CALCULUS OF VARIATIONS
MA 4311 SOLUTION MANUAL

B. Neta
Department of Mathematics
Naval Postgraduate School
Code MA/Nd
Monterey, California 93943

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Credits

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CHAPTER 1

1 Functions of n Variables

Problems

1. Use the method of Lagrange Multipliers to solve the problem

$$\begin{aligned}&\text{minimize} \quad f = x^2 + y^2 + z^2\\ &\text{subject to} \quad \phi = xy + 1 - z = 0\end{aligned}$$

2. Show that

$$\max_{\lambda}\left|\frac{\lambda}{\cosh\lambda}\right| = \frac{\lambda_0}{\cosh\lambda_0}$$

where λ_0 is the positive root of

$$\cosh\lambda - \lambda\sinh\lambda = 0.$$

Sketch to show λ_0.

3. Of all rectangular parallelepipeds which have sides parallel to the coordinate planes, and which are inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

determine the dimensions of that one which has the largest volume.

4. Of all parabolas which pass through the points (0,0) and (1,1), determine that one which, when rotated about the x-axis, generates a solid of revolution with least possible volume between $x = 0$ and $x = 1$. [Notice that the equation may be taken in the form $y = x + cx(1 - x)$, when c is to be determined.

5. a. If $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ is a real vector, and $\mathbf{A}$ is a real symmetric matrix of order n, show that the requirement that

$$F \equiv \mathbf{x}^T\mathbf{A}\mathbf{x} - \lambda\mathbf{x}^T\mathbf{x}$$

be stationary, for a prescibed $\mathbf{A}$, takes the form

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}.$$

Deduce that the requirement that the quadratic form

$$\alpha \equiv \mathbf{x}^T\mathbf{A}\mathbf{x}$$

be stationary, subject to the constraint

$$\beta \equiv \mathbf{x}^T\mathbf{x} = \text{constant},$$

leads to the requirement

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x},$$

where λ is a constant to be determined. [Notice that the same is true of the requirement that β is stationary, subject to the constraint that $\alpha =$ constant, with a suitable definition of λ.]

b. Show that, if we write

$$\lambda = \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \equiv \frac{\alpha}{\beta},$$

the requirement that λ be stationary leads again to the matrix equation

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}.$$

[Notice that the requirement $d\lambda = 0$ can be written as

$$\frac{\beta d\alpha - \alpha d\beta}{\beta^2} = 0$$

or

$$\frac{d\alpha - \lambda d\beta}{\beta} = 0]$$

Deduce that stationary values of the ratio

$$\frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$$

are characteristic numbers of the symmetric matrix $\mathbf{A}$.

1. $f = x^2 + y^2 + z^2$

$\varphi = xy + 1 - z = 0$

$F = f + \lambda\varphi = x^2 + y^2 + z^2 + \lambda(xy + 1 - z)$

$$\frac{\partial F}{\partial x} = 2x + \lambda y = 0 \tag{1}$$

$$\frac{\partial F}{\partial y} = 2y + \lambda x = 0 \tag{2}$$

$$\frac{\partial F}{\partial z} = 2z - \lambda = 0 \tag{3}$$

$$\frac{\partial F}{\partial \lambda} = xy + 1 - z = 0 \tag{4}$$

$$(3) \quad \Rightarrow \quad \lambda = 2z \tag{5}$$

$$(4) \quad \Rightarrow \quad z = xy + 1 \tag{6}$$

$$(5) \text{ and } (16) \quad \Rightarrow \quad \lambda = 2(xy + 1) \tag{7}$$

Substitute (7) in (1) - (2)

$$\Rightarrow 2x \; + \; 2(xy + 1)y = 0 \tag{8}$$

$$2y + 2(xy + 1)x = 0 \tag{9}$$

$$\left.\begin{array}{r} x + xy^2 + y = 0 \\ y + x^2y + x = 0 \end{array}\right\} -$$

$$xy(y - x) = 0 \tag{10}$$

$\Rightarrow x = 0 \quad \text{or} \quad y = 0 \quad \text{or} \quad x = y$

$$\underline{x=0} \underset{(7)}{\Rightarrow} \lambda = 2 \underset{(5)}{\Rightarrow} z = 1 \;, y = 0 \quad \text{by}(1)$$

$$\underline{y=0} \underset{(7)}{\Rightarrow} \lambda = 2 \underset{(5)}{\Rightarrow} z = 1 \;, x = 0 \quad \text{by}(1)$$

$$\underline{x=y} \underset{(7)}{\Rightarrow} \lambda = 2 \underset{(5)}{\Rightarrow} z = -1 \;, \underset{(6)}{\Rightarrow} xy = -2$$

$$\Rightarrow x^2 = -2$$

Not possible

So the only possibility

$$x = y = 0 \quad z = 1 \quad \lambda = 2$$

$\Rightarrow \quad f = 1$

2. Find max $\left|\dfrac{\lambda}{\cosh\lambda}\right|$

Differentiate

$$\frac{d}{d\lambda}\left(\frac{\lambda}{\cosh\lambda}\right) = \frac{\cosh\lambda - \lambda\sinh\lambda}{\cosh^2\lambda} = 0$$

Since $\cosh\lambda \neq 0$,

$$\cosh\lambda - \lambda\sinh\lambda = 0$$

The positive root is λ_0

Thus the function at λ_0 becomes

$$\frac{\lambda_0}{\cosh\lambda_0}$$

No need for absolute value since $\lambda_0 > 0$

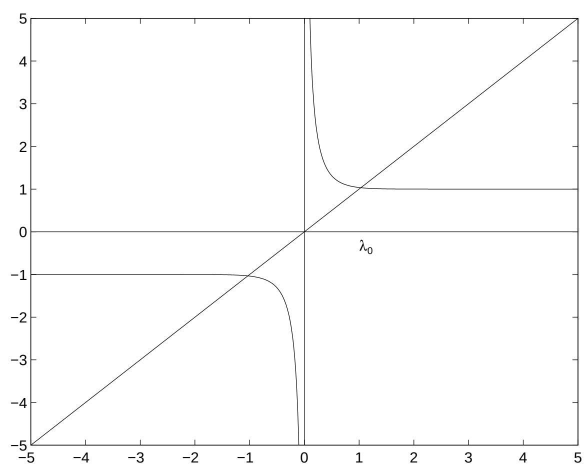

Figure 1:

3. max xyz $s.t.$ $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

Write F $= xyz + \lambda\,(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1)$, then

$$0 = F_x = yz + \frac{2\lambda x}{a^2} \tag{1}$$

$$0 = F_y = xz + \frac{2\lambda y}{b^2} \tag{2}$$

$$0 = F_z = xy + \frac{2\lambda z}{c^2} \tag{3}$$

$$0 = F_\lambda = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \tag{4}$$

If any of x, y or z are zero then the volume is zero and not the max. Therefore $x \neq 0$, $y \neq 0$, $z \neq 0$ so

$$0 = -xF_x + yF_y = \frac{-2\lambda x^2}{a^2} + \frac{2\lambda y^2}{b^2} \Rightarrow \frac{y^2}{b^2} = \frac{x^2}{a^2} \tag{5}$$

Also

$$0 = -zF_z + yF_y = \frac{-2\lambda x^2}{a^2} + \frac{2\lambda y^2}{b^2} \Rightarrow \frac{y^2}{b^2} = \frac{z^2}{c^2} \tag{6}$$

Then by (4) $\dfrac{3y^2}{b^2} = 1 \Rightarrow y^2 = \dfrac{b^2}{3}$ taking only the (+) square root (length) $y = \dfrac{b}{\sqrt{3}}$

$x = \dfrac{a}{\sqrt{3}}$, $z = \dfrac{c}{\sqrt{3}}$ by (5), (6) respectfully.

The largest volume parallelepiped inside the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ has dimension $\dfrac{a}{\sqrt{3}} \times \dfrac{b}{\sqrt{3}} \times \dfrac{c}{\sqrt{3}}$

4. $\varphi = -y + x + cx(1-x)$

Volume $V = \int_0^1 \pi y^2 dx$

$$\min V = \pi \int_0^1 [x + cx(1-x)]^2 \, dx$$

$$\frac{dV(c)}{dc} = \pi \int_0^1 2 \left[x + cx(1-x)\right] x(1-x) dx = 0$$

$$2\pi \int_0^1 x^2(1-x)dx + 2\pi c \int_0^1 x^2(1-x)^2 dx = 0$$

$$2\pi \left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right] \Big|_0^1 + 2\pi c \left[\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right] \Big|_0^1 = 0$$

$$2\pi (\frac{1}{3} - \frac{1}{4}) \qquad + 2\pi c \left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right] = 0$$

$$2\pi \cdot \frac{1}{12} + 2\pi c \frac{1}{30} = 0$$

$$c = -\frac{15}{6} = -\frac{5}{\underline{2}}$$

$$y = x - \frac{5}{2}x(1-x)$$

$$V(c) = \pi \int_0^1 \left[x^2 + 2cx^2(1-x) + c^2x^2(1-x)^2\right] \, dx$$

$$= \pi \left\{\frac{1}{3} + 2e\frac{1}{12} + c^2\frac{1}{30}\right\}$$

$$V(c = -5/2) = \frac{3}{24}\pi$$

5. $F = x^T Ax - \lambda x^T x$

$$= \sum_{i,j} A_{ij} x_i x_j - \lambda \sum_i x_i^2$$

$$\frac{\partial F}{\partial x_k} = \sum_j A_{kj}\, x_j + \sum_i A_{ik} x_i - 2\lambda x_k = 0 \qquad k = 1,\, 2, \ldots, n$$

$$\Rightarrow Ax + A^T x - 2\lambda x = 0$$

Since A is symmetric

$$Ax = \lambda x$$

min $F = x^T Ax + \lambda(x^T x - c)$

implies (by differentiating with respect to x_k, $k = 1, \ldots, n$)

$$Ax = \lambda x$$

b. $\lambda = \dfrac{x^T Ax}{x^T x} = \dfrac{\alpha}{\beta}$

To minimize λ we require

$$d\lambda = \frac{\beta d\alpha - \alpha d\beta}{\beta^2} = 0$$

Divide by β

$$\frac{d\alpha - \frac{\alpha}{\beta} d\beta}{\beta} = 0$$

or

$$\frac{d\alpha - \lambda d\beta}{\beta} = 0$$

CHAPTER 2

2 Examples, Notation

Problems

1. For the integral

$$I = \int_{x_1}^{x_2} f(x, y, y')\, dx$$

with

$$f = y^{1/2}\left(1 + y'^2\right)$$

write the first and second variations $I'(0)$, and $I''(0)$.

2. Consider the functional

$$J(y) = \int_0^1 (1 + x)(y')^2 dx$$

where y is twice continuously differentiable and $y(0) = 0$ and $y(1) = 1$. Of all functions of the form

$$y(x) = x + c_1 x(1 - x) + c_2 x^2(1 - x),$$

where c_1 and c_2 are constants, find the one that minimizes J.

1. $f = y^{1/2}(1 + y'^2)$

$$f_y = \frac{1}{2}y^{-1/2}(1 + y'^2)$$

$$f_{y'} = 2y'y^{1/2}$$

$$I'(0) = \int_{x_1}^{x_2} \left[\frac{1}{2}y^{-1/2}(1 + y'^2)\eta + 2y'y^{1/2}\eta'\right] dx$$

$$f_{yy} = -\frac{1}{4}y^{-3/2}(1 + y'^2)$$

$$f_{yy'} = y^{-1/2}y'$$

$$f_{y'y'} = 2y^{1/2}$$

$$I''(0) = \int_{x_1}^{x_2} \left[-\frac{1}{4}y^{-3/2}(1 + y'^2)\eta^2 + 2y^{-1/2}y'\eta\eta' + 2y^{1/2}\eta'^2\right] dx$$

2. We are given that,after expanding, $y(x) = (c_1+1)x + (c_2 - c_1)x^2 - c_2x^3$.Then we also have that;

$$y'(x) = (c_1+1) + 2(c_2 - c_1)x - 3c_2x^2$$

and that;

$$\begin{aligned}(y'(x))^2 &= (c_1+1)^2 + 4x(c_1+1)(c_2-c_1) - 6x^2c_2(c_1+1) \\ &\quad +4x^2(c_2-c_1)^2 - 12x^3c_2(c_2-c_1) + 9c_2^2x^4\end{aligned}$$

Therefore, we now can integrate $J(y)$ and get a solution in terms of c_1 and c_2;

$$\begin{aligned}\textstyle\int_0^1 (1+x)(y')^2dx &= \tfrac{3}{2}(c_1+1)^2 + \tfrac{10}{3}(c_1+1)(c_2-c_1) \\ &\quad -\tfrac{14}{4}c_2(c_1+1) + \tfrac{7}{3}(c_2-c_1)^2 \\ &\quad -\tfrac{27}{5}c_2(c_2-c_1) + \tfrac{99}{30}c_2^2\end{aligned}$$

To get the minimum, we want to solve $J_{c_1} = 0$ and $J_{c_2} = 0$. After taking these partial derivatives and simplifying we get;

$$J_{c_2} = \frac{17}{30}c_1 + \frac{7}{15}c_2 - \frac{1}{6} = 0$$

and

$$J_{c_1} = c_1 + \frac{17}{30}c_2 - \frac{1}{3} = 0$$

Putting this in matrix form, we want to solve;

$$\begin{bmatrix} \frac{17}{30} & \frac{7}{15} \\ 1 & \frac{17}{30} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{3} \end{bmatrix}$$

Using Cramer's rule, we have that;

$$c_1 = \frac{\begin{vmatrix} \frac{1}{6} & \frac{7}{15} \\ \frac{1}{3} & \frac{17}{30} \end{vmatrix}}{\begin{vmatrix} \frac{17}{30} & \frac{7}{15} \\ 1 & \frac{17}{30} \end{vmatrix}} = \frac{55}{131} \approx .42$$

and

$$c_2 = \frac{\begin{vmatrix} \frac{17}{30} & \frac{1}{6} \\ 1 & \frac{1}{3} \end{vmatrix}}{\begin{vmatrix} \frac{17}{30} & \frac{7}{15} \\ 1 & \frac{17}{30} \end{vmatrix}} = \frac{-20}{131} \approx -.15$$

Therefore, we have that the $y(x)$ which minimizes $J(y)$ is;

$$\begin{aligned}y(x) &= \tfrac{186}{131}x + \tfrac{-77}{131}x^2 + \tfrac{20}{131}x^3 \\ &\approx 1.42x - .57x^2 + .15x^3\end{aligned}$$

Using a technique found in Chapter 3, it can be shown that the extremal of the $J(y)$ is;

$$y(x) = \frac{1}{ln2} ln(1 + x)$$

which, after expanding about $x = 0$ is represented as;

$$\begin{aligned} y(x) &= \tfrac{1}{ln2}x - \tfrac{1}{2ln2}x^2 + \tfrac{1}{3ln2}x^3 + R(x) \\ &\approx 1.44x - .72x^2 + .48x^3 + R(x) \end{aligned}$$

So we can see that the form for $y(x)$ given in the problem is similar to the series representation gotten using a different method.

CHAPTER 3

3 First Results

Problems

1. Find the extremals of

$$I = \int_{x_1}^{x_2} F(x, y, y')\, dx$$

for each case

a. $F = (y')^2 - k^2 y^2 \qquad (k \text{ constant})$

b. $F = (y')^2 + 2y$

c. $F = (y')^2 + 4xy'$

d. $F = (y')^2 + yy' + y^2$

e. $F = x\,(y')^2 - yy' + y$

f. $F = a(x)\,(y')^2 - b(x)y^2$

g. $F = (y')^2 + k^2 \cos y$

2. Solve the problem minimize $I = \displaystyle\int_a^b \left[(y')^2 - y^2\right]\, dx$

with

$$y(a) = y_a, \qquad y(b) = y_b.$$

What happens if $b - a = n\pi$?

3. Show that if $F = y^2 + 2xyy'$, then I has the same value for all curves joining the endpoints.

4. A geodesic on a given surface is a curve, lying on that surface, along which distance between two points is as small as possible. On a plane, a geodesic is a straight line. Determine equations of geodesics on the following surfaces:

a. Right circular cylinder. [Take $ds^2 = a^2 d\theta^2 + dz^2$ and minimize $\displaystyle\int \sqrt{a^2 + \left(\frac{dz}{d\theta}\right)^2}\, d\theta$ or $\displaystyle\int \sqrt{a^2 \left(\frac{d\theta}{dz}\right)^2 + 1}\, dz$]

b. Right circular cone. [Use spherical coordinates with $ds^2 = dr^2 + r^2 \sin^2 \alpha d\theta^2$.]

c. Sphere. [Use spherical coordinates with $ds^2 = a^2 \sin^2 \phi d\theta^2 + a^2 d\phi^2$.]

d. Surface of revolution. [Write $x = r\cos\theta$, $y = r\sin\theta$, $z = f(r)$. Express the desired relation between r and θ in terms of an integral.]

