left(\frac{1+\cos(2t)}{2}\right) = \left[\frac{1}{2}t + \frac{1}{4}\sin(2t)\right]\Big|_0^{2\pi} = \pi$$

so

$$\int_C \mathbf{F}\cdot d\mathbf{R} = \pi \qquad \text{exactly as before}$$

17. $\phi(x,y,z) = xyz + 5$ the surface S is $x^2+y^2+z^2=9$
Therefore $\nabla\phi = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and the normal to the sphere

$$\mathbf{n} = \frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{\sqrt{\underbrace{4x^2+4y^2+4z^2}_{=4(9)}}} = \frac{2(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})}{2(3)} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{3} \quad \text{since } x^2+y^2+z^2=9$$

$$\nabla\phi\cdot\mathbf{n} = xyz$$

Using spherical coordinates

$$\int_0^{2\pi}\int_0^{\pi} \underbrace{3\sin\phi\cos\theta}_{=x}\underbrace{3\sin\phi\sin\theta}_{=y}\underbrace{3\cos\phi}_{=z}\underbrace{9\sin\phi d\phi d\theta}_{dS} = 3^5\int_0^{2\pi}\underbrace{\int_0^{\pi}\sin^3\phi\cos\phi d\phi}_{=0,\text{ see below}}\sin\theta\cos\theta d\theta$$

Use $u=\sin\phi,\ du=\cos\phi d\phi$ so

$$\int_0^{\pi}\sin^3\phi\cos\phi d\phi = \frac{\sin^4\phi}{4}\Big|_0^{\pi} = 0$$

If we use the divergence theorem, $\nabla\cdot(\nabla\phi) = 0$ and so the answer is again zero.

4.10

Problem: 1

1. Given $S_t = x^2 + y^2 + z^2 = (vt)^2, \;\; z \geq 0$ and $\mathbf{F}(\mathbf{R}, \mathbf{t}) = \mathbf{R}t$. Find the time derivative of

$$\Phi(t) = \int\int_{S_t} \mathbf{F}(\mathbf{R}, \mathbf{t}) \cdot d\mathbf{S}$$

$$\frac{d\Phi}{dt} = \int\int_{S_t} \left(\underbrace{\frac{\partial \mathbf{F}}{\partial t}}_{=\mathbf{R}} + \underbrace{\nabla \cdot \mathbf{F}}_{=3t} \mathbf{v} \right) \cdot d\mathbf{S} + \int_{C_t} \underbrace{\mathbf{F} \times \mathbf{v}}_{\mathbf{R}\times\mathbf{v}t} \cdot d\mathbf{R}$$

Now

$$\Phi(t) = \int_0^{2\pi} \int_0^{\pi/2} (vt^2 \sin\phi \cos\theta \mathbf{i} + vt^2 \sin\phi \sin\theta \mathbf{j} + vt^2 \cos\phi \mathbf{k}) \cdot \left(\frac{\partial \mathbf{R}}{\partial \phi} \times \frac{\partial \mathbf{R}}{\partial \theta} \right) d\phi d\theta$$

In spherical coordinates $x = vt \sin\phi \cos\theta, \;\; y = vt \sin\phi \sin\theta, \;\; z = vt \cos\phi$ and therefore

$$\frac{\partial \mathbf{R}}{\partial \phi} = vt \cos\phi \cos\theta \mathbf{i} + vt \cos\phi \sin\theta \mathbf{j} - vt \sin\phi \mathbf{k}$$

$$\frac{\partial \mathbf{R}}{\partial \theta} = -vt \sin\phi \sin\theta \mathbf{i} + vt \sin\phi \cos\theta \mathbf{j}$$

$$\frac{\partial \mathbf{R}}{\partial \phi} \times \frac{\partial \mathbf{R}}{\partial \theta} = v^2 t^2 (\sin^2\phi \cos\theta \mathbf{i} + \sin^2\phi \sin\theta \mathbf{j} + \sin\phi \cos\phi \mathbf{k})$$

and the integrand becomes

$$v^3 t^4 \left(\underbrace{\sin^3\phi \cos^2\theta + \sin^3\phi \sin^2\theta}_{\sin^3\phi} + \sin\phi \cos^2\phi \right) d\phi d\theta =$$