5. Determine the stationary function associated with the integral

$$I = \int_0^1 (y')^2 \, f(x) \, ds$$

when $y(0) = 0$ and $y(1) = 1$, where

$$f(x) = \begin{cases} -1 & 0 \le x < \frac{1}{4} \\ 1 & \frac{1}{4} < x \le 1 \end{cases}$$

6. Find the extremals

a. $J(y) = \int_0^1 y' dx, \qquad y(0) = 0, \; y(1) = 1.$

b. $J(y) = \int_0^1 yy' dx, \qquad y(0) = 0, \; y(1) = 1.$

c. $J(y) = \int_0^1 xyy' dx, \qquad y(0) = 0, \; y(1) = 1.$

7. Find extremals for

a. $J(y) = \int_0^1 \frac{y'^2}{x^3} dx,$

b. $J(y) = \int_0^1 \left(y^2 + (y')^2 + 2ye^x \right) dx.$

8. Obtain the necessary condition for a function y to be a local minimum of the functional

$$J(y) = \int \int_R K(s,t) y(s) y(t) ds dt + \int_a^b y^2 dt - 2 \int_a^b y(t) f(t) dt$$

where $K(s,t)$ is a given continuous function of s and t on the square R, for which $a \le s, t \le b$, $K(s,t)$ is symmetric and $f(t)$ is continuous.

Hint: the answer is a Fredholm integral equation.

9. Find the extremal for

$$J(y) = \int_0^1 (1+x)(y')^2 dx, \qquad y(0) = 0, \; y(1) = 1.$$

What is the extremal if the boundary condition at $x = 1$ is changed to $y'(1) = 0$?

10. Find the extremals

$$J(y) = \int_a^b \left(x^2 (y')^2 + y^2 \right) dx.$$

1. $\int^{x_2} F(x,y,y')dx$ Find the externals.

a. $F(y,y') = (y')^2 - k^2y^2$ k = constant

by Euler's equation $F - y'F_{y'} = c$ so,

$$(y')^2 - (ky)^2 - y'(2y') = -(y')^2 - (ky)^2 = c$$

$$\Rightarrow (y')^2 = -(ky)^2 - c \quad \Rightarrow \quad y' = \pm(-(ky)^2 - c)^{1/2}$$

$$\frac{dy}{(-(ky)^2 - c)^{1/2}} = \pm dx \quad \Rightarrow \quad \frac{dy}{[(ky)^2 + c]^{1/2}} = \pm i dx$$

Using the fact that $\int \frac{du}{\sqrt{u^2 + a^2}} = \ln|u + \sqrt{u^2 + a^2}|$ we get

$$\int \frac{dy}{((ky)^2 + c)^{1/2}} = \ln|ky + \sqrt{(ky)^2 + c}| = \int \pm i dx = \pm ix$$

$$ky + \sqrt{(ky)^2 + c} = e^{\pm ix}$$

Let's try another way using $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a}$

$$\int \frac{dy}{[\sqrt{-c}^2 - (ky)^2]^{1/2}} = \int \pm dx$$

$$\Rightarrow \sin^{-1}\frac{ky}{\sqrt{-c}} = \pm x \quad \Rightarrow \quad \frac{ky}{\sqrt{-c}} = \sin(\pm x) = \pm \sin x$$

$$\boxed{y = \pm\frac{\sqrt{-c}}{k}\sin x} \quad c < 0$$

b. $F(y, y') = (y')^2 + 2y$

$$F - y'F_{y'} = (y')^2 + 2y - y'(2y') = -(y')^2 + 2y = c$$

$$\Rightarrow (y')^2 = 2y - c \quad \Rightarrow \quad \frac{dy}{\sqrt{2y-c}} = \pm dx$$

$$\Rightarrow (2y-c)^{1/2} = \pm x \quad \Rightarrow \quad 2y - c = x^2$$

$$\boxed{y = \frac{1}{2}(x^2 + c)}$$

c. $F(x, y') = 4xy' + (y')^2$

use $F_{y'} = c \quad \Rightarrow \quad 4x + 2y' = c$

$$\Rightarrow \quad y' = \frac{1}{2}(c - 4x) \quad \Rightarrow \quad \boxed{y = \frac{1}{2}x(c - 2x)}$$

d. $F = y'^2 + yy' + y^2$

$F - y'F_{y'} = c$ see (21)

$F_{y'} = 2y' + y$

$\Rightarrow F - y'F_{y'} = y'^2 + yy' + y^2 - y'(2y' + y)$

$= -y'^2 + y^2$

$\Rightarrow -y'^2 + y^2 = c$

$y'^2 = y^2 - c$

$y' = \pm\sqrt{y^2 - c}$

$$\int \frac{dy}{\pm\sqrt{y^2 - c}} = \int dx$$

$$\pm(arc\ \cosh\ \frac{y}{\sqrt{c}} + c_2) = x$$

can also be written as $\ln|\,y + \sqrt{y^2 - c}\,|$

$$arc\ \cosh\ \frac{y}{\sqrt{c}} = \pm x - c_2$$

$$\cosh(\pm\, x - c_2) = \frac{y}{\sqrt{c}}$$

$$\boxed{y = \sqrt{c}\ \cosh(\pm\, x - c_2)}$$

e. $F = x\,y'^2 - yy' + y$

$$\frac{d}{dx}F_{y'} = F_y \qquad \text{see (12)}$$

$$F_y = -y' + 1$$

$$F_{y'} = 2xy' - y$$

$$\frac{d}{dx}(2xy' - y) = -y' + 1$$

$$2y' + 2xy'' - y' = -y' + 1$$

$$2xy'' + 2y' = 1$$

$$(2xy')' = 1$$

$$2xy' = c_1$$

$$y' = \frac{c_1}{2x}$$

$$dy = \frac{c_1}{2}\ \frac{dx}{x}$$

$$\boxed{y = \frac{c_1}{2}\ln|\,x\,| + c_2}$$

f. $F = a(x)y'^2 - b(x)y^2$

$$F_y = -2b(x)y$$

$$F_{y'} = 2a(x)y'$$

$$\frac{d}{dx}F_{y'} = F_y \quad \Rightarrow \quad (2a(x)y')' = -2by$$

$$a(x)y'' + a'y' + by = 0$$

Linear nonconstant coefficients. Can be Solved !

g. $F = y'^2 + k^2 \cos y$

$$F - y' F y' = c$$

$$F y' = 2y'$$

$$y'^2 + k^2 \cos\ y - 2y'^2 = c_1$$

$$-y'^2 + k^2 \cos\ y = c_1$$

$$y'^2 = c_1 - k^2 \cos\ y$$

$$\frac{dy}{dx} = \pm\sqrt{c_1 - k^2 \cos\ y}$$

$$\frac{dy}{\sqrt{c_1 - k^2\ \cos\ y}} = \pm dx$$

$$x + c_2 = \pm \int \frac{dy}{\sqrt{c_1 - k^2\ \cos\ y}}$$

2. $F = y'^2 - y^2$

From problem 1a with $k = 1$ we have

$$y = \pm\sqrt{-c}\sin x, \quad c < 0$$

$$y(a) = y_a \quad \Rightarrow \quad y_a = \pm\sqrt{-c}\sin a$$

$$y(b) = y_b \quad \Rightarrow \quad y_b = \pm\sqrt{-c}\sin b$$

$$\frac{\sin\ b}{\sin\ a} = \frac{y_b}{y_a} \quad \text{to get a solution.}$$

The solution is not unique: $y = \dfrac{y_b}{\sin\ b}\sin x = \dfrac{y_a}{\sin\ a}\sin x$

If $b = a + n\pi$ then $y_b = \pm\sqrt{-c}\sin(a + n\pi)$

$$= \pm\sqrt{-c}\sin a$$

then $y_b = y_a$ for a solution !

otherwise no solution.

3. If $F = y^2 + 2xyy'$, show I has the same values for all curves joining the end-points.

Using Euler's equation (12) in Chapter 3, we need only show

$$\frac{d}{dx} F_{y'}(x) = F_y(x) \quad x, \leq x \leq x_2.$$

$F_{y'} = 2xy$, $F_y = 2y + 2xy'$

$\dfrac{d}{dx} F_{y'}(x) = \dfrac{d}{dx}(2xy) = 2xy' + 2y$

which is F_y

Note that

$$F = \frac{d}{dx}(xy^2)$$

$\Rightarrow \quad \int_{x_1}^{x_2} F = xy^2 \Big|_{x_1}^{x_2}$ independent of curve.

4. a. Right circular cylinder

$$\min \int \sqrt{a^2 + \left(\frac{dz}{d\theta}\right)^2}\, d\theta$$

$$F(\theta,\, z,\, z') = \sqrt{a^2 + z'^2}$$

$$F - z' F_{z'} = c$$

$$F_{z'} = \frac{1}{2}(a^2 + z'^2)^{-1/2} \cdot 2z'$$

$$\sqrt{a^2 + z'^2} - z'^2 \frac{1}{\sqrt{a^2 + z'^2}} = c_1$$

$$a^2 + z'^2 - z'^2 = c_1 \sqrt{a^2 + z'^2}$$

$$\sqrt{a^2 + z'^2} = \frac{a^2}{c_1}$$

$$a^2 + z'^2 = \left(\frac{a^2}{c_1}\right)^2$$

$$z'^2 = \left(\frac{a^2}{c_1}\right)^2 - a^2$$

$$z' = \pm \sqrt{\left(\frac{a^2}{c_1}\right)^2 - a^2}$$

$$z = \pm \sqrt{\left(\frac{a^2}{c_1}\right)^2 - a^2}\,\theta + c_2$$

2 parameter family of helical lines.

4. b. Right circular cone

$$\int \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2 \sin^2 \alpha} \; d\theta$$

$$F(\theta,\, r,\, r') = \sqrt{r'^2 + r^2 \sin^2 \alpha}$$

No dependence on θ , thus we can use (21)

$$F_{r'} = \frac{1}{2}(r'^2 + r^2 \sin^2 \alpha)^{-1/2} \cdot 2r'$$

$$F - r' F_{r'} = c_1$$

$$\Rightarrow \quad \sqrt{r'^2 + r^2 \sin^2 \alpha} - \frac{r'^2}{\sqrt{r'^2 + r^2 \sin^2 \alpha}} = c_1$$

$$r'^2 + r^2 \sin^2 \alpha - r'^2 = c_1 \sqrt{r'^2 + r^2 \sin^2 \alpha}$$

$$r'^2 + r^2 \sin^2 \alpha = \left(\frac{\sin^2 \alpha}{c_1}\; r^2\right)^2$$

$$r'^2 = r^2 \sin^2 \alpha \; \left[\frac{r^2 \sin^2 \alpha}{c_1^2} - 1\right]$$

$$r' = \pm r \frac{\sin \alpha}{c_1} \sqrt{r^2\, sin^2\, \alpha - c_1^2}$$

$$\frac{dr}{r \sin \alpha \sqrt{r^2\, sin^2\, \alpha - c_1^2}} = \pm \frac{d\theta}{c_1}$$

Let $\rho = r \sin \alpha$

$$\int \frac{d\rho / \sin \alpha}{\rho \sqrt{\rho^2 - c_1^2}} \; \pm \int \frac{d\theta}{c_1}$$

$$\frac{1}{\sin \alpha} \; \sec^{-1} \frac{r \sin \alpha}{c_1} + c_2 c_1 \; = \pm\, \theta$$

$$r \sin \alpha = c_1 \sec\left[(\pm\, \theta \; - \; c_2 c_1) \sin \alpha\right]$$

4. c. Sphere

$$\int \sqrt{a^2 \left(\frac{d\phi}{d\theta}\right)^2 \; + a^2 \, \sin^2 \, \phi} \, d\theta$$

$$F(\phi, \, \phi') = \sqrt{a^2 \sin^2 \phi + a^2 \phi'^2}$$

$$F - \phi' \; F_{\phi'} = c_1$$

$$\Rightarrow \quad \sqrt{a^2 \, \sin^2 \, \phi \, + \, a^2 \, \phi'^2} - \frac{\phi' \, (2a^2 \, \phi')}{\sqrt{a^2 \, \sin^2 \, \phi \, + \, a^2 \, \phi'^2}} = c_1$$

$$a^2 \, \sin^2 \, \phi \, + \, a^2 \, \phi'^2 \, - \, a^2 \, \phi'^2 = c_1 \sqrt{a^2 \, \sin^2 \, \phi \, + \, a^2 \, \phi'^2}$$

$$\phi'^2 \; = \; \sin^2 \, \phi \, \left[\left(\frac{a \sin \phi}{c_1} \right)^2 \, - \, 1 \right]$$

$$\phi' \; = \; \pm \, \sin \phi \sqrt{\frac{a}{c_1} \sin^2 \, \phi \, - \, 1}$$

$$\frac{d \, \phi}{\sin \phi \sqrt{\frac{a}{c_1} \sin^2 \, \phi \, - \, 1}} \; = \; \pm \, d\theta$$

4. d. Surface is given as

$\vec{r}(\rho\,\theta)$

in parametric form

$$x = \rho \cos \theta$$
$$y = \rho \sin \theta$$
$$z = f(\rho)$$

The length

$$L(\rho,\,\theta) = \int_{t_0}^{t_1} \sqrt{\vec{r}_\rho \cdot \vec{r}_\rho\, \rho'^2 + 2\,\vec{r}_\rho \cdot \vec{r}_\theta\, \rho'\, \theta' + \vec{r}_\theta \cdot \vec{r}_\theta\, \theta'^2}\; dt$$

$$\vec{r}_\rho = \cos\vartheta\, i + \sin\vartheta\, j + f'(\rho)\, k$$

$$\vec{r}_\theta = -\rho \sin\vartheta\, i + \rho\cos\vartheta\, j$$

$$\vec{r}_\rho \cdot \vec{r}_\rho = \cos^2\vartheta + \sin^2\vartheta + f'^2(\rho) = 1 + [f'(\rho)]^2$$

$$\vec{r}_\rho \cdot \vec{r}_\vartheta = 0$$

$$\vec{r}_\vartheta \cdot \vec{r}_\vartheta = \rho^2$$

$$\Rightarrow \quad L = \int_{t_0}^{t_1} \sqrt{(1 + [f'(\rho)]^2)\, \rho'^1 + \rho^2\, \vartheta'^2}\; dt$$

or

$$L = \int \sqrt{(1 + [f'(\rho)]^2) \left(\frac{d\rho}{d\vartheta}\right)^2 + \rho^2}\; d\vartheta$$

So F is a function of ρ and $\dfrac{d\rho}{d\vartheta}$

$$F - \rho'\, F_{\rho'} = c_1$$

$$F_{\rho'} = \frac{1}{2} \left\{ (1 + [f'(\rho)]^2) \left(\frac{d\rho}{d\vartheta}\right)^2 + \rho^2 \right\}^{-1/2} 2\,(1 + [f'(\rho)]^2)\, \rho'$$

$$\sqrt{(1 + [f'(\rho)]^2)\, \rho'^2 + \rho^2} - (1 + [f'(\rho)]^2)\, \frac{\rho'^2}{\sqrt{(1 + [f'(\rho)]^2)\, \rho'^2 + \rho^2}} = c_1$$

$$\rho^2 = c_1 \sqrt{(1 + [f'(\rho)]^2)\, \rho'^2 + \rho^2}$$

$$\left(\frac{\rho^2}{c_1}\right)^2 = (1 + [f'(\rho)]^2)\,\rho'^2 + \rho^2$$

$$\rho' = \pm\sqrt{\frac{\left(\frac{\rho^2}{c_1}\right)^2 - \rho^2}{1 + [f'(\rho)]^2}}$$

5. $F = f(x)\, y'^2$

Using (12) $\quad \dfrac{d}{dx} F_{y'} \;=\; F_y$

$$F_y \;=\; 0$$

$$F_{y'} \;=\; 2f(x)\, y'$$

$$\Rightarrow \quad \frac{d}{dx}\,(2f(x)\, y') \;=\; 0$$

$$f(x)\, y' \;=\; c$$

$$y' \;=\; \frac{c}{f(x)}$$

$$\int dy \;=\; \int \;\frac{c}{f(x)}\; dx$$

$$y \;=\; \int \;\frac{c}{f(x)}\; dx \;+\; k$$

using $y(0) \;=\; 0$

$$y(x) \;=\; \int_0^x \;\frac{c}{f(\zeta)}\; d\zeta$$

$$y(1) \;=\; 1 \;= \quad \Rightarrow \quad \int_0^1 \;\frac{c}{f(\zeta)}\; d\zeta \;=\; 1$$

Substituting for f: $\quad -\displaystyle\int_0^{1/4} c\, d\zeta \;+\; \int_{1/4}^1 c\, d\zeta \;=\; 1$

$$-c\,\frac{1}{4} \;+\; c\left(1 \;-\; \frac{1}{4}\right) \;=\; 1$$

$$\frac{1}{2}\, c \;=\; 1$$

$$\boxed{c = 2}$$

$$\boxed{y(x) \;=\; \int_0^x \frac{2}{f(\zeta)} d\zeta}$$

6. a. $J(y) = \int_0^1 y' dx, \qquad y(0) = 0,\ y(1) = 1$

Euler's equation in this case is

$$\frac{d}{dx} 1 \ = \ 0$$

which is satisfied for all y. Clearly that y should also satisfy the boundary conditions, i.e. $y \ = \ x$.

Looking at this problem from another point of view, notice that $J(y)$ can be computed directly and we have (after using the boundary condition),

$$J(y) \ = \ 1$$

Since this value is constant, the functional is minimzed by any y that satisfies the boundary conditions.

b. $J(y) = \int_0^1 yy' dx, \qquad y(0) = 0,\ y(1) = 1$

Euler's equation in this case is

$$\frac{d}{dx} y \ = \ y'$$

which is the identity $y' \ = \ y'$ which is satisfied for all y. Clearly that y should also satisfy the boundary conditions, i.e. $y \ = \ x$.