$$v^3 t^4 \sin\phi \underbrace{\left(\sin^2\phi + \cos^2\phi \right)}_{=1} d\phi d\theta =$$

$$v^3 t^4 \sin\phi d\phi d\theta$$

Substitute in $\Phi(t)$, we have

$$\Phi(t) = \int_0^{2\pi} \int_0^{\pi/2} v^3 t^4 \sin\phi d\phi d\theta = v^3 t^4 2\pi \underbrace{\left(-\cos\phi \right) \Big|_0^{\pi/2}}_{=1} = 2\pi v^3 t^4$$

Differentiating

$$\frac{d\Phi}{dt} = 8\pi v^3 t^3$$

Now let us compute the surface and line integrals.

$$\int\int_{S_t} (\mathbf{R} + 3t\mathbf{v}) \cdot d\mathbf{S}$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \left(\underbrace{vt\sin\phi\cos\theta\mathbf{i} + vt\sin\phi\sin\theta\mathbf{j} + vt\cos\phi\mathbf{k}}_{\mathbf{R}}\right) \cdot \left(\frac{\partial\mathbf{R}}{\partial\phi} \times \frac{\partial\mathbf{R}}{\partial\theta}\right) d\phi d\theta$$

$$+ \int_0^{2\pi}\int_0^{\pi/2} \left(3\underbrace{t\mathbf{v}}_{=\mathbf{R}}\right) \cdot \left(\frac{\partial\mathbf{R}}{\partial\phi} \times \frac{\partial\mathbf{R}}{\partial\theta}\right) d\phi d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \left[v^3t^3\sin\phi + 3\underbrace{\mathbf{R} \cdot \left(\frac{\partial\mathbf{R}}{\partial\phi} \times \frac{\partial\mathbf{R}}{\partial\theta}\right)}_{= v^3t^3\sin\phi}\right] d\phi d\theta$$

$$= 4v^3t^3\int_0^{2\pi}\int_0^{\pi/2} \sin\phi d\phi d\theta = 8v^3t^3\pi$$

$$\int_{C_t} \left(\underbrace{\mathbf{R} \times \underbrace{\mathbf{v}}_{= \frac{d\mathbf{R}}{dt}}}_{=0}\right) t \cdot d\mathbf{R} = 0$$

Therefore

$$\frac{d\Phi}{dt} = \int\int_{S_t} \left(\underbrace{\frac{\partial\mathbf{F}}{\partial t}}_{=\mathbf{R}} + \underbrace{\nabla \cdot \mathbf{F}}_{=3t}\mathbf{v}\right) \cdot d\mathbf{S} + \int_{C_t} \underbrace{\mathbf{F} \times \mathbf{v}}_{\mathbf{R}\times\mathbf{v}t} \cdot d\mathbf{R} = 8v^3t^3\pi + 0 = 8v^3t^3\pi$$

5.1

Problems: 6–9

6. $\mathbf{F} = x^3\mathbf{i} + yx\mathbf{j} - x^3\mathbf{k}$

$$\nabla \cdot \mathbf{F} = 3x^2 + x$$

At the point $(3, 1, -2)$ the divergence is $\nabla \cdot \mathbf{F} = 3(3^2) + 3 = 30$.

7. $\mathbf{F} = 3x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\nabla \cdot \mathbf{F} = 3 + 1 + 1 = 5$$

This is the flux per unit volume and so the flux is $5v$, where v is the volume.

8. $\mathbf{F} = 3x^2\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\nabla \cdot \mathbf{F} = 6x + 1 + 1 = 6x + 2$$

The flux depends on x and so it does depend on location.