Looking at this problem from another point of view, notice that $J(y)$ can be computed directly and we have (after using the boundary condition),

$$J(y) \ = \ \frac{1}{2}$$

Since this value is constant, the functional is minimzed by any y that satisfies the boundary conditions.

c. $J(y) = \int_0^1 xyy' dx, \qquad y(0) = 0,\ y(1) = 1$

Euler's equation in this case is

$$\frac{d}{dx} xy \ = \ xy'$$

which is

$$y + xy' \ = \ xy'$$

or

$$y \ = \ 0.$$

Clearly that y could NOT satisfy the boundary conditions.

7.

a. $J(y) = \int_0^1 \frac{(y')^2}{x^3} dx$

$F = \frac{(y')^2}{x^3}$

Euler equation $\frac{d}{dx} F_{y'} = F_y$

$F_{y'} = \frac{2y'}{x^3}$

$F_y = 0$

Integrate Euler's equation $F_{y'} = c \Longrightarrow \frac{2y'}{x^3} = c$

$2y' = cx^3$

$y' = \frac{cx^3}{2}$

$\boxed{\Longrightarrow y = \frac{cx^4}{8} + b}$

b. $J(y) = \int_0^1 (y^2 + (y')^2 + 2ye^x) dx$

$F = y^2 + (y')^2 + 2ye^x$

$F_x = 2ye^x$

$F_y = 2y + 2e^x$

$F_{y'} = 2y'$

Euler equation $\frac{d}{dx} F_{y'} = F_y$

$\frac{d}{dx} 2y' = 2y + 2e^x$

$y'' - 2y = e^x$

Solve the homogeneous;

$y'' - 2y = 0 \Longrightarrow y = c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x}$

Find a particular solution of the nonhomogeneous;

$y'' - 2y = e^x \Longrightarrow y = 2e^x$

Therefore the general solution of the nonhomogeneous is:

$\boxed{y = c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x} + 2e^x}$

8. Obtain the necessary condition for a function y to be a local minimum of the functional:

$$J(y) = \int_a^b \int_a^b K(s,t)y(s)y(t)dsdt + \int_a^b y(t)^2 dt - 2\int_a^b y(t)f(t)dt$$

Find the first variation of J,

$$\begin{aligned} J(y+\varepsilon\eta) &= \int_a^b \int_a^b K(s,t)[y(s)+\varepsilon\eta(s)][y(t)+\varepsilon\eta(t)]dsdt + \int_a^b [y(t)+\varepsilon\eta(t)]^2 dt \\ &\quad -2\int_a^b [y(t)+\varepsilon\eta(t)]f(t)dt \end{aligned}$$

Then,

$$\begin{aligned} \frac{d}{d\varepsilon}J(y+\varepsilon\eta) &= \int_a^b \int_a^b \{K(s,t)[y(s)+\varepsilon\eta(s)]\eta(t) + K(s,t)[y(t)+\varepsilon\eta(t)]\eta(s)\}\, dsdt \\ &\quad +2\int_a^b [y(t)+\varepsilon\eta(t)]\eta(t)dt - 2\int_a^b \eta(t)f(t)dt \end{aligned}$$

Now letting $\varepsilon = 0$, we have,

$$\begin{aligned} \left.\frac{d}{d\varepsilon}J(y+\varepsilon\eta)\right|_{\varepsilon=0} &= \int_a^b \left\{\int_a^b K(s,t)y(s)ds\right\}\eta(t)dt + \int_a^b \left\{\int_a^b K(s,t)y(t)dt\right\}\eta(s)ds \\ &\quad +2\int_a^b y(t)\eta(t)dt - 2\int_a^b f(t)\eta(t)dt \end{aligned}$$

Since the limits of s and t are constants, we can interchange s for t, and vice versa, in the second of four terms above,

$$= \int_a^b \left\{\int_a^b K(s,t)y(s)ds\right\}\eta(t)dt + \int_a^b \left\{\int_a^b K(t,s)y(s)ds\right\}\eta(t)dt + 2\int_a^b y(t)\eta(t)dt - 2\int_a^b f(t)\eta(t)dt$$

Combining the first two terms and factoring out an $\eta(t)dt$ yields:

$$= \int_a^b \left\{\int_a^b [K(s,t)+K(t,s)]\, y(s)ds + 2y(t) - 2f(t)\right\}\eta(t)dt$$

Setting this equal to 0 implies:

$$\frac{1}{2}\int_a^b [K(s,t)+K(t,s)]\, y(s)ds + y(t) = f(t)$$

Which is a Fredholm equation.

9. Given $F = (1+x)(y')^2$. It is easy to find that

$F_{y'} = 2y'(1+x)$

$F_y = 0$

Therefore $\dfrac{d}{dx}F_{y'} = 0 \Longrightarrow \dfrac{d}{dx}y'(1+x) = 0$

Integrating both sides we obtain,

$y'(1+x) = c_1 \Longrightarrow y' = \dfrac{c_1}{(1+x)}$

Integrating again leads to

$y = c_1 \ln(1+x) + c_2$

Now applying the boundary conditions,

$y(0) = 0 \Longrightarrow c_1 \ln(1+0) + c_2 = 0 \Longrightarrow c_2 = 0$

$y(1) = 1 \Longrightarrow c_1 \ln(1+1) = 1 \Longrightarrow c_1 = \dfrac{1}{\ln 2}$

Therefore the final solution is

$y = \dfrac{\ln(1+x)}{\ln 2}$

It is easy to show that in that case the functional $J(y)$ is $\dfrac{1}{\ln 2}$.

If our boundary condition at $x = 1$ was $y'(1) = 0$, then

$y = c_1 \ln(1+x)$ and $y' = \dfrac{c_1}{1+x}$

Then $y'(1) = \dfrac{c_1}{1+1} = 0 \Longrightarrow c_1 = 0$

and we get the trivial solution.

10. Find the extremal: $J(y) = \int_a^b (x^2 y'^2 + y^2)\, dx$

$$F = x^2 (y')^2 + y^2 \qquad \begin{aligned} F_{y'} &= 2x^2 y' \\ F_y &= 2y \\ \frac{d}{dx} F_{y'} &= 2x^2 y'' + 4xy' \end{aligned}$$

Euler's Equation: $\dfrac{d}{dx} F_{y'} - F_y = 0$

$$\begin{aligned} 2x^2 y'' + 4xy' - 2y &= 0 \\ x^2 y'' + 2xy' - y &= 0 \end{aligned}$$

This is an Euler equation Thus $(a,\, b)$ must not contain the origin.

$$\text{Let} \quad \begin{aligned} y &= x^r \\ y' &= r x^{r-1} \\ y'' &= r(r-1)\, x^{r-2} \end{aligned}$$

Substituting,

$$(r^2 - r)\, x^r + 2r x^r - x^r = 0$$

$$(r^2 - r + 2r - 1)\, x^r = 0$$

$$r^2 - r + 2r - 1 = 0$$

$$r^2 + r - 1 = 0$$

$$r = \frac{-1 \pm \sqrt{1 + 4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

$$y = \left| x \right|^{\frac{-1 - \sqrt{5}}{2}} \quad , \quad y = \left| x \right|^{\frac{-1 + \sqrt{5}}{2}}$$

$$y(x) = c_1 \left| x \right|^{-1.618} + c_2 \left| x \right|^{0.618} \quad \text{for} \quad a \le x \le b$$

CHAPTER 4

4 Variable End-Point Problems

Problems

1. Solve the problem minimize $I = \int_0^{x_1} \left[y^2 - (y')^2\right] dx$

   with left end point fixed and $y(x_1)$ is along the curve

$$x_1 = \frac{\pi}{4}.$$

2. Find the extremals for

$$I = \int_0^1 \left[\frac{1}{2}(y')^2 + yy' + y' + y\right] dx$$

where end values of y are free.

3. Solve the Euler-Lagrange equation for

$$I = \int_a^b y\sqrt{1 + (y')^2}\, dx$$

where

$$y(a) = A, \qquad y(b) = B.$$

   b. Investigate the special case when

$$a = -b, \qquad A = B$$

and show that depending upon the relative size of b, B there may be none, one or two candidate curves that satisfy the requisite endpoints conditions.

4. Solve the Euler-Lagrange equation associated with

$$I = \int_a^b \left[y^2 - yy' + (y')^2\right] dx$$

5. What is the relevant Euler-Lagrange equation associated with

$$I = \int_0^1 \left[y^2 + 2xy + (y')^2\right] dx$$

6. Investigate all possibilities with regard to tranversality for the problem

$$\min \int_a^b \sqrt{1-(y')^2}\, dx$$

7. Determine the stationary functions associated with the integral

$$I = \int_0^1 \left[(y')^2 - 2\alpha y y' - 2\beta y'\right] dx$$

where α and β are constants, in each of the following situations:

a. The end conditions $y(0) = 0$ and $y(1) = 1$ are preassigned.

b. Only the end conditions $y(0) = 0$ is preassigned.

c. Only the end conditions $y(1) = 1$ is preassigned.

d. No end conditions are preassigned.

8. Determine the natural boundary conditions associated with the determination of extremals in each of the cases considered in Problem 1 of Chapter 3.

9. Find the curves for which the functional

$$I = \int_0^{x_1} \frac{\sqrt{1+y'^2}}{y} dx$$

with $y(0) = 0$ can have extrema, if

a. The point (x_1, y_1) can vary along the line $y = x - 5$.

b. The point (x_1, y_1) can vary along the circle $(x-9)^2 + y^2 = 9$.

10. If F depends upon x_2, show that the transversality condition must be replaced by

$$\left[F + (\phi' - y')\frac{\partial F}{\partial y'}\right]\Bigg|_{x=x_2} + \int_{x_1}^{x_2} \frac{\partial F}{\partial x_2}\, dx = 0.$$

11. Find an extremal for

$$J(y) = \int_1^e \left(\frac{1}{2}x^2(y')^2 - \frac{1}{8}y^2\right) dx, \qquad y(1) = 1,\ y(e) \text{ is unspecified.}$$

12. Find an extremal for

$$J(y) = \int_0^1 (y')^2 dx + y(1)^2, \qquad y(0) = 1, y(1) \text{ is unspecified.}$$

1. $F = y^2 - (y')^2$

$$F_y - \frac{d}{dx} F_{y'} \;, = \; 0$$

$$2y - \frac{d}{dx}(-2y') = 0$$

$$y'' + y = 0$$

$$y = A\cos x + B\sin x$$

using $\quad y(0) = 0$

$$\underline{y = B\sin x}$$

Now for the transversity condition

$$F + (\varphi' - y')F_{y'}\Big|_{x=\pi/4} = 0$$

$$\uparrow$$

slope of curve

Since the curve is $x = \dfrac{\pi}{4}$ (vertical line, slope is infinite) we should rewrite the condition

$$F \;\underbrace{\frac{1}{\varphi'}}_{=0} \;+ (1 \;-\; \underbrace{\frac{y'}{\varphi'}}_{=0})\, F_{y'} \;=\; 0$$

$$F_{y'}\Big|_{x=\pi/4} = \; 0$$

$$-2y'\Big|_{x=\pi/4} = \;0 \quad \Rightarrow \quad -2B\;\cos\frac{\pi}{4} \;=\; 0 \quad \Rightarrow \quad \underline{B \;=\; 0}$$

$$\Rightarrow \quad y \;\equiv\; 0.$$

2. $F = \frac{1}{2}y'^2 + yy' + y' + y$

$$F_y - \frac{d}{dx}F_{y'} = 0$$

$$F_y = y' + 1$$

$$F_{y'} = y' + y + 1$$

$$F_y - \frac{d}{dx}F_{y'} = y' + 1 - (y' + y + 1)' = 0$$

$$y' + 1 - y'' - y' = 0$$

$$y'' - 1 = 0$$

$$y_H = Ax + B$$

$$y_P = \frac{1}{2}x^2$$

$$\boxed{y = Ax + B + \frac{1}{2}x^2}$$

Free ends at $x = 0$, $x = 1$

$$F + (\varphi' - y')F_{y'}\Big|_{x=0} = 0$$

$$F + (\varphi' - y')F_{y'}\Big|_{x=1} = 0$$

The free ends are on vertical lines $x = 0$, $x = 1$

$$F_{y'}\Big|_{x=0} = 0 \quad \Rightarrow \quad y'(0) + y(0) + 1 = 0$$

$$A + B + 1 = 0$$

$$F_{y'}\Big|_{x=1} = 0 \quad \Rightarrow \quad y'(1) + y(1) + 1 = 0$$

$$2A + B + \frac{5}{2} = 0$$

$$\left.\begin{array}{l} A+B+1=0 \\[2ex] \underline{2A+B+\dfrac{5}{2}=0} \end{array}\right\} -$$

$$A \quad +3/2=0 \quad \Rightarrow \quad A=-3/2$$
$$B=-A-1=1/2$$

$$\boxed{y=-\frac{3}{2}x+\frac{1}{2}+\frac{1}{2}x^2}$$

3. $F = y\sqrt{1+y'^2}$

$$F_y = \sqrt{1+y'^2}$$

$$F_{y'} = y\frac{1}{2}(1+y'^2)^{-1/2}\,2y'$$

$$F_{y'} = \frac{yy'}{\sqrt{1+y'^2}}$$

$$F - y'F_{y'} = c_1$$

$$y\sqrt{1+y'^2} - \frac{yy'^2}{\sqrt{1+y'^2}} = c_1$$

$$y(1+y'^2) - yy'^2 = c_1\sqrt{1+y'^2}$$

$$\frac{y^2}{c_1^2} = 1 + y'^2$$

$$y' = \pm\sqrt{\frac{y^2}{c_1^2} - 1}$$

$$\int \frac{c_1\,dy}{\sqrt{y^2 - c_1^2}} = \pm\int dx$$

$$c_1\,arc\,\cosh\,\frac{y}{c_1} + c_2 = \pm\,x$$

OR $c_1\,\ln\left|y + \sqrt{y^2 - c_1^2}\right| + c_2 = \pm\,x$

$$arc\,\cosh\,\frac{y}{c_1} = \frac{\pm\,x - c_2}{c_1}$$

$$\boxed{c_1\,\cosh\,\frac{\pm\,x - c_2}{c_1} = y}$$

$$y(a) = A \quad \Rightarrow \quad c_1\cosh\frac{\pm\,a - c_2}{c_1} = A$$

$$y(b) = B \quad \Rightarrow \quad c_1\cosh\frac{\pm\,b - c_2}{c_1} = B$$

$$\left.\begin{aligned} \pm\,a - c_2 &= c_1\,arc\,\cosh\tfrac{A}{c_1} \\ \pm\,b - c_2 &= c_1\,arc\,\cosh\tfrac{B}{c_1} \end{aligned}\right\} -$$

$$\pm a \mp b = c_1 \left\{ arc\ \cosh \frac{A}{c_1} - arc\ \cosh \frac{B}{c_1} \right\} \tag{1}$$

This gives c_1, then

$$c_2 = \pm a - c_1\ arc\ \cosh\ \frac{A}{c_1}. \tag{2}$$

$$\begin{array}{lcc} \text{If} & a = -b, & A = B \\ & \Downarrow & \Downarrow \\ & \text{zero on left} & \text{zero on right} \\ & \text{of (1)} & \text{of (1)} \end{array}$$

Thus we cannot specify c_1, based on that free c_1, we can get c_2 using (2). Thus we have a one parameter family.