9. a. Sphere of radius 2 with a hole in the center of spherical shape and radius 1. The normal is in the direction of r on the outer sphere and $-r$ on the hole.

b. Use spherical coordinates with $\mathbf{n} = \dfrac{\mathbf{r}}{||\mathbf{r}||}$ on the outer sphere and $\mathbf{n} = -\dfrac{\mathbf{r}}{||\mathbf{r}||}$ on the hole.

c. They are equal. The only source is in the origin and flow inside ($\mathbf{n}$ is pointing inside) and out the outer sphere. No other sources to influence the outcome.

d. No difference.

e. $\displaystyle\int\int_S \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}} \cdot \mathbf{n}dS$ where S is the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{16} = 1$

$$\begin{aligned}
\frac{\partial \mathbf{F}_1}{\partial x} &= \frac{1(x^2 + y^2 + z^2)^{3/2} - \left[\frac{3}{2}(x^2 + y^2 + z^2)^{1/2}(2x)\right] x}{\left[(x^2 + y^2 + z^2)^{3/2}\right]^2} \\
&= \frac{(x^2 + y^2 + z^2)^{3/2} - 3x^2(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3} \\
&= \frac{x^2 + y^2 + z^2 - 3x^2}{(x^2 + y^2 + z^2)^{5/2}}
\end{aligned}$$

By symmetry

$$\frac{\partial \mathbf{F}_2}{\partial y} = \frac{x^2 + y^2 + z^2 - 3y^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\frac{\partial \mathbf{F}_3}{\partial z} = \frac{x^2 + y^2 + z^2 - 3z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

and so

$$\frac{\partial \mathbf{F}_1}{\partial x} + \frac{\partial \mathbf{F}_2}{\partial y} + \frac{\partial \mathbf{F}_3}{\partial z} = \frac{3(x^2+y^2+z^2) - 3x^2 - 3y^2 - 3z^2}{(x^2+y^2+z^2)^{5/2}} = 0$$

But $\mathbf{F}$ is not continuous at the origin, so we should cut out a small sphere centered at the origin.

$$\int\int_S \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} \cdot \mathbf{n} dS + \int\int_{Ball} \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} \cdot \mathbf{n} dS = \int\int\int_D \underbrace{\nabla \cdot \mathbf{F}}_{=0} dV$$

Now what is the second integral on the left hand side? The outward normal to the ball is

$$\mathbf{n} = -\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$$

so

$$\mathbf{F} \cdot \mathbf{n} = -\frac{1}{x^2+y^2+z^2} = -\frac{1}{r^2}$$

Using spherical coordinates

$$\int\int_{Ball} \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} \cdot \mathbf{n} dS = -\int_0^\pi \int_0^{2\pi} \frac{1}{r^2} r^2 \sin\phi \, d\theta d\phi = -2\pi(-\cos\phi)\Big|_0^\pi = -2\pi(1+1) = -4\pi$$

$$\int\int_S \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} \cdot \mathbf{n} dS = -\underbrace{\int\int_{Ball} \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} \cdot \mathbf{n} dS}_{=-4\pi} = 4\pi$$

5.2

Problems: 1, 2, 5, 8a

1. $\phi = xyz + 5$

$$\nabla\phi = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$

$$\int\int_S \left[\frac{1}{\mathbf{R}}\nabla\phi - \phi\nabla\left(\frac{1}{\mathbf{R}}\right)\right] \cdot d\mathbf{S} = 4\pi \underbrace{\phi(0,0,0)}_{=5} + \int\int\int_D \frac{\nabla^2\phi}{\mathbf{R}}\, dV, \text{ using (5.15)}$$

The volume integral vanishes since $\nabla^2\phi = 0$ and

$$\int\int_S \left[\frac{1}{\mathbf{R}}\nabla\phi - \phi\nabla\left(\frac{1}{\mathbf{R}}\right)\right] \cdot d\mathbf{S} = 20\pi$$

2. a. $\phi = x^2 + y^2 - 2z^2 + 4$ so $\nabla^2\phi = 2 + 2 - 4 = 0$ and $\phi(0,0,0) = 4$ therefore

$$\int\int_S \left[-\frac{1}{\mathbf{R}}\nabla\phi + \phi\nabla\left(\frac{1}{\mathbf{R}}\right)\right] \cdot d\mathbf{S} = -4\pi(4) = -16\pi$$

b. $\phi = x^2 - z^2 + 5$ so $\nabla^2\phi = 2 - 2 = 0$ and $\phi(0,0,0) = 5$ therefore

$$\int\int_S \left[-\frac{1}{\mathbf{R}}\nabla\phi + \phi\nabla\left(\frac{1}{\mathbf{R}}\right)\right] \cdot d\mathbf{S} = -4\pi(5) = -20\pi$$

8. a. Prove that $\int\int\int_D \nabla\phi \cdot \nabla \times \mathbf{F} dV = \int\int_S \mathbf{F} \times \nabla\phi \cdot d\mathbf{S}$. Using (3.36)

$$\mathbf{G} \cdot \nabla \times \mathbf{F} = \mathbf{F} \cdot \nabla \times \mathbf{G} + \nabla \cdot (\mathbf{F} \times \mathbf{G})$$

where we take $\mathbf{G} = \nabla\phi$. Recall that by (3.40) $\nabla \times (\nabla\phi) = 0$ so the above becomes

$$(\nabla\phi) \cdot (\nabla \times \mathbf{F}) = \mathbf{F} \cdot \underbrace{\nabla \times (\nabla\phi)}_{=0} + \nabla \cdot (\mathbf{F} \times \nabla\phi)$$

Now integrate and use the divergence theorem

$$\int\int\int_D \nabla\phi \cdot (\nabla \times \mathbf{F}) dV = \int\int\int_D \nabla \cdot (\mathbf{F} \times \nabla\phi) dV = \int\int_S \mathbf{F} \times (\nabla\phi) \cdot d\mathbf{S}$$

5.4

Problems: 4, 5, 7, 9, 12

4. Derive (5.49).

$$\int_C \frac{1}{2}(xdy - ydx) = A$$

Let $\mathbf{F} = -\frac{1}{2}y\mathbf{i} + \frac{1}{2}x\mathbf{j}$ then $\dfrac{\partial \mathbf{F}_1}{\partial y} = -\dfrac{1}{2}$ and $\dfrac{\partial \mathbf{F}_2}{\partial x} = \dfrac{1}{2}$

$$\int_C (\mathbf{F}_1 dx + \mathbf{F}_2 dy) = \int_C \left(-\frac{1}{2}ydx + \frac{1}{2}xdy\right) = \frac{1}{2}\int_C (xdy - ydx)$$

On the other hand

$$\int\int_D \left(\frac{\partial \mathbf{F}_2}{\partial x} - \frac{\partial \mathbf{F}_1}{\partial y}\right) dxdy = \int\int_D \left(\frac{1}{2} + \frac{1}{2}\right) dxdy = \int\int_D dxdy = A$$

5. $\mathbf{R} = x\mathbf{i} + y\mathbf{j}$ then $d\mathbf{R} = dx\mathbf{i} + dy\mathbf{j}$

a. Evaluate $||\mathbf{R} \times (\mathbf{R} + d\mathbf{R})||$

$$\mathbf{R} \times (\mathbf{R} + d\mathbf{R}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & 0 \\ x+dx & y+dy & 0 \end{vmatrix} = [x(y+dy) - y(x+dx)]\,\mathbf{k} = (xdy - ydx)\mathbf{k}$$

The magnitude of this is $||\mathbf{R} \times (\mathbf{R} + d\mathbf{R})|| = |xdy - ydx|$.

b. The integrand of (5.49) is $\dfrac{1}{2}(xdy - ydx) = \dfrac{1}{2}\underbrace{||\mathbf{R} \times (\mathbf{R} + d\mathbf{R})||}_{=\text{area of parallelogram}} = \text{area of a triangle}$

7. C is given by $x^2 + y^2 = 9$ and $\mathbf{F} = y\mathbf{i} - 3x\mathbf{j}$

$$\int_C (ydx - 3xdy) = \int\int_D (-3-1)dxdy = -4\int_0^{2\pi}\int_0^3 rdrd\theta = -4(2\pi)\frac{1}{2}r^2\Big|_0^3 = -4\pi(9) = -36\pi$$

9. a. Compute directly the integral $\displaystyle\int_C (4y^3dx - 2x^2dy)$ where the domain is a square bounded by $x = \pm 1$ and $y = \pm 1$.