4. $F = y^2 - yy' + (y')^2$

$$F - y' F_{y'} = c_1$$

$$y^2 - yy' + y'^2 - y'(-y + 2y') = c_1$$

$$y^2 - (y')^2 = c_1$$

$$(y')^2 = y^2 - c_1$$

$$y' = \pm\sqrt{y^2 - c_1}$$

$$\frac{dy}{\sqrt{y^2 - c_1}} = \pm\, dx$$

$$arc\ \cosh \frac{y}{\sqrt{c_1}} + c_2 = \pm\, x$$

5. $F = y^2 + 2xy + (y')^2$

$$F_y - \frac{d}{dx} F_{y'} = 0$$

$$2y + 2x - \frac{d}{dx}(2y') = 0$$

$$-2y'' + 2y = -2x$$

$$y'' - y = x$$

$$y'' - y = 0 \quad \Rightarrow \quad y_h = Ae^x + Be^{-x}$$

$$y_p = Cx + D$$

$$-Cx - D = x$$

$$C = -1 \quad D = 0$$

$$y_p = -x$$

$$\boxed{y = Ae^x + Be^{-x} - x}$$

6. $F = \sqrt{1 - (y')^2}$

$$F_{y'} = \frac{-2y'}{2\sqrt{1-(y')^2}} \quad \Rightarrow \quad \boxed{\mathrm{y = Ax + B}}$$

$$y' = A$$

$$F + (\phi' - y')\, F_{y'} \Big|_{x=a} = 0$$

$$F + (\psi' - y')\, F_{y'} \Big|_{x=b} = 0$$

$$\sqrt{1 - (y')^2} + (\phi' - y') \frac{-\, y'}{\sqrt{1 - (y')^2}} \Big|_{x=a,\, b} = 0$$

$$1 - (y')^2 + (y')^2 - \phi' y' = 0$$

$$\phi' = \frac{1}{y'}\Big|_{x=a} \quad \Rightarrow \quad \phi' = \frac{1}{A}$$

$$\psi' = \frac{1}{y'}\Big|_{x=b} \quad \Rightarrow \quad \psi' = \frac{1}{A}$$

Therefore if both end points are free then the slopes are the same

$$\varphi'(a) = \psi'(b) = \frac{1}{A}$$

7. $F = (y')^2 - 2\alpha\, yy' - 2\beta\, y'$

a. $y(0) = 0$
$y(1) = 1$

$$F_y - \frac{d}{dx} F_{y'} = 0$$

$$-2\alpha y' - \frac{d}{dx}(2y' - 2\alpha y - 2\beta) = 0$$

$$-2\alpha y' - 2y'' + 2\alpha y' = 0$$

$$y'' = 0$$

$$y = Ax + B$$

$$y(0) = 0 \quad \Rightarrow \quad B = 0$$

$$y(1) = 1 \quad \Rightarrow \quad A = 1$$

$$\Rightarrow \quad \boxed{y = x}$$

b. If only $y(0) = 0 \quad \Rightarrow \quad \boxed{y = Ax}$

Transversality condition at $x = 1$

$$F_{y'}\Big|_{x=1} = 0$$

imples

$$2y'(1) - 2\alpha y(1) - 2\beta = 0$$

Substituting for y

$$2A - 2\alpha A - 2\beta = 0$$
$$A = \frac{\beta}{1-\alpha}$$

Thus the solution is

$$y = \frac{\beta}{1-\alpha} x.$$

c. $y(1) = 1$ only

$$y = Ax + B$$

$$y(1) = 1 \quad \Rightarrow \quad A + B = 1 \quad \Rightarrow \quad B = 1 - A$$

$$\boxed{y = Ax + 1 - A}$$

$$F_{y'}\Big|_{x=0} = 0$$

$$2y'(0) - 2\alpha y(0) - 2\beta = 0$$

$$A - \alpha(1 - A) - \beta = 0$$

$$A(1 + \alpha) = \alpha + \beta$$

$$\boxed{A = \frac{\alpha + \beta}{\alpha + 1}}$$

d. No end conditions

$$y = Ax + B$$

$$y' = A$$

$$2y'(0) - 2\alpha\, y(0) - 2\beta = 0$$

$$2y'(1) - 2\alpha\, y(1) - 2\beta = 0$$

$$\left.\begin{aligned} 2A - 2\alpha\, B \qquad\quad - 2\beta &= 0 \\ 2A - 2\alpha\,(A + B) - 2\beta &= 0 \end{aligned}\right\} -$$

$$2\alpha\, A = 0 \quad \Rightarrow \quad \boxed{A = 0}$$

$$+\,2\alpha B = 2A - 2\beta = -\,2\beta$$

$$\boxed{B = -\frac{\beta}{\alpha}}$$

8. Natural Boundary conditions are

$$F_{y'}\Big|_{x=a,\,b} = 0$$

a. For

$$F = y'^2 - k^2 y^2 \qquad \text{(1a of chapter 3)}$$

$$F_{y'} = 2y'$$

$$\boxed{\begin{aligned} y'(a) &= 0 \\ y'(b) &= 0 \end{aligned}}$$

b. For $F = y'^2 + 2y$ exactly the same

c. For $F = y'^2 + 4xy'$

$$F_{y'} = 2y' + 4x$$

$$\boxed{\begin{aligned} y'(a) + 2a &= 0 \\ y'(b) + 2b &= 0 \end{aligned}}$$

d. $F = y'^2 + yy' + y^2$

$$F_{y'} = 2y' + y$$

$$\boxed{\begin{array}{c} 2y'(a) + y(a) = 0 \\ 2y'(b) + y(b) = 0 \end{array}}$$

e. $F = x\, y'^2 - yy' + y$

$$F_{y'} = 2xy' - y$$

$$\boxed{\begin{array}{c} 2ay'(a) - y(a) = 0 \\ 2by'(b) - y(b) = 0 \end{array}}$$

f. $F = a(x)\, y'^2 - b(x)y^2$

$$F_{y'} = 2a(x)\, y'$$

$$2a(\alpha)\, y'(\alpha) = 0$$

$$2a(\beta)\, y'(\beta) = 0$$

Divide by $2a(\alpha)$ or $2a(\beta)$ to get same as part a.

g. $F = y'^2 + k^2 \cos y$

$$F_{y'} = 2y'$$

same as part a

9. $F = \dfrac{\sqrt{1+y'^2}}{y}$

$$F_y - \frac{d}{dx} F_{y'} = 0$$

$$F_y = -\frac{\sqrt{1+y'^2}}{y^2}$$

$$F_{y'} = \frac{y'}{y\sqrt{1+y'^2}}$$

$$\frac{d}{dx} F_{y'} = \frac{y''y\sqrt{1+y'^2} - y'\left[y'\sqrt{1+y'^2} + y\,\frac{y'y''}{\sqrt{1+y'^2}}\right]}{y^2(1+y'^2)}$$

$$F_y - \frac{d}{dx} F_{y'} = -\frac{\sqrt{1+y'^2}}{y^2} - \frac{y''y(1+y'^2) - y'^2(1+y'^2) - yy'^2y''}{y^2(1+y'^2)\sqrt{1+y'^2}} = 0$$

$$-(1+y'^2)^2 - [y''y - y'^2 - y'^4] = 0$$

$$-1 - 2y'^2 - y'^4 - y''y + y'^2 + y'^4 = 0$$

$$yy'' + y'^2 = -1$$

$$(yy')' = -1$$

$$yy' = -x + c_1$$

$$ydy = (-x + c_1)\,dx$$

$$\frac{1}{2}y^2 = -\frac{x^2}{2} + c_1 x + c_2$$

$$y^2 = -x^2 + 2c_1 x + 2c_2$$

$$y(0) = 0 \quad \Rightarrow \quad c_2 = 0$$

a. Transversality condition: $\phi' = 1$

$$\left[F + (1-y')\,F_{y'}\right]\Big|_{x=x_1} = 0$$

$$\left[\frac{\sqrt{1+y'^2}}{y} + \frac{(1-y')\,y'}{y\sqrt{1+y'^2}}\right]\Bigg|_{x=x_1} = 0$$

$$(1 + y'^2 + y' - y'^2)\Big|_{x=x_1} = 0$$

$$\Rightarrow \quad 1 + y'(x_1) = 0 \quad \Rightarrow \quad y'(x_1) = -1$$

Since $y^2 = -x^2 + 2c_1 x$

$$2yy' = -2x + 2c_1$$

at $x = x_1$

$$\underbrace{2y(x_1)}_{x_1 - 5}\underbrace{y'(x_1)}_{= -1} = -2x_1 + 2c_1$$

on
the line

$$-2(x_1 - 5) = -2x_1 + 2c_1$$

$$\Rightarrow \boxed{c_1 = 5}$$

$$y^2 = -x^2 + 10x \quad \text{or} \quad \boxed{y = \pm\sqrt{10x - x^2}}$$

b. On $(x - 9)^2 + y^2 = 9$

The slope ϕ' is computed

$$2(x - 9) + 2yy' = 0$$

$$yy' = -(x - 9)$$

At $x = x_1$ $\quad \phi'(x_1) = -\dfrac{x_1 - 9}{y(x_1)}$

Remember that at x_1 $y(x_1)$ from the solution:

$$y(x_1)^2 = -x_1^2 + 2c_1 x_1$$

is the same as from the circle

$$y(x_1)^2 + (x_1 - 9)^2 = 9$$

$$\Rightarrow \quad -x_1^2 + 2c_1 x_1 = 9 - (x_1 - 9)^2$$

$$\Rightarrow \quad -x_1^2 + 2c_1 x_1 = 9 - x_1^2 + 18x_1 - 81$$

$$\boxed{c_1 x_1 = 9x_1 - 36} \qquad (*)$$

Substituting in the transversality condition

$$\left[F + (\phi' - y')\,F_{y'}\right]\Big|_{x_1} = 0$$

we have

$$1 + \phi'(x_1) \underbrace{y'(x_1)}_{\left.\frac{-x + c_1}{y}\right|_{x_1}} = 0$$

$$1 - \frac{x_1 - 9}{y(x_1)} \frac{-x_1 + c_1}{y(x_1)} = 0$$

$$1 + \frac{(x_1 - 9)(x_1 - c_1)}{y^2(x_1)} = 0$$

$$\underbrace{y^2(x_1)}_{9-(x_1-9)^2} + (x_1 - 9)(x_1 - c_1) = 0$$

$$9 - (x_1 - 9)^2 + (x_1 - 9)(x_1 - c_1) = 0$$

$$9 - (x_1 - 9)\,[x_1 - 9 - x_1 + c_1)] = 0$$

$$\boxed{9 - (x_1 - 9)(c_1 - 9) = 0}$$

Solve this with (*)

$$c_1\, x_1 = 9x_1 - 36$$

to get:

$$c_1 = 9 - \frac{36}{x_1}$$

$$9 - (x_1 - 9)(-\frac{36}{x_1}) = 0$$

$$9 + 36 = \frac{9.36}{x_1}$$

$$x_1 = \frac{9 \cdot 36}{45} \quad \Rightarrow \quad \boxed{x_1 = \frac{36}{5}} \quad \Rightarrow \quad c_1 = 9 - 5 = 4$$

$$\Rightarrow \quad \boxed{y^2 = -x^2 + 8x}$$

10. In this case equation (8) will have another term resulting from the dependence of F on $x_2(\epsilon)$, that is

$$\int_{x_1(0)}^{x_2(0)} \frac{\partial F}{\partial x_2}\, dx$$

11. In this problem, one boundary is variable and the line along which this variable point moves is given by $y(e) = y_2$ which implies that ϕ is the line $x = e$. First we satisfy *Euler's first equation.* Since $F = \frac{1}{2}x^2(y')^2 - \frac{1}{8}y^2$, we have

$$F_y - \frac{d}{dx}F_{y'} = 0$$

and so,

$$\begin{aligned} 0 &= -\tfrac{1}{4}y - \tfrac{d}{dx}(x^2y') &= -\tfrac{1}{4}y - (2xy' + x^2y'') \\ & &= x^2y'' + 2xy' + \tfrac{1}{4}y \end{aligned}$$

Therefore

$$x^2y'' + 2xy' + \frac{1}{4}y \ = \ 0$$

This is a Cauchy-Euler equation with assumed solution of the form $y = x^r$. Plugging this in and simplifying results in the following equation for r;

$$r^2 + (2-1)r + \frac{1}{4} = 0$$

which has two identical real roots, $r_1 = r_2 = -\frac{1}{2}$ and therefore the solution to the differential equation is;

$$y(x) = c_1x^{-\frac{1}{2}} + c_2x^{-\frac{1}{2}}\ln x$$

The initial condition $y(1) = 1$ implies that $c_1 = 1$. The solution is then

$$y \ = \ x^{-1/2} + c_2x^{-1/2}\ln x.$$

To get the other constant, we have to consider the *transversality condition.* Therefore we need to solve;

$$F + (\phi' - y')F_{y'}|_{x=e} = 0$$

Which means we solve the following (note that ϕ is a vertical line);

$$\begin{aligned} -\tfrac{F}{\phi'} + (1 - \tfrac{y'}{\phi'})F_{y'}\Big|_{x=e} &= F_{y'}|_{x=e} \\ &= x^2y'|_{x=e} \\ &= 0 \end{aligned}$$

which implies that $y'(e) = 0$ is our *natural boundary condition.*

$$y' \ = \ -\frac{1}{2}x^{-3/2} - \frac{1}{2}c_2x^{-3/2}\ln x + c_2x^{-3/2}$$

With this natural boundary condition we get that $c_2 = 1$, and therefore the solution is;

$$y(x) = x^{-\frac{1}{2}}(1 + \ln x)$$

12. Find an extremal for $J(y) = \int_0^1 (y')^2 dx + y(1)^2$, where $y(0) = 1$, $y(1)$ is unspecified.

$F = (y')^2 + y(1)^2$,

$F_y = 0, \qquad F_{y'} = 2y'$.

Notice that since y(1) is unspecified, the right hand value is on the vertical line $x = 1$. By the Fundamental Lemma, an extremal solution, y, must satisfy the Euler equation $F_y - \frac{d}{dx} F_{y'} = 0$.

$0 - \frac{d}{dx} 2y' = 0$,

$-2y'' = 0$,

$y'' = 0$.

Solving this ordinary differential equation via standard integration results in the following:

$y = Ax + B$.

Given the fixed left endpoint equation, $y(0) = 1$, this extremal solution can be further refined to the following:

$y = Ax + 1$.

Additionally, y must satisfy a natural boundary condition at $y(1)$. In this case where y(1) is part of the functional to minimize, we substitute the solution $y = Ax + 1$ into the functional to get:

$$I(A) = \int_0^1 A^2 dx + (A+1)^2 \; = \; A^2 + (A+1)^2$$

Differentiating I with respect to A and setting the derivative to zero (necessary condition for a minimum), we have

$$2A + 2(A+1) \; = \; 0$$

Therefore

$$A = -\frac{1}{2}$$

and the solution is

$$y \; = \; -\frac{1}{2}x + 1.$$

CHAPTER 5

5 Higher Dimensional Problems and Another Proof of the Second Euler Equation

Problems

1. A particle moves on the surface $\phi(x, y, z) = 0$ from the point (x_1, y_1, z_1) to the point (x_2, y_2, z_2) in the time T. Show that if it moves in such a way that the integral of its kinetic energy over that time is a minimum, its coordinates must also satisfy the equations

$$\frac{\ddot{x}}{\phi_x} = \frac{\ddot{y}}{\phi_y} = \frac{\ddot{z}}{\phi_z}.$$

2. Specialize problem 1 in the case when the particle moves on the unit sphere, from $(0, 0, 1)$ to $(0, 0, -1)$, in time T.

3. Determine the equation of the shortest arc in the first quadrant, which passes through the points $(0, 0)$ and $(1, 0)$ and encloses a prescribed area A with the x-axis, where $A \leq \frac{\pi}{8}$.

4. Finish the example on page 51. What if $L = \frac{\pi}{2}$?

5. Solve the following variational problem by finding extremals satisfying the conditions

$$J(y_1, y_2) = \int_0^{\frac{\pi}{4}} \left(4y_1^2 + y_2^2 + y_1' y_2'\right) dx$$

$$y_1(0) = 1,\ y_1\left(\frac{\pi}{4}\right) = 0,\ \ y_2(0) = 0,\ y_2\left(\frac{\pi}{4}\right) = 1.$$

6. Solve the isoparametric problem

$$J(y) = \int_0^1 \left((y')^2 + x^2\right) dx,\ \ y(0) = y(1) = 0,$$

and

$$\int_0^1 y^2 dx = 2.$$

7. Derive a necessary condition for the isoparametric problem
Minimize

$$I(y_1, y_2) = \int_a^b L(x, y_1, y_2, y_1', y_2') dx$$

subject to

$$\int_a^b G(x, y_1, y_2, y_1', y_2')dx \;=\; C$$

and

$$y_1(a) = A_1, \quad y_2(a) = A_2, \quad y_1(b) = B_1, \quad y_2(b) = B_2$$

where C, A_1, A_2, B_1, and B_2 are constants.

8. Use the results of the previous problem to maximize

$$I(x, y) \;=\; \int_{t_0}^{t_1} (x\dot{y} - y\dot{x})dt$$

subject to

$$\int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2}dt \;=\; 1.$$

Show that I represents the area enclosed by a curve with parametric equations $x = x(t)$, $y = y(y)$ and the contraint fixes the length of the curve.