On C_1 defined by $y = -1$ we have $dy = 0$ and

$$\int_{C_1} (4y^3dx - 2x^2dy) = \int_{-1}^1 4(-1)^3dx = \int_{-1}^1 (-4)dx = -4x\Big|_{-1}^1 = -8$$

On C_2 defined by $x = 1$ we have $dx = 0$ and

$$\int_{C_2} (4y^3dx - 2x^2dy) = \int_{-1}^1 (-2dy) = -2y\Big|_{-1}^1 = -4$$

On C_3 defined by $y = 1$ we have $dy = 0$ and

$$\int_{C_3} (4y^3 dx - 2x^2 dy) = \int_1^{-1} (4dx) = 4x\Big|_1^{-1} = -8$$

On C_4 defined by $x = -1$ we have $dx = 0$ and

$$\int_{C_4} (4y^3 dx - 2x^2 dy) = \int_1^{-1} (-2dy) = -2y\Big|_1^{-1} = 4$$

Therefore $\int_C (4y^3 dx - 2x^2 dy) = -8 - 4 - 8 + 4 = -16$

b. Using Green's theorem $\int_C (4y^3 dx - 2x^2 dy) = \int\int_D (-4x - 12y^2) dx dy$.
This integral can be written as

$$\begin{aligned}
\int_{-1}^{1}\int_{-1}^{1} (-4x - 12y^2) dx dy &= \int_{-1}^{1} \left(-4\frac{x^2}{2} - 12xy^2\right)\Big|_{-1}^{1} dy \\
&= \int_{-1}^{1} \left(-2 - 12y^2 - (-2 + 12y^2)\right) dy \\
&= -24\int_{-1}^{1} y^2 dy = -24\frac{y^3}{3}\Big|_{-1}^{1} \\
&= -24\left(\frac{1}{3} + \frac{1}{3}\right) = -16
\end{aligned}$$

Same as in part a.

c. By symmetry. As we noted in part a. the contribution of $\int_C (-2x^2) dy$ cancel and we are left with $\int_C 4y^3 dx$. The integral on $y = 1$ goes in the opposite direction to the integral on $y = -1$ and so the contributions add up. Each is $\int_C 4(-1)^3 dx = -4x\Big|_{-1}^{1} = -8$ and the total is -16.

12. Use Green's theorem to find the area inside $x = \dfrac{t}{1+t^3}$, $y = \dfrac{t^2}{1+t^3}$, $0 \le t < \infty$

$$
\begin{aligned}
A = \frac{1}{2}\int_C (xdy - ydx) &= \frac{1}{2}\int_0^\infty \left(\underbrace{\frac{t}{1+t^3}\frac{2t(1+t^3) - 3t^2(t^2)}{(1+t^3)^2}}_{= \frac{t(2t-t^4)}{(1+t^3)^3}} - \underbrace{\frac{t^2}{1+t^3}\frac{1+t^3-3t^2(t)}{(1+t^3)^2}}_{= \frac{t^2(1-2t^3)}{(1+t^3)^3}} \right) dt \\
&= \frac{1}{2}\int_0^\infty \frac{2t^2 - t^5 - t^2 + 2t^5}{(1+t^3)^3}\, dt \\
&= \frac{1}{2}\int_0^\infty \frac{t^2(1+t^3)}{(1+t^3)^3}\, dt \\
&= \frac{1}{2}\int_0^\infty \frac{t^2}{(1+t^3)^2}\, dt \qquad \text{use substitution } u = 1+t^3 \\
&= \frac{1}{2}\left(\frac{1}{3}\int_1^\infty \frac{du}{u^2}\right) \\
&= -\frac{1}{6u}\Big|_1^\infty = 0 + \frac{1}{6} \\
&= \frac{1}{6}
\end{aligned}
$$