9. Find extremals of the isoparametric problem

$$I(y) \;=\; \int_0^\pi (y')^2 dx, \qquad y(0) = y(\pi) = 0,$$

subject to

$$\int_0^\pi y^2 dx \;=\; 1.$$

1. Kinetic energy E is given by

$$E = \int_0^T \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\,dt$$

The problem is to minimize E subject to

$$\varphi(x, y, z) = 0$$

Let $\quad F(x, y, z) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \lambda\varphi(x, y, z)$

Using (65)

$$F_{y_j} - \frac{d}{dt}F_{y'_j} = 0 \qquad j = 1, 2, 3$$

$$\lambda\varphi_x - \frac{d}{dt}\dot{x} = 0 \quad \Rightarrow \quad \frac{\ddot{x}}{\varphi_x} = \lambda$$

$$\lambda\varphi_y - \frac{d}{dt}\dot{y} = 0 \quad \Rightarrow \quad \frac{\ddot{y}}{\varphi_y} = \lambda$$

$$\lambda\varphi_z - \frac{d}{dt}\dot{z} = 0 \quad \Rightarrow \quad \frac{\ddot{z}}{\varphi_z} = \lambda$$

$$\Rightarrow \quad \frac{\ddot{x}}{\varphi_x} = \frac{\ddot{y}}{\varphi_y} = \frac{\ddot{z}}{\varphi_z}$$

2. If $\varphi \equiv x^2 + y^2 + z^2 - 1 = 0$

then $\dfrac{\ddot{x}}{2x} = \dfrac{\ddot{y}}{2y} = \dfrac{\ddot{z}}{2z} = -\lambda$

$$\ddot{x} + 2\lambda x = 0$$

$$\ddot{y} + 2\lambda y = 0$$

$$\ddot{z} + 2\lambda z = 0$$

Solving
$$\begin{aligned} x &= A \cos \sqrt{2\lambda}\, t + B \sin \sqrt{2\lambda}\, t \\ y &= C \cos \sqrt{2\lambda}\, t + D \sin \sqrt{2\lambda}\, t \\ z &= E \cos \sqrt{2\lambda}\, t + G \sin \sqrt{2\lambda}\, t \end{aligned}$$

Use the boundary condition at $t = 0$

$$x(0) = y(0) = 0 \quad z(0) = 1$$

$$\Rightarrow A = 0$$

$$C = 0$$

$$E = 1$$

Therefore the solution becomes

$$x = B \sin \sqrt{2\lambda}\, t$$

$$y = D \sin \sqrt{2\lambda}\, t$$

$$z = \cos \sqrt{2\lambda\, t} + G \sin \sqrt{2\lambda}\, t$$

The boundary condition at $t = T$

$$x(T) = 0$$

$$y(T) = 0$$

$$z(T) = -1$$

$$\Rightarrow B \sin \sqrt{2\lambda\, t} = 0 \quad \Rightarrow \quad \sqrt{2\lambda\, t} = n\pi$$

$$\sqrt{2\lambda} = \frac{n\pi}{T}$$

same conclusion for y

$$\Rightarrow \quad x = B \sin \frac{n\pi}{T}\, t$$

$$y = D \sin \frac{n\pi}{T} t$$

$$z = \cos \frac{n\pi}{T} t + G \sin \frac{n\pi}{T} t$$

Now use $z(T) = -1$

$$\Rightarrow \quad -1 = \cos n\pi + G \underbrace{\sin n\pi}_{=0}$$

$\Rightarrow$ <u>n is odd</u>

$$\left.\begin{aligned} x &= B \sin \frac{n\pi}{T} t \\ y &= D \sin \frac{n\pi}{T} t \\ z &= \cos \frac{n\pi}{T} t + G \sin \frac{n\pi}{T} t \end{aligned}\right\} \text{n = odd}$$

Now substitute in the kinetic energy integral

$$\begin{aligned} E &= \int_0^T \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\, dt \\ &= \frac{1}{2} \int_0^T \left\{ \left(\frac{n\pi}{T} B \right)^2 \cos^2 \frac{n\pi}{T} t + \left(\frac{n\pi}{T} D \right)^2 \cos^2 \frac{n\pi}{T} t \right. \\ &\quad \left. + \left(- \sin \frac{n\pi}{T} t + G \cos \frac{n\pi}{T} t \right)^2 \left(\frac{n\pi}{T} \right)^2 \right\} dt \\ &= \frac{1}{2} \int_0^T \left\{ \left(\frac{n\pi}{T} \right)^2 \left(B^2 + D^2 + G^2 \right) \cos^2 \frac{n\pi}{T} t \right. \\ &\quad + \left(\frac{n\pi}{T} \right)^2 \sin^2 \left(\frac{n\pi}{T} \right) t \\ &\quad \left. -2G \left(\frac{n\pi}{T} \right)^2 \sin \frac{n\pi}{T} t \cos \frac{n\pi}{T} t \right\} dt \end{aligned}$$

$$\int_0^T \sin \frac{2n\pi}{T} t \, dt = \frac{T}{2n\pi} \cos \frac{2n\pi}{T} t \Big|_0^T = 0$$

$$\int_0^T \underbrace{\sin^2 \frac{n\pi}{T} t}_{\frac{-\cos 2\frac{n\pi}{T} t \,+\, 1}{2}} \, dt \;=\; -\frac{1}{2}\,\frac{T}{2n\pi}\,\sin\,\frac{2n\pi}{T} t \,+\, \frac{1}{2} t \,\Big|_0^T = \frac{T}{2}$$

$$\int_0^T \underbrace{\cos^2 \frac{n\pi}{T} t}_{\frac{\cos \frac{2n\pi}{T} t \,+\, 1}{2}} \, dt \;=\; \frac{T}{2}$$

$$E \;=\; \frac{1}{2}\left(\frac{n\pi}{T}\right)^2 (B^2 \,+\, D^2 \,+\, G^2 \,+\, 1)\,\frac{T}{2}$$

Clearly E increases with n, thus the minimum is for $n = 1$.

Therefore the solution is

$$x \;=\; B\,\sin\,\frac{\pi}{T} t$$

$$y \;=\; D\,\sin\,\frac{\pi}{T} t$$

$$z \;=\; \cos\,\sin\,\frac{\pi}{T} t \,+\, G\,\sin\,\frac{\pi}{T} t$$

3. Min $\quad L = \int_0^1 \sqrt{1 + y'^2}\, dx$

subject to $A = \int_0^1 y\, dx \leq \pi/8$

$$F = \sqrt{1 + y'^2} + \lambda y$$

$$F - y' F_{y'} = c_1$$

$$\lambda y + \sqrt{1 + y'^2} - y' \frac{y'}{\sqrt{1 + y'^2}} = c_1$$

$$\lambda y \sqrt{1 + y'^2} + 1 + y'^2 - y'^2 = c_1 \sqrt{1 + y'^2}$$

$$(\lambda y - c_1) \sqrt{1 + y'^2} = -1$$

$$1 + y'^2 = \frac{1}{(c_1 - \lambda y)^2}$$

$$y' = \pm i \sqrt{- \frac{1}{(c_1 - \lambda y)^2} + 1}$$

$$\frac{dy}{\sqrt{\frac{-1}{(c_1 - \lambda y)^2} + 1}} = \pm i\, dx$$

$$\int \frac{(c_1 - \lambda y)\, dy}{\sqrt{(c_1 - \lambda y)^2 + 1}} = \pm i \int dx$$

Use substitution

$$u = (c_1 - \lambda y)^2 - 1$$

$$\Rightarrow \quad du = 2(c_1 - \lambda y)(-\lambda)\, dy$$

$$\Rightarrow \quad (c_1 - \lambda y)\, dy = -\frac{du}{2\lambda}$$

$$-\frac{1}{2} \int \frac{du}{u^{1/2}} = \pm i \int dx$$

$$-\frac{1}{2\lambda} \frac{u^{1/2}}{1/2} = \pm i\, x + c_2$$

substitute for u

$$-\sqrt{(c_1 - \lambda y)^2 - 1} = (\pm i\, x + c_2)\, \lambda$$

square both sides

$$(c_1 - \lambda y)^2 - 1 = \lambda^2(\pm i\, x + c_2)^2$$

$$\left(-\frac{c_1}{\lambda} + y\right)^2 - \frac{1}{\lambda^2} = (-x^2 \pm 2ix\, c_2 + c_2^2)$$

$$\left(y - \frac{c_1}{\lambda}\right)^2 - \frac{1}{\lambda^2} = - \underbrace{x^2 \mp 2i\, c_2\, x - c_2^2}_{(x+D)^2}$$

$$\boxed{\left(y - \frac{c_1}{\lambda}\right)^2 + (x+D)^2 = \frac{1}{\lambda^2}}$$

We need the curve to go thru (0, 0) and (1, 0)

$$\left.\begin{array}{ll} x = y = 0 \quad \Rightarrow & \left(-\frac{c_1}{\lambda}\right)^2 + D^2 = \frac{1}{\lambda^2} \\[2ex] x = 1,\, y = 0 \quad \Rightarrow & \left(-\frac{c_1}{\lambda}\right)^2 + (1+D)^2 = \frac{1}{\lambda^2} \end{array}\right\} -$$

$$D^2 - (1+D)^2 = 0$$

$$D^2 - 1 - 2D - D^2 = 0$$

$$2D = -1$$

$$\boxed{D = -1/2} \quad \Rightarrow \quad \left(y - \frac{c_1}{\lambda}\right)^2 + \left(x - \frac{1}{2}\right)^2 = \frac{1}{\lambda^2}$$

Let

$$k = -\frac{c_1}{\lambda}$$

then the equation is

$$\left(x - \frac{1}{2}\right)^2 + (y + k)^2 = k^2 + \frac{1}{4}$$

To find k_1 we use the area A

$$A = \int_0^1 y\, dx = \int_0^1 \left[+ \sqrt{k^2 + \frac{1}{4} - \left(x - \frac{1}{2}\right)^2} - k \right] dx$$

use:

$$\int \sqrt{a^2 - u^2}\, du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\, arc \sin \frac{u}{a}$$

where

$$a^2 = k^2 + \frac{1}{4}$$

$$u = x - \frac{1}{2}$$

$$A = \frac{x - \frac{1}{2}}{2}\sqrt{k^2 + \frac{1}{4} - \left(x - \frac{1}{2}\right)^2} + \frac{\left(k^2 + \frac{1}{4}\right)}{2}\, arc \sin \frac{x - 1/2}{\sqrt{k^2 + \frac{1}{4}}} - kx \Big|_0^1$$

$$= \frac{1}{4}\sqrt{k^2 + \frac{1}{4} - \frac{1}{4}} + \frac{\left(k^2 + \frac{1}{4}\right)}{2}\, arc \sin \frac{1/2}{\sqrt{k^2 + \frac{1}{4}}} - k$$

$$- \left\{ -\frac{1}{4}\sqrt{k^2 + \frac{1}{4} - \frac{1}{4}} + \frac{\left(k^2 + \frac{1}{4}\right)}{2}\, arc \sin \frac{1/2}{\sqrt{k^2 + \frac{1}{4}}} \right\}$$

$$A = \frac{1}{2}k - k + \left(k^2 + 1/4\right)\, arc \sin \frac{1}{\sqrt{4k^2 + 1}}$$

$$A + \frac{1}{2}k = \frac{(4k^2 + 1)}{4}\, arc \sin \frac{1}{\sqrt{4k^2 + 1}}$$

$$4A + 2k = (4k^2 + 1)\, arc \sin \frac{1}{\sqrt{4k^2 + 1}} = (4k^2 + 1)\, arc\ cot\, 2k$$

So:

$$4A + 2k = (4k^2 + 1)\, arc\ cot\, 2k$$

and

$$\left(x - \frac{1}{2}\right)^2 + (y + k)^2 = k^2 + \frac{1}{4}$$

4. $c_2^2 + c_1^2 = \lambda^2$ at $(0, 0)$

$$c_2 + (1 - c_1)^2 = \lambda^2 \qquad \text{at} \quad (1, 0)$$

subtract

$$c_1^2 - (1 - c_1)^2 = 0$$

$$c_1^2 - 1 + 2c_1 - c_1^2 = 0$$

$$\boxed{c_1 = 1/2}$$

Now use (34):

Since $y' = \tan\theta$

$$L = \int_0^1 \sec\theta \, dx$$

since

$$sin\,\theta = \frac{x - c_1}{\lambda}$$

$$dx = \lambda \cos\theta \, d\theta$$

$$x = 0 \quad \Rightarrow \quad \sin\theta_1 = -\frac{c_1}{\lambda} = -\frac{1}{2\lambda}$$

$$x = 1 \quad \Rightarrow \quad \sin\theta_2 = \frac{1 - c_1}{\lambda} = \frac{1}{2\lambda}$$

$$\Rightarrow \quad L = \lambda \int_{\theta_1}^{\theta_2} \sec\theta \cos\theta \, d\theta = \lambda(\theta_2 - \theta_1) = 2\lambda \, arc \sin \frac{1}{2\lambda}$$

$$\frac{L}{2\lambda} = arc \sin \frac{1}{2\lambda}$$

Suppose we sketch the two sides as a function of $\dfrac{1}{2\lambda}$

λ_0 is the value such that

$$\frac{L}{2\lambda_0} = arc \sin \frac{1}{2\lambda_0}$$

λ_0 is a function of L !

$$c_2^2 + \frac{1}{4} = \lambda_0^2(L)$$

$$\boxed{c_2^2 = \lambda_0^2(L) - \frac{1}{4}}$$

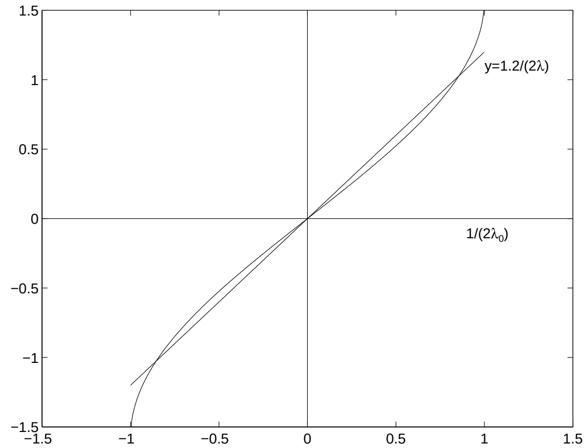

Figure 2:

$$L = \pi/2 \quad \Rightarrow \quad \lambda = 1/2$$

$$c_2^2 = \frac{1}{4} - \frac{1}{4} = 0$$

The curve is then $y^2 + (x - \frac{1}{2})^2 = \frac{1}{4}$

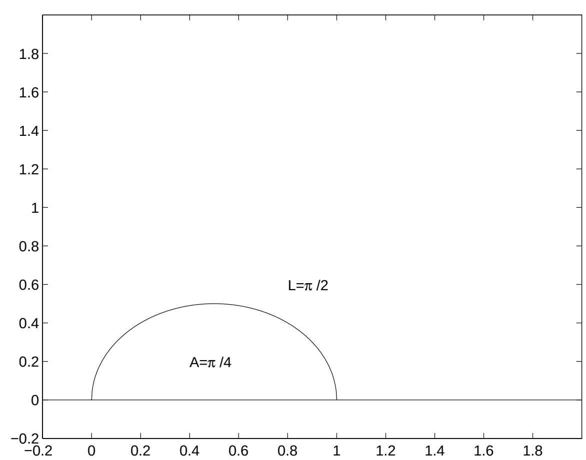

Figure 3:

5. Solve the following variational problem by finding the extremals satisfying the conditions:

$$J(y_1, y_2) = \int_0^{\pi/4} (4y_1^2 + y_2^2 + y_1' y_2')dx$$

$$y_1(0) = 1,\ y_1(\pi/4) = 0,\ y_2(0) = 1,\ y_2(\pi/4) = 1$$

Vary each variable independently by choosing η_1 and η_2 in $C^2[0, \pi/4]$, satisfying:

$$\eta_1(0) = \eta_2(0) = \eta_1(\pi/4) = \eta_2(\pi/4) = 0$$

Form a one parameter admissible pair of functions:

$$y_1 + \varepsilon\eta_1 \text{ and } y_2 + \varepsilon\eta_2$$

Yielding two Euler equations of the form:

$$F_{y_1} - \frac{d}{dx}F_{y_1'} = 0 \text{ and } F_{y_2} - \frac{d}{dx}F_{y_2'} = 0$$

For our problem:

$$F = 4y_1^2 + y_2^2 + y_1' y_2'$$

Taking the partials of F yields:

$$\begin{aligned} F_{y_1} &= 8y_1 \\ F_{y_2} &= 2y_2 \\ F_{y_1'} &= y_2' \\ F_{y_2'} &= y_1' \end{aligned}$$

Substituting the partials with respect to y_1 into the Euler equation:

$$\begin{aligned} 8y_1 - \frac{d}{dx}y_2' &= 0 \\ y_2'' &= 8y_1 \end{aligned}$$

Substituting the partials with respect to y_2 into the Euler equation

$$\begin{aligned} 2y_2 - \frac{d}{dx}y_1' &= 0 \\ y_1'' &= 2y_2 \end{aligned}$$

Solving for y_2 and substituting into the first, second order equation:

$$y_2 = \frac{1}{2}y_1'' \Longrightarrow y_1'''' = 16y_1$$

Since this is a 4^{th} order, homogeneous, constant coefficient, differential equation, we can assume a solution of the form

$$y_1 = e^{rx}$$

Now substituting into $y_1'''' = 16y_1$ gives:

$$\begin{aligned} r^4e^{rx} &= 16e^{rx} \\ r^4 &= 16 \\ r^2 &= \pm 4 \\ r &= \pm 2, \pm 2i \end{aligned}$$

This yields a homogeneous solution of:

$$\begin{aligned} y_1 &= C_1e^{2x} + C_2e^{-2x} + \overline{C}_3e^{2ix} + \overline{C}_4e^{-2ix} \\ &= C_1e^{2x} + C_2e^{-2x} + C_3\cos 2x + C_4\sin 2x \end{aligned}$$

Now using the result from above:

$$\begin{aligned} y_2 &= \frac{1}{2}y_1'' \\ &= \frac{1}{2}\frac{d}{dx}\left(2C_1e^{2x} - 2C_2e^{-2x} - 2C_3\sin 2x + 2C_4\cos 2x\right) \\ &= \frac{1}{2}\left(4C_1e^{2x} + 4C_2e^{-2x} - 4C_3\cos 2x - 4C_4\sin 2x\right) \\ &= 2C_1e^{2x} + 2C_2e^{-2x} - 2C_3\cos 2x - 2C_4\sin 2x \end{aligned}$$

Applying the initial conditions:

$$\begin{aligned} y_1(0) &= 1 \Longrightarrow C_1 + C_2 + C_3 = 1 \\ y_1(\frac{\pi}{4}) &= 0 \Longrightarrow C_1e^{\frac{\pi}{2}} + C_2e^{-\frac{\pi}{2}} + C_4 = 0 \\ y_2(0) &= 0 \Longrightarrow C_1 + C_2 - C_3 = 0 \\ y_2(\frac{\pi}{4}) &= 1 \Longrightarrow C_1e^{\frac{\pi}{2}} + C_2e^{-\frac{\pi}{2}} - C_4 = \frac{1}{2} \end{aligned}$$

We now have 4 equations with 4 unknowns

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ e^{\frac{\pi}{2}} & e^{-\frac{\pi}{2}} & 0 & 1 \\ e^{\frac{\pi}{2}} & e^{-\frac{\pi}{2}} & 0 & -1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

Performing Gaussian Elimination on the augmented matrix:

$$\begin{array}{c} (-1) \\ \hookrightarrow \\ (-1) \\ \hookrightarrow \end{array} \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 & 0 \\ e^{\frac{\pi}{2}} & e^{-\frac{\pi}{2}} & 0 & 1 & 0 \\ e^{\frac{\pi}{2}} & e^{-\frac{\pi}{2}} & 0 & -1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 & -1 \\ e^{\frac{\pi}{2}} & e^{-\frac{\pi}{2}} & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & \frac{1}{2} \end{bmatrix}$$

The augmented matrix yields:

$$\begin{aligned} C_1 + C_2 + C_3 &= 1 \\ -2C_3 &= -1 \Longrightarrow C_3 = \frac{1}{2} \\ -2C_4 &= \frac{1}{2} \Longrightarrow C_4 = -\frac{1}{4} \\ C_1 e^{\frac{\pi}{2}} + C_2 e^{-\frac{\pi}{2}} + C_4 &= 0 \end{aligned}$$

Substituting C_3 and C_4 into the first and fourth equations gives:

$$\begin{aligned} C_1 + C_2 &= \frac{1}{2} \Longrightarrow C_1 = \frac{1}{2} - C_2 \\ C_1 e^{\frac{\pi}{2}} + C_2 e^{-\frac{\pi}{2}} &= \frac{1}{4} \text{ and } C_1 = \frac{1}{2} - C_2 \Longrightarrow C_2 = \frac{\frac{1}{4} e^{-\frac{\pi}{2}} - \frac{1}{2}}{e^{-\pi} - 1} = .4683 \Longrightarrow C_1 = .0317 \end{aligned}$$

Finally:

$$\begin{aligned} y_1 &= .0317 e^{2x} + .4683 e^{-2x} + \frac{1}{2} \cos 2x - \frac{1}{4} \sin 2x \\ y_2 &= .0634 e^{2x} + .9366 e^{-2x} - \cos 2x + \frac{1}{2} \sin 2x \end{aligned}$$

6. The problem is solved using the Lagrangian technique.

$$L = \int_0^1 ((y')^2 + x^2)dx + \lambda \int_0^1 (y^2 - 2)dx$$

$$L = F + \lambda G = (y')^2 + x^2 + \lambda(y^2 - 2)$$

where

$F = (y')^2 + x^2$ and $G = y^2 - 2$

$L_y = 2\lambda y$ and $L_{y'} = 2y'$

Now we use Euler's Equation to obtain

$$\frac{d}{dx}(2y') = 2\lambda y$$

$$y'' = \lambda y$$

Solving for y

$$y = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

Applying the initial conditions,

$$y(0) = A\cos(\sqrt{\lambda}0) + B\sin(\sqrt{\lambda}0) \Longrightarrow A = 0$$

$$y(1) = B\sin(\sqrt{\lambda}) = 0$$

If $B = 0$ then we get the trivial solution. Therefore, we want $\sin(\sqrt{\lambda}) = 0$.

This implies that $\sqrt{\lambda} = n\pi$, $n = 1, 2, 3, \ldots$

Now we solve for B using our constraint.

$$y = B\sin(n\pi x)$$

$$\int_0^1 y^2 dx = \int_0^1 B^2 \sin^2(n\pi x)dx = 2$$

$$B^2\left[\frac{x}{2} - \frac{\sin 2\pi x}{4\pi}\right]_0^1 = 2 \Longrightarrow B^2\left[(\frac{1}{2} - 0) - 0\right] = 2$$

$B^2 = 4$ or $B = \pm 2$.

Therfore, our final solution is

$$y = \pm 2\sin(n\pi x),\ n = 1, 2, 3, \ldots$$

7. Derive a necessary condition for the isoperimetric problem.

Minimize $\quad I(y_1, y_2) = \int_a^b L(x, y_1, y_2, y_1', y_2')\,dx$

subject to $\quad \int_a^b G(x, y_1, y_2, y_1', y_2')\,dx = C$

and $\quad y_1(a) = A_1$, $y_2(a) = A_2$, $y_1(b) = B_1$, $y_2(b) = B_2$

where $\quad A_1, A_2, B_1, B_2$, and C are constants.

Assume L and G are twice continuously differentiable functions. The fact that $\int_a^b G(x, y_1, y_2, y_1', y_2')\,dx = C$ is called an isoperimetric constraint.
Let $W = \int_a^b G(x, y_1, y_2, y_1', y_2')\,dx$

We must embed an assumed local minimum $y(x)$ in a family of admissible functions with respect to which we carry out the extremization. Introduce a two-parameter family

$$z_i = y_i(x) + \varepsilon_i \eta_i(x) \qquad i = 1, 2$$

where $\eta_1, \eta_2 \; \epsilon\, C^2(a, b)$ and

$$\eta_i(a) = \eta_i(b) = 0 \qquad i = 1, 2 \tag{11}$$

and ε_1 , ε_2 are real parameters ranging over intervals containing the orign. Assume W does not have an extremum at y_i then for any choice of η_1 and η_2 there will be values of ε_1 and ε_2 in the neighborhood of $(0, 0)$, for which $W(z) = C$.

Evaluating I and W at z gives

$$J(\varepsilon_1, \varepsilon_2) = \int_a^b L(x, z_1, z_2, z_1', z_2')\,dx \text{ and } V(\varepsilon_1, \varepsilon_2) = \int_a^b G(x, z_1, z_2, z_1', z_2')\,dx$$

Since y is a local minimum subject to V, the point $(\varepsilon_1, \varepsilon_2) = (0, 0)$ must be a local minimum for $J(\varepsilon_1, \varepsilon_2)$ subject to the constraint $V(\varepsilon_1, \varepsilon_2) = C$.

This is just a differential calculus problem and so the Lagrange multiplier rule may be applied. There must exist a constant λ such that

$$\frac{\partial J^*}{\partial \varepsilon_1} = \frac{\partial J^*}{\partial \varepsilon_2} = 0 \quad \text{at} \quad (\varepsilon_1, \varepsilon_2) = (0, 0) \tag{12}$$

where J^* is defined by

$$J^* = J + \lambda V = \int_a^b L^*(x, z_1, z_2, z_1', z_2')\,dx$$

with

$$L^* = L + \lambda G$$

We now calculate the derivatives in (12), afterward setting $\varepsilon_1 = \varepsilon_2 = 0$. Accordingly,

$$\frac{\partial J^*}{\partial \varepsilon_i}(0,\, 0) = \int_a^b \left[L_y^*(x,\, y_1,\, y_2,\, y_1',\, y_2')\,\eta_i + L_{y'}^*(x,\, y_1,\, y_2,\, y_1',\, y_2')\,\eta_i'\right] dx \quad i = 1,\, 2$$

Integrating the second term by parts (as in the notes) and applying the conditions of (11) gives

$$\frac{\partial J^*}{\partial \varepsilon_i}(0,\, 0) = \int_a^b [L_y^*(x,\, y_1,\, y_2,\, y_1',\, y_2') - \frac{d}{dx} L_{y'}^*(x,\, y_1,\, y_2,\, y_1',\, y_2')]\,\eta_i'\, dx \quad i = 1,\, 2$$

Therefore from (12), and because of the arbitrary character of η_1 or η_2, the Fundamental Lemma implies

$$L_y^*(x,\, y_1,\, y_2,\, y_1',\, y_2') - \frac{d}{dx} L_{y'}^*(x,\, y_1,\, y_2,\, y_1',\, y_2') = 0$$

Which is a necessary condition for an extremum.

8. Let the two dimensional position vector $\vec{R}$ be $\vec{R} = x\vec{i} + y\vec{j}$, then the velocity vector $\vec{v} = \dot{x}\vec{i} + \dot{y}\vec{j}$. From vector calculus it is known that the triple $\vec{a} \cdot \vec{b} \times \vec{c}$ gives the volume of the parallelepiped whose edges are these three vectors. If one of the vectors is of length unity then the volume is the same as the area of the parallelogram whose edges are the other 2 vectors. Now lets take $\vec{a} = \vec{k}$, $\vec{b} = \vec{R}$ and $\vec{c} = \vec{v}$. Computing the triple, we have $x\dot{y} - \dot{x}y$ which is the integrand in I. The second integral gives the length of the curve from t_0 to t_1 (see definition of arc length in any Calculus book).

To use the previous problem, let

$$L(t, x, y, \dot{x}, \dot{y}) = x\dot{y} - \dot{x}y$$

$$G(t, x, y, \dot{x}, \dot{y}) = \sqrt{\dot{x}^2 + \dot{y}^2}$$

then

$$\begin{array}{lll} L_x = \dot{y} & & L_y = -\dot{x} \\ G_x = 0 & & G_y = 0 \\ L_{\dot{x}} = -y & & L_{\dot{y}} = x \\ G_{\dot{x}} = \dfrac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} & & G_{\dot{y}} = \dfrac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \end{array}$$

Substituting in the Euler equations, we end up with the two equations:

$$\dot{y}\left\{2 - \lambda\frac{\ddot{x}\dot{y} - \dot{x}\ddot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}\right\} = 0$$

$$\dot{x}\left\{-2 + \lambda\frac{\ddot{x}\dot{y} - \dot{x}\ddot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}\right\} = 0$$

Case 1: $\dot{y} = 0$
Substituting this in the second equation, yields $\dot{x} = 0$.
Thus the solution is $x = c_1, \quad y = c_2$

Case 2: $\dot{x} = 0$, then the first one yields $\dot{y} = 0$ and we have the same solution.

Case 3: $\dot{x} \neq 0$, and $\dot{y} \neq 0$
In this case the term in the braces is zero, or

$$\frac{2}{\lambda}(\dot{x}^2 + \dot{y}^2)^{3/2} = \ddot{x}\dot{y} - \dot{x}\ddot{y}$$

The right hand side can be written as $\dot{y}^2 \dfrac{d}{dt}\left(\dfrac{\dot{x}}{\dot{y}}\right)$.

Now let $u = \dfrac{\dot{x}}{\dot{y}}$, we get

$$\frac{du}{(1 + u^2)^{3/2}} = \frac{2}{\lambda}dy$$

For this we use the trigonometric substitution $u = \tan\theta$. This gives the following:

$$\frac{\dot{x}}{\dot{y}} = \left(\frac{2}{\lambda}y + c\right)\sqrt{1 + (\frac{\dot{x}}{\dot{y}})^2}$$

Simplifying we get

$$dx = \frac{\frac{2}{\lambda}y + c}{\sqrt{1 - (\frac{2}{\lambda}y + c)^2}}dy$$

Substitute $v = 1 - (\dfrac{2}{\lambda}y + c)^2$ and we get

$$\left(y + \frac{\lambda c}{2}\right)^2 + \left(x + \frac{\lambda k}{2}\right)^2 = \left(\frac{\lambda}{2}\right)^2$$

which is the equation of a circle.

9. Let $\bar{F} = F + \lambda G = (y')^2 + \lambda y^2$. Then Euler's first equation gives;

$$\begin{aligned} 2\lambda y - \tfrac{d}{dx}(2y') = 0 \;\; &\Rightarrow \;\; 2\lambda y - 2y'' = 0 \\ &\Rightarrow \;\; y'' - \lambda y = 0 \\ &\Rightarrow \;\; r^2 - \lambda = 0 \\ &\Rightarrow \;\; r = \pm\sqrt{\lambda} \end{aligned}$$

Where we are substituting the assumed solution form of $y = e^{rx}$ into the differential equation to get an equation for r. Note that $\lambda = 0$ and $\lambda > 0$ both lead to trivial solutions for $y(x)$ and there would be no way to satisfy the condition that $\int_o^\pi y^2 dx = 1$. Therefore, assume that $\lambda < 0$. We then have that the solution has the form;

$$y(x) = c_1 cos(\sqrt{-\lambda}x) + c_2 sin(\sqrt{-\lambda}x)$$

The initial conditions result in $c_1 = 0$ and $c_2 sin(\sqrt{-\lambda}\pi) = 0$. Since $c_2 = 0$ would give us the trivial solution again, it must be that $\sqrt{-\lambda}\pi = n\pi$, where $n = 1, 2, \ldots$. This implies that $-\lambda = n^2$ or eqivalently $\lambda = -n^2$, $n = 1, 2, \ldots$.

We now use this solution and the requirement $\int_o^\pi y^2 dx = 1$ to solve for the constant c_2. Therefore, we have;

$$\begin{aligned} \int_0^\pi c_2^2 sin^2(nx)dx &= \int_0^{n\pi} \frac{c_2^2}{n} sin^2 u du \\ &= \frac{c_2^2}{n}\left(\frac{u}{2} - \frac{sin(2u)}{4}\right)\Bigg|_0^{n\pi} \\ &= \frac{c_2^2\pi}{2} - \frac{sin(2n\pi)}{4} \\ &= \frac{c_2^2\pi}{2} \\ &= 1, \quad \text{for } n = 1, 2, \ldots \end{aligned}$$

After solving for the constant we have that;

$$y(x) = \pm\sqrt{\frac{2}{\pi}} sin(nx), \quad n = 1, 2, \ldots$$

If we now plug this solution into the equation $\displaystyle\int_0^\pi (y')^2 dx$ we get that $I(y) = n^2$ which implies we should choose $n = 1$ to minimize $I(y)$. Therefore, our final solution is;

$$y(x) = \pm\sqrt{\frac{2}{\pi}} sin(x)$$

CHAPTER 6

6 Integrals Involving More Than One Independent Variable

Problem

1. Find all minimal surfaces whose equations have the form $z = \phi(x) + \psi(y)$.

2. Derive the Euler equation and obtain the natural boundary conditions of the problem

$$\delta \int\int_R \left[\alpha(x,y)u_x^2 + \beta(x,y)u_y^2 - \gamma(x,y)u^2\right] dxdy = 0.$$

In particular, show that if $\beta(x,y) = \alpha(x,y)$ the natural boundary condition takes the form

$$\alpha \frac{\partial u}{\partial n}\delta u = 0$$

where $\dfrac{\partial u}{\partial n}$ is the normal derivative of u.

3. Determine the natural boundary condition for the multiple integral problem

$$I(u) = \int\int_R L(x,y,u,u_x,u_y)dxdy, \quad u\epsilon C^2(R), \quad u \text{ unspecified on the boundary of } R$$

4. Find the Euler equations corresponding to the following functionals

   a. $I(u) = \int\int_R (x^2u_x^2 + y^2u_y^2)dxdy$

   b. $I(u) = \int\int_R (u_t^2 - c^2u_x^2)dxdt, c$ is constant

1. $z = \phi(x) + \psi(y)$

$$S = \int\int_R \sqrt{1 + z_x^2 + z_y^2}\, dx\, dy$$

$$= \int\int_R \sqrt{1 + \phi'^2(x) + \psi'^2(y)}\, dx\, dy$$

$$\frac{\partial}{\partial x}\left(\frac{\phi'(x)}{\sqrt{1 + \phi'^2 + \psi'^2}}\right) + \frac{\partial}{\partial y}\left(\frac{\psi'(y)}{\sqrt{1 + \phi'^2 + \psi'^2}}\right) = 0$$

Differentiate and multiply by $1 + \phi'^2 + \psi'^2$

$$\phi''(x)\sqrt{1 + \phi'^2 + \psi'^2} - \phi'^2\,\phi''\,[\sqrt{1 + \phi'^2 + \psi'^2}]^{-1/2} +$$

$$\psi''(y)\sqrt{1 + \phi'^2 + \psi'^2} - \psi'^2\,\psi''\,[\sqrt{1 + \phi'^2 + \psi'^2}]^{-1/2} = 0$$

Expand and collect terms

$$\phi''(x)\sqrt{1 + \psi'^2 + (y)} + \psi''(y)\,[\sqrt{1 + \phi'^2 + (x)}] = 0$$

Separate the variables

$$\frac{\phi''(x)}{1 + \phi'^2 + (y)} = -\frac{\psi''(y)}{1 + \psi'^2 + (x)}$$

One possibility is

$$\phi''(x) = \psi''(y) = 0 \quad \Rightarrow \quad \begin{aligned} \phi(x) &= Ax + \alpha \\ \psi(y) &= By + \beta \end{aligned}$$

$$\Rightarrow z = Ax + By + C \quad \text{which is a plane}$$

The other possibility is that each side is a constant (left hand side is a function of only x and the right hand side depends only on y)

$$\frac{\phi''(x)}{1 + \phi'^2(x)} = \lambda = -\frac{\psi''(y)}{1 + \psi'^2 + (y)}$$