5.5

Problems: 1–3, 7

1. Given $\mathbf{F} = 3y\mathbf{i} + (5-2x)\mathbf{j} + (z^2-2)\mathbf{k}$

   a. $\nabla \cdot \mathbf{F} = 0 + 0 + 2z = 2z$

   b. $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3y & 5-2x & z^2-2 \end{vmatrix} = (0-0)\mathbf{i} - (0-0)\mathbf{j} + (-2-3)\mathbf{k} = -5\mathbf{k}$

   c. $\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$ where S the open upper hemisphere of radius 2. This can be computed by

$$\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS + \int\int_{disk} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \int\int\int_D \underbrace{\nabla \cdot (\nabla \times \mathbf{F})}_{=0} dV$$

so

$$\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = -\int\int_{disk} (-5\mathbf{k}) \cdot \underbrace{\mathbf{n}}_{=-\mathbf{k}} dS = -5 \underbrace{\int\int_{disk} dS}_{=4\pi} = -20\pi$$

2. Given $\nabla \times \mathbf{F} = 2y\mathbf{i} - 2z\mathbf{j} + 3\mathbf{k}$

   a. $\int\int\int_D \underbrace{\nabla \cdot (\nabla \times \mathbf{F})}_{=0} dV = 0$ on the other hand

$$\int\int\int_D \nabla \cdot (\nabla \times \mathbf{F}) dV = \int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS + \int\int_{disk} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$$

Therefore

$$\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = -\int\int_{disk} \underbrace{(\nabla \times \mathbf{F}) \cdot \underbrace{\mathbf{n}}_{=-\mathbf{k}}}_{=-3} dS = -\int\int_{disk} -3 dS = 3(9\pi) = 27\pi$$

   b. $\int\int\int_D \underbrace{\nabla \cdot (\nabla \times \mathbf{F})}_{=0} dV = 0$ on the other hand

$$\int\int\int_D \nabla \cdot (\nabla \times \mathbf{F}) dV = \int\int_{Sphere} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$$

Therefore

$$\int\int_{Sphere} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = 0$$

3.

$$\int\int_S \nabla\phi \times \nabla\psi \cdot d\mathbf{S} = \int\int_S \nabla \times (\phi \nabla \psi) \cdot d\mathbf{S}$$

Now

$$\nabla\phi \times \nabla\psi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial\phi}{\partial x} & \dfrac{\partial\phi}{\partial y} & \dfrac{\partial\phi}{\partial z} \\ \\ \dfrac{\partial\psi}{\partial x} & \dfrac{\partial\psi}{\partial y} & \dfrac{\partial\psi}{\partial z} \end{vmatrix} = \left(\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial z} - \frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial y}\right)\mathbf{i} + \ldots$$

The component in the x direction of $\nabla \times (\phi\nabla\psi)$ is

$$\begin{aligned} \nabla \times (\phi\nabla\psi) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \\ \phi\dfrac{\partial\psi}{\partial x} & \phi\dfrac{\partial\psi}{\partial y} & \phi\dfrac{\partial\psi}{\partial z} \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(\phi\frac{\partial\psi}{\partial z}) - \frac{\partial}{\partial z}(\phi\frac{\partial\psi}{\partial y})\right)\mathbf{i} + \ldots \\ &= \left(\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial z} + \phi\frac{\partial^2\psi}{\partial z\partial y} - \frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial y} - \phi\frac{\partial^2\psi}{\partial z\partial y}\right)\mathbf{i} + \ldots \end{aligned}$$

Note that the second derivative terms cancel and we get the same answer for the component in the x direction. Similarly for the other components.

By Stokes' theorem $\int\int_S \nabla \times (\phi\nabla\psi) \cdot d\mathbf{S} = \int_C \phi(\nabla\psi) \cdot d\mathbf{R}$

7. $\mathbf{F} = \nabla\phi$

$$\int_C \mathbf{F} \cdot d\mathbf{R} = \phi(b) - \phi(a) = 0, \qquad \text{for closed curves}$$

a. $\int\int_S \nabla \times \nabla\phi \cdot \mathbf{n} dS = \int_C \nabla\phi \cdot d\mathbf{R} = 0$

b. $\mathbf{n} \cdot \nabla \times (\nabla\phi) = 0,$ because of part a.

c. Therefore $\nabla \times \nabla\phi = \mathbf{0}$

d. The identity in Section 3.8 (3.40) is $\nabla \times \nabla\phi = \mathbf{0}$