Let $\quad \xi = \phi'(x) \quad$ then

$$\frac{\xi'}{1 + \xi^2} = \lambda$$

$$\frac{d\xi}{1 + \xi^2} = \lambda\, dx$$

$$arc \tan \xi = \lambda x + c_1$$

$$\xi = \tan(\lambda x + c_1)$$

Integrate again

$$\phi(x) = \int \tan(\lambda x + c_1)\,dx$$

$$\phi(x) = -\frac{1}{\lambda}\ln\Big|\cos(\lambda x + c_1)\Big| + c_2$$

$$e^{(c_2 - \phi(x))} = \cos(\lambda x + c_1)) \tag{1}$$

Similarly for $\psi(y)$ (sign is different !)

$$\psi(y) = \frac{1}{\lambda}\ln\Big|\cos(\lambda y - D_1)\Big| + D_2$$

$$e^{\lambda(\psi(y) - D_2)} = \cos(\lambda y - D_1)$$

Divide equation (2) by equation (1)

$$e^{\lambda(-c_2 - D_2 + \psi(y) + \phi(x))} = \frac{\cos(\lambda y - D_1)}{\cos(\lambda x + c_1)}$$

using $z = \phi(x) + \psi(y)$ we have

$$e^{\lambda(-c_2 - D_2)}\,e^{\lambda z} = \frac{\cos(\lambda y - D_1)}{\cos(\lambda x + c_1)}$$

If we let (x_0, y_0, z_0) be on the surface, we find

$$e^{\lambda(z - z_0)} = \frac{\cos(\lambda y - D_1)}{\cos(\lambda x + c_1)}\,\frac{\cos(\lambda x_0 + c_1)}{\cos(\lambda y_0 - D_1)}$$

2. $F = \alpha(x, y)\, u_x^2 + \beta(x, y)\, u_y^2 - \gamma(x, y)\, u^2$

$$-F_u + \frac{\partial}{\partial x} F_{u_x} + \frac{\partial}{\partial y} F_{u_y} = 0 \quad \text{(see equation 11)}$$

$$F_{u_x} = 2\alpha(x, y)\, u_x$$

$$F_{u_y} = 2\beta(x, y)\, u_y$$

$$F_u = -2\gamma(x, y)\, u$$

$$\Rightarrow \quad \frac{\partial}{\partial x}(\alpha(x, y)\, u_x) + \frac{\partial}{\partial y}(\beta(x, y)\, u_y) + \gamma(x, y)\, u = 0$$

The natural boundary conditions come from the boundary integral

$$F_{u_x} \cos \nu + F_{u_y} \sin \nu = 0$$

$$(\alpha(x, y)\, u_x \cos \nu + \beta(x, y)\, u_y \sin \nu) = 0$$

If $\alpha(x, y) = \beta(x, y)$ then

$$\alpha(x, y)\, \underbrace{(u_x \cos \nu + u_y \sin \nu)}_{\underbrace{\nabla u \cdot \vec{n}}_{= \frac{\partial u}{\partial n}}} = 0$$

$$\Rightarrow \frac{\partial u}{\partial n} = 0$$

3. Determine the natural boundary condition for the muliple integral problem

$$I(u) = \int\int_R L(x, y, u, u_x, u_y)dxdy, u \in C^2(R),$$

u unspecified on the boundary of R.

Let $u(x, y)$ be a minimizing function (among the admissible functions) for $I(u)$. Consider the one-parameter family of functions $u(\varepsilon) = u(x, y) + \varepsilon\eta(x, y)$ where $\eta \in C^2$ over R and $\eta(x, y) = 0$ on the boundary of R. Then if

$$I(\varepsilon) = \int\int_R L(x, y, u + \varepsilon\eta, u_x + \varepsilon\eta_x, u_y + \varepsilon\eta_y)dxdy,$$

a necessary condition for a minimum is $I'(0) = 0$.

Now, $I'(0) = \int\int_R (\eta L_u + \eta_x L_{u_x} + \eta_y L_{u_y})dxdy$, where the arguments in the partial derivatives of L are the elements (x, y, u, u_x, u_y) of the minimizing function u. Thus,

$$I'(0) = \int\int_R \eta(L_u - \frac{\partial}{\partial x}L_{u_x} - \frac{\partial}{\partial y}L_{u_y})dxdy + \int\int_R (\frac{\partial}{\partial x}(\eta L_{u_x}) + \frac{\partial}{\partial y}(\eta L_{u_y}))dxdy.$$

The second integral in this equation is equal to (by Green's Theorem)

$$\oint_{\partial R} \eta(\ell L_{u_x} + mL_{u_y})ds$$

where ℓ and m are the direction cosines of the outward normal to ∂R and ds is the arc length of the ∂R . But, since $\eta(x, y) = 0$ on ∂R, this integral vanishes. Thus, the condition $I'(0) = 0$ which holds for all admissible $\eta(x, y)$ reduces to

$$\int\int_R \eta(L_u - \frac{\partial}{\partial x}L_{u_x} - \frac{\partial}{\partial y}L_{u_y})dxdy = 0.$$

Therefore, $L_u - \frac{\partial}{\partial x}L_{u_x} - \frac{\partial}{\partial y}L_{u_y} = 0$ at all points of R. This is the Euler-Lagrange equation (11) for the two dimensional problem.

Now consider the problem

$$I(u) = \int\int_R L(x, y, u, u_x, u_y)dxdy = \int_c^d \int_a^b L(x, y, u, u_x, u_y)dxdy$$

where all or or a portion of the ∂R is unspecified. This condition is analogous to the single integral variable endpoint problem discussed previously. Recall the line integral presented above:

$\oint_{\partial R} \eta(\ell L_{u_x} + mL_{u_y})ds$ where ℓ and m are the direction cosines of the outward normal to ∂R and ds is the arc length of the ∂R . Recall that in the case where u is given on ∂R (analogous to fixed endpoint) this integral vanishes since $\eta(x, y) = 0$ on ∂R. However, in the case where on all or a portion of ∂R u is unspecified, $\eta(x, y) \neq 0$. Therefore, the natural boundary condition which must hold on ∂R is $\ell L_{u_x} + mL_{u_y} = 0$ where ℓ and m are the direction cosines of the outward normal to ∂R.

4. Euler's equation

$$\frac{\partial}{\partial x}F_{u_x} \; + \; \frac{\partial}{\partial y}F_{u_y} \; - \; F_u \; = \; 0$$

a. $F \; = \; x^2 u_x^2 + y^2 u_y^2$
Differentiate and substitute in Euler's equation, we have

$$2xu_x + x^2 u_{xx} + 2yu_y + y^2 u_{yy} = 0$$

b. $F \; = \; u_t^2 - c^2 u_x^2$
Differentiate and substitute in Euler's equation, we have

$$u_{tt} - c^2 u_{xx} = 0$$

which is the wave equation.

CHAPTER 7

7 Examples of Numerical Techniques

Problems

1. Find the minimal arc $y(x)$ that solves, minimize $I = \int_0^{x_1} \left[y^2 - (y')^2\right] dx$

   a. Using the indirect (fixed end point) method when $x_1 = 1$.

   b. Using the indirect (variable end point) method with $y(0)$=1 and $y(x_1) = Y_1 = x_2 - \frac{\pi}{4}$.

2. Find the minimal arc $y(x)$ that solves, minimize $I = \int_0^1 \left[\frac{1}{2}(y')^2 + yy' + y' + y\right] dx$ where $y(0) = 1$ and $y(1) = 2$.

3. Solve the problem, minimze $I = \int_0^{x_1} \left[y^2 - yy' + (y')^2\right] dx$

   a. Using the indirect (fixed end point) method when $x_1 = 1$.

   b. Using the indirect (variable end point) method with $y(0)$=1 and $y(x_1) = Y_1 = x_2 - 1$.

4. Solve for the minimal arc $y(x)$:

$$I = \int_0^1 \left[y^2 + 2xy + 2y'\right] dx$$

   where $y(0) = 0$ and $y(1) = 1$.

1.

a. Here is the Matlab function defining all the derivatives required

```
% odef.m
function xdot=odef(t,x)
% fy1fy1 - fy'y' (2nd partial wrt y' y')
% fy1y   - fy'y  (2nd partial wrt y' y)
% fy       - fy  (1st partial wrt y)
% fy1x   - fy'x  (2nd partial wrt y' x)
fy1y1  = -2;
fy1y   = 0;
fy       = 2*x(1);
fy1x   = 0;
rhs2=[-fy1y/fy1y1,(fy-fy1x)/fy1y1];
xdot=[x(2),rhs2(1)*x(2)+rhs2(2)]';
```

The graph of the solution is given in the following figure

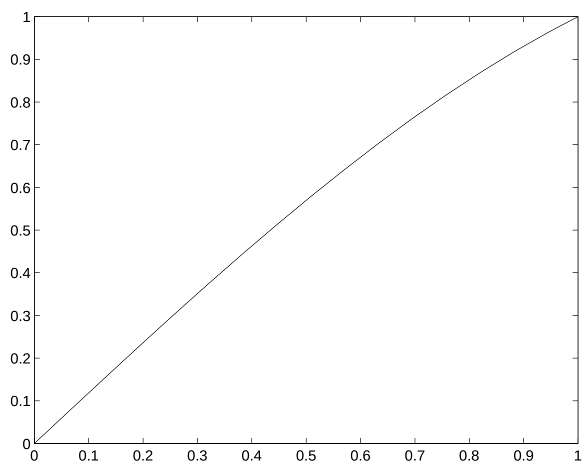

Figure 4:

2. First we give the modified finput.m

```
% function VALUE = FINPUT(x,y,yprime,num) returns the value of the
% functions F(x,y,y'), Fy(x,y,y'), Fy'(x,y,y') for a given num.
% num defines which function you want to evaluate:
%       1 for F, 2 for Fy, 3 for Fy'.

if nargin < 4, error('Four arguments are required'), break, end
if (num < 1) | (num > 3)
  error('num must be between 1 and 3'), break
end

if num == 1, value = .5*yp^2+yp*y+yp+y; end     % F
if num == 2, value = yp+1; end                                   % Fy
if num == 3, value = yp+y+1; end                               % Fy'
```

The boundary conditions are given in the main program dmethod.m (see lecture notes).

The graph of the solution (using direct method) follows

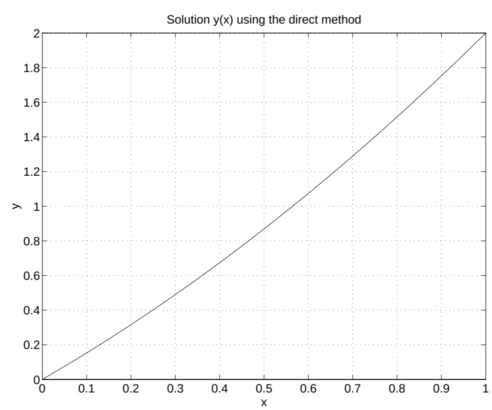

Figure 5:

3.

a. Here is the Matlab function defining all the derivatives required

```
% odef.m
function xdot=odef(t,x)
% fy1fy1 - fy'y' (2nd partial wrt y' y')
% fy1y   - fy'y  (2nd partial wrt y' y)
% fy       - fy  (1st partial wrt y)
% fy1x   - fy'x  (2nd partial wrt y' x)
fy1y1  = 2;
fy1y   = -1;
fy       = 2*x(1)-x(2);
fy1x   = 0;
rhs2=[-fy1y/fy1y1,(fy-fy1x)/fy1y1];
xdot=[x(2),rhs2(1)*x(2)+rhs2(2)]';
```

The graph of the solution is given in the following figure

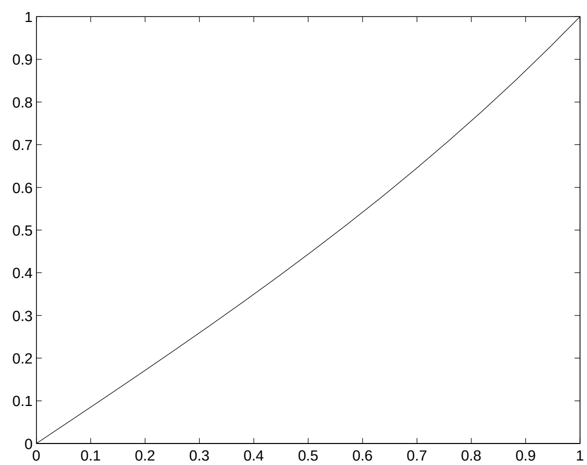

Figure 6:

4. First we give the modified finput.m

```
% function VALUE = FINPUT(x,y,yprime,num) returns the value of the
% functions F(x,y,y'), Fy(x,y,y'), Fy'(x,y,y') for a given num.
% num defines which function you want to evaluate:
%       1 for F, 2 for Fy, 3 for Fy'.

if nargin < 4, error('Four arguments are required'), break, end
if (num < 1) | (num > 3)
  error('num must be between 1 and 3'), break
end

if num == 1, value = y^2+2*x*y+2*yp; end     % F
if num == 2, value = 2*y+2*x; end                % Fy
if num == 3, value = 2; end                         % Fy'
```

The boundary conditions are given in the main program dmethod.m (see lecture notes).

The graph of the solution (using direct method) follows

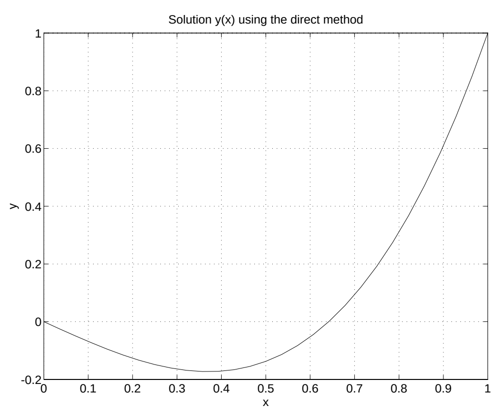

Figure 7:

CHAPTER 8

8 The Rayleigh-Ritz Method

Problems

1. Write a MAPLE program for the Rayleigh-Ritz approximation to minimize the integral

$$I = \int_0^1 \left[(y')^2 - y^2 - 2xy \right] dx$$

$$y(0) = 1$$

$$y(1) = 2.$$

Plot the graph of y_0, y_1, y_2 and the exact solution.

2. Solve the same problem using finite differences.

1.

```
with(plots):
phi0:= 1+x:
y0 :=phi0:
p0:=plot(y0,x=0..1,color=yellow,style=point):
phi0:= 1+x:phi1:= a1*x*(1-x):
y1 :=phi0 + phi1:
dy1 :=diff(y1,x):
f := (dy1^2 - y1^2 - 2*x*y1):
w := int(f,x=0..1):
dw := diff(w,a1):
a1:= fsolve(dw=0,a1):
p1:=plot(y1,x=0..1,color=green,style=point):
phi0:= 1+x:phi1:= b1*x*(1-x):phi2 := b2*x*x*(1-x):
y2 :=phi0 + phi1 + phi2:
dy2 :=diff(y2,x):
f := (dy2^2 - y2^2 - 2*x*y2):
w := int(f,x=0..1):
dw1 := diff(w,b1):
c_1:=solve(dw1=0,b1):
dw2 := diff(w,b2):
c_2:=solve(dw2=0,b1):
b3:= c_1-c_2:
b2:=solve(b3=0,b2):
b1:=c_1:
p2:=plot(y2,x=0..1,color=cyan,style=point):
phi0:= 1+x:
phi1:= c1*x*(1-x):
phi2 := c2*x*x*(1-x):
phi3 := c3*x*x*x*(1-x):
y3 :=phi0 + phi1 + phi2 + phi3:
dy3 :=diff(y3,x):
f := (dy3^2 - y3^2 - 2*x*y3):
w := int(f,x=0..1):
dw1 := diff(w,c1):
c_1:=solve(dw1=0,c1):
dw2 := diff(w,c2):
c_2:=solve(dw2=0,c1):
dw3 := diff(w,c3):
c_3:=solve(dw3=0,c1):
a1:= c_1 - c_2:
a_1:=solve(a1=0,c2):
a2:= c_3 - c_2:
```

```
a_2:=solve(a2=0,c2):
b1:= a_1 - a_2:
c3:=solve(b1=0,c3):
c2:=a_1:
c1:=c_1:
p3:=plot(y3,x=0..1,color=blue,style=point):
y:= cos(x) +((3-cos(1))/sin(1))*sin(x) - x:
p:=plot(y,x=0..1,color=red,style=line):
display({p,p0,p1,p2,p3});
```

Note: Delete p2 or p3 (or both) if you want to make the True versus Approximations more noticable.

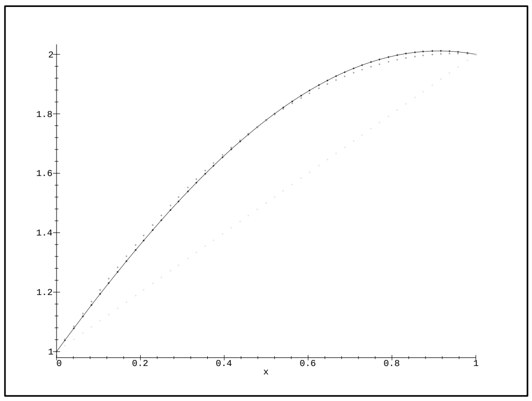

Figure 8:

2.

```
F =dy^2-y^2-2*y*x;
with(plots):
f = (((y[i+1]-y[i])/delx)^2 - y[i]^2 - 2*x[i]*y[i]);
phi1 :=sum((((y1[i+1]-y1[i])/delx1)^2 - y1[i]^2 - 2*x1[i]*y1[i])*delx1,'i'=0..1):
dy1[0] := diff(phi1,y1[0]):
dy1[1] := diff(phi1,y1[1]):
dy1[2] := diff(phi1,y1[2]):
x1[0]:=0:
x1[1]:=.5:
x1[2]:=1:
delx1 := 1/2:
y1[0] := 1:
y1[2]:=2:
y1[1]:=solve(dy1[1]=0,y1[1]):
p1:=array(1..6,[x1[0],y1[0],x1[1],y1[1],x1[2],y1[2]]):
p1:=plot(p1):

phi2 :=sum((((y2[i+1]-y2[i])/delx2)^2 - y2[i]^2 - 2*x2[i]*y2[i])*delx2,'i'=0..2):
dy2[0] := diff(phi2,y2[0]):
dy2[1] := diff(phi2,y2[1]):
dy2[2] := diff(phi2,y2[2]):
dy2[3] := diff(phi2,y2[3]):
x2[0]:=0:
x2[1]:=1/3:
x2[2]:=2/3:
x2[3]:=1:
delx2 := 1/3:
y2[0] := 1:
y2[3]:=2:
d2[2]:=solve(dy2[2]=0,y2[2]):
d2[1]:=solve(dy2[1]=0,y2[2]):
d2[3] :=d2[2]-d2[1]:
y2[1]:= solve(d2[3]=0,y2[1]):
y2[2]:=d2[2]:
p2:=array(1..8,[x2[0],y2[0],x2[1],y2[1],x2[2],y2[2],x2[3],y2[3]]):
p2:=plot(p2):

phi3 :=sum((((y3[i+1]-y3[i])/delx3)^2 - y3[i]^2 - 2*x3[i]*y3[i])*delx3,'i'=0..3):
dy3[0] := diff(phi3,y3[0]):
dy3[1] := diff(phi3,y3[1]):
dy3[2] := diff(phi3,y3[2]):dy3[3] := diff(phi3,y3[3]):
```

```
dy3[4] := diff(phi3,y3[4]):
x3[0]:=0:
x3[1]:=1/4:
x3[2]:=1/2:
x3[3]:=3/4:
x3[4]:=1:
delx3 := 1/4:
y3[0] := 1:
y3[4]:=2:
d3[1]:=solve(dy3[1]=0,y3[2]):
d3[2]:=solve(dy3[2]=0,y3[2]):
d3[3]:=solve(dy3[3]=0,y3[2]):
d3[1] :=d3[2]-d3[1]:d3[3] :=d3[2]-d3[3]:
d3[1]:=solve(d3[1]=0,y3[3]):
d3[3]:=solve(d3[3]=0,y3[3]):
d3[1]:= d3[1]-d3[3]:
y3[1]:= solve(d3[1]=0,y3[1]):
y3[3]:=d3[3]:
y3[2]:=d3[2]:
p3:=array(1..10,[x3[0],y3[0],x3[1],y3[1],x3[2],y3[2],x3[3],y3[3],x3[4],y3[4]]):
p3:=plot(p3):

phi4 :=sum((((y4[i+1]-y4[i])/delx4)^2 - y4[i]^2 - 2*x4[i]*y4[i])*delx4,'i'=0..4):
dy4[0] := diff(phi4,y4[0]):
dy4[1] := diff(phi4,y4[1]):
dy4[2] := diff(phi4,y4[2]):
dy4[3] := diff(phi4,y4[3]):
dy4[4] := diff(phi4,y4[4]):
dy4[5] := diff(phi4,y4[5]):
x4[0]:=0:
x4[1]:=1/5:
x4[2]:=2/5:
x4[3]:=3/5:
x4[4]:=4/5:
x4[5]:=1:
delx4 := 1/5:
y4[0] := 1:
y4[5]:=2:
d4[1]:=solve(dy4[1]=0,y4[2]):
d4[2]:=solve(dy4[2]=0,y4[3]):
d4[3]:=solve(dy4[3]=0,y4[4]):
d4[4]:=solve(dy4[4]=0,y4[4]):
d4[3]:= d4[3]-d4[4]:
d4[3]:=solve(d4[3]=0,y4[3]):
```

```
 d4[2]:=d4[2]-d4[3]:
 d4[2]:=solve(d4[2]=0,y4[2]):
 d4[1]:=d4[1]-d4[2]:
 y4[1]:=solve(d4[1]=0,y4[1]):
 y4[2]:=d4[2]:
 y4[3]:=d4[3]:
 y4[4]:=d4[4]:
 p4:=array(1..12,[x4[0],y4[0],x4[1],y4[1],x4[2],y4[2],x4[3],y4[3],x4[4],y4[4],
x4[5],y4[5]]):
 p4:=plot(p4):

 y:= cos(x) +((3-cos(1))/sin(1))*sin(x) - x:
 p:=plot(y,x=0..1,color=red,style=line):
 display({p,p1,p2,p3,p4});
```

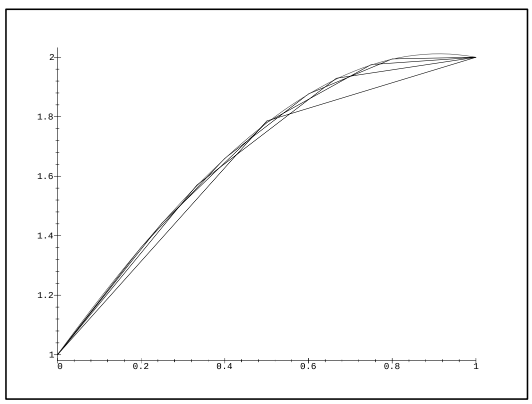

Figure 9:

CHAPTER 9

9 Hamilton's Principle

Problems

1. If ℓ is not preassigned, show that the stationary functions corresponding to the problem

$$\delta \int_0^1 y'^2 \, dx = 0$$

subject to

$$y(0) = 2, \qquad y(\ell) = \sin \ell$$

are of the form $y = 2 + 2x \cos \ell$, where ℓ satisfies the transcendental equation

$$2 + 2\ell \cos \ell - \sin \ell = 0.$$

Also verify that the smallest positive value of ℓ is between $\dfrac{\pi}{2}$ and $\dfrac{3\pi}{4}$.

2. If ℓ is not preassigned, show that the stationary functions corresponding to the problem

$$\delta \int_0^1 \left[y'^2 + 4(y - \ell) \right] dx = 0$$

subject to

$$y(0) = 2, \qquad y(\ell) = \ell^2$$

are of the form $y = x^2 - 2\dfrac{x}{\ell} + 2$, where ℓ is one of the two real roots of the quartic equation $2\ell^4 - \ell^3 - 1 = 0$.

3. A particle of mass m is falling vertically, under the action of gravity. If y is distance measured downward and no resistive forces are present.

   a. Show that the Lagrangian function is

$$L = T - V = m \left(\frac{1}{2} \dot{y}^2 + gy \right) + \text{constant}$$

and verify that the Euler equation of the problem

$$\delta \int_{t_1}^{t_2} L \, dt = 0$$

is the proper equation of motion of the particle.

   b. Use the momentum $p = m\dot{y}$ to write the Hamiltonian of the system.

   c. Show that

$$\frac{\partial}{\partial p} H = \phi = \dot{y}$$
$$\frac{\partial}{\partial y} H = -\dot{p}$$

4. A particle of mass m is moving vertically, under the action of gravity and a resistive force numerically equal to k times the displacement y from an equilibrium position. Show that the equation of Hamilton's principle is of the form

$$\delta \int_{t_1}^{t_2} \left(\frac{1}{2} m \dot{y}^2 + mgy - \frac{1}{2} k y^2 \right) dt = 0,$$

and obtain the Euler equation.

5. A particle of mass m is moving vertically, under the action of gravity and a resistive force numerically equal to c times its velocity $\dot{y}$. Show that the equation of Hamilton's principle is of the form

$$\delta \int_{t_1}^{t_2} \left(\frac{1}{2} m \dot{y}^2 + mgy \right) dt - \int_{t_1}^{t_2} c \dot{y} \delta y \, dt = 0.$$

6. Three masses are connected in series to a fixed support, by linear springs. Assuming that only the spring forces are present, show that the Lagrangian function of the system is

$$L = \frac{1}{2} \left[m_1 \dot{x}_1^2 + m_2 \dot{x}_2^2 + m_3 \dot{x}_3^2 - k_1 x_1^2 - k_2 (x_2 - x_1)^2 - k_3 (x_3 - x_2)^2 \right] + \text{constant},$$

where the x_i represent displacements from equilibrium and k_i are the spring constants.

1. If ℓ is not preassigned, show that the stationary functions corresponding to the problem

$$\delta \int_0^\ell (\ y')^2 dx = 0$$

Subject to

$$\mathrm{y}(0) = 2 \text{ and } y(\ell) = sin\ell$$

Are equal to,

$$y = 2 + 2x\cos\ell$$

Using the Euler equation $L_y - \frac{d}{dt}L_{y'} = 0$ with

$$\begin{aligned} L &= (y')^2 \\ L_y &= 0 \\ L_{y'} &= 2y' \end{aligned}$$

We get the 2nd order ODE

$$\begin{aligned} -2y'' &= 0 \\ y'' &= 0 \end{aligned}$$

Integrating twice, we have

$$y = Ax + B$$

Using our initial conditions to solve for for A and B,

$$\begin{aligned} y(0) &= 2 = A(0) + B \Longrightarrow B = 2 \\ y(\ell) &= \sin\ell = A\ell + 2 \Longrightarrow A = \frac{\sin\ell - 2}{\ell} \end{aligned}$$

Substituting A and B into our original equation gives,

$$y = \left(\frac{\sin\ell - 2}{\ell}\right) x + 2$$

Now, because we have a variable right hand end point, we must satisfy the following transversality condition:

$$F + (\Phi' - y')F_{y'}|_{x=\ell} = 0$$

Where,

$$\begin{aligned}
F &= (y')^2 \\
F_{y'} &= \frac{2\sin(\ell)-4}{\ell} \\
\Phi' &= \cos\ell \\
y' &= \frac{\sin(\ell)-2}{\ell}
\end{aligned}$$

Therefore,

$$\begin{aligned}
[y'(\ell)]^2 + \left(\cos(\ell) - \frac{\sin(\ell)-2}{\ell}\right) 2y' &= 0 \\
[y'(\ell)]^2 + \left(\cos(\ell) - \frac{\sin(\ell)-2}{\ell}\right) 2\left(\frac{\sin(\ell)-2}{\ell}\right) &= 0 \\
\left(\frac{\sin(\ell)-2}{\ell}\right)^2 + \left(\cos(\ell) - \frac{\sin(\ell)-2}{\ell}\right) 2\left(\frac{\sin(\ell)-2}{\ell}\right) &= 0 \\
\left(\frac{\sin(\ell)-2}{\ell}\right) + 2\left(\cos(\ell) - \frac{\sin(\ell)-2}{\ell}\right) &= 0 \\
\sin(\ell) - 2 + 2\ell\cos(\ell) - 2\sin(\ell) + 4 &= 0 \\
2 + 2\ell\cos(\ell) - \sin(\ell) &= 0
\end{aligned}$$

Which is our transversality condition. Since ℓ satisfies the transcendental equation above, we have,

$$\frac{\sin\ell - 2}{\ell} = 2\cos\ell$$

Substituting this back into the equation for y yields,

$$y = 2 + 2x\cos\ell$$

Which is what we wanted to show.

To verify that the smallest positive value of ℓ is between $\frac{\pi}{2}$ and $\frac{3\pi}{4}$, we must first solve the transcendental equation for ℓ.

$$\begin{aligned}
2 + 2\ell\cos\ell - \sin\ell &= 0 \\
2\ell &= \frac{\sin\ell}{\cos\ell} - \frac{2}{\cos\ell} \\
\ell &= \frac{1}{2}\tan\ell - \sec\ell
\end{aligned}$$

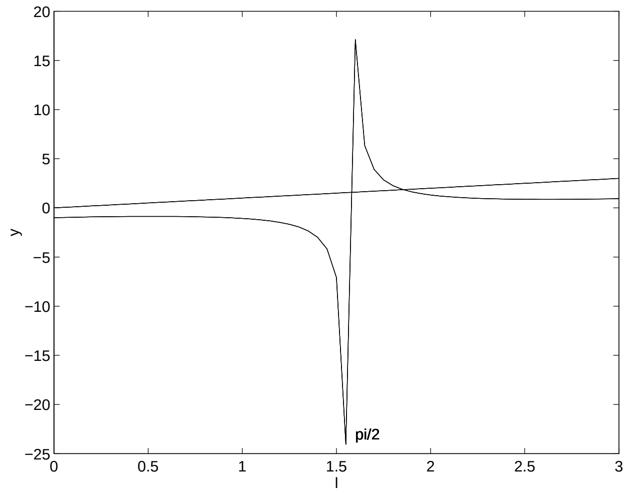

Figure 10: Plot of $y = \ell$ and $y = \dfrac{1}{2}\tan(\ell) - \sec(\ell)$

Then plot the curves,

$$\begin{aligned} y &= \ell \\ y &= \frac{1}{2}\tan\ell - \sec\ell \end{aligned}$$

between 0 and Pi, to see where they intersect.
Since they appear to intersect at approximately $\frac{\pi}{2}$, lets verify the limits of $y = \frac{1}{2}\tan\ell - \sec\ell$ analytically.

$$\begin{aligned} & \lim_{l \longrightarrow \frac{\pi}{2}} \frac{1}{2}\tan\ell - \sec\ell \\ = \; & \frac{1}{2}\frac{\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}} - \frac{1}{\cos\frac{\pi}{2}} \\ = \; & \frac{1}{2}\frac{\sin\frac{\pi}{2} - 2}{\cos\frac{\pi}{2}} \\ = \; & \frac{-1}{0} \\ = \; & \pm\infty \end{aligned}$$

Which agrees with the plot . Therefore, $\frac{\pi}{2}$ is the smallest value of ℓ

2.

$$\delta \int_0^1 \left[(y')^2 + 4(y-\ell)\right] dx = 0$$

subject to

$$y(0) = 2, y(\ell) = \ell^2$$

Since $L = (y')^2 + 4(y-\ell)$
we have $L_y = 4$ and $L_{y'} = 2y^{'}$
Thus Euler's equation: $L_y - \dfrac{d}{dx}L_{y'} = 0$ becomes $\dfrac{d}{dx}2y^{'} = 4$
Integrating leads to

$$y^{'} = 2x + \frac{c_1}{2}$$

Integrating again $y = x^2 + \dfrac{c_1}{2}x + c_2$
Now use the left end condition: $y(0) = 2 = 0 + 0 + c_2$
At $x = \ell$ we have: $y(\ell) = \ell^2 = \ell^2 + \dfrac{c_1}{2}\ell + 2 \Longrightarrow c_1 = -\dfrac{4}{\ell}$
Thus the solution is: $y = x^2 - \dfrac{2}{\ell}x + 2$

Let's differentiate y for the transversality condition: $y^{'} = 2x - \dfrac{2}{\ell}$
Now we apply the transversality condition
$L + (\phi^{'} - y^{'})L_{y'}\Big|_{x=\ell} = 0$ where $\phi = \ell^2$ and $\phi^{'} = 2\ell$
Now substituting for ϕ, L, $L_{y'}$, y and y' and evaluating at $x = \ell$, we obtain

$(2\ell - \dfrac{2}{\ell})^2 + 4(\ell^2 - \dfrac{2}{\ell}\ell + 2 - \ell) + (2\ell - (2\ell - \dfrac{2}{\ell}))2(2\ell - \dfrac{2}{\ell}) = 0$

$4\ell^2 - 8 + \dfrac{4}{\ell^2} + 4(\ell^2 - \ell) + \dfrac{4}{\ell}(2\ell - \dfrac{2}{\ell}) = 0$

$4\ell^2 - 8 + \dfrac{4}{\ell^2} + 4\ell^2 - 4\ell + 8 - \dfrac{8}{\ell^2} = 0$

$8\ell^2 - 4\ell - \dfrac{4}{\ell^2} = 0$

$2\ell^4 - \ell^3 - 1 = 0$
Therefore the final solution is
$\boxed{y = x^2 - \dfrac{2}{\ell}x + 2}$
where ℓ is one of the two real roots of $2\ell^4 - \ell^3 - 1 = 0$.

3. First, using Newton's Second Law of Motion, a particle with mass m with position vector y is acted on by a force of gravity. Summing the forces gives

$$m\ddot{y} - F = 0$$

Taking the downward direction of y to be positive, $F = mgy$. Thus

$$m\ddot{y} + mgy = 0$$

From Eqn (9) and the definition of $T = \frac{1}{2}m\dot{y}^2$, we obtain

$$\int_{t_1}^{t_2} (\delta T \ + F \cdot dy)\ dt = 0$$

From Eqn (10),

$$\int_{t_1}^{t_2} (m\dot{y} \cdot \delta y + F \cdot \delta y)\ dt = 0$$

Defining the potential energy as

$$F \cdot \delta y = -\delta V = mgy \cdot \delta y$$

gives

$$\int_{t_1}^{t_2} \delta(T - V)\ dt = 0$$

or

$$\int_{t_1}^{t_2} \delta(\frac{1}{2}m\dot{y}^2 - mgy)\ dt = 0$$

If we define the Lagrangian L as $L \equiv T - V$, we obtain the result

$$L = m(\frac{1}{2}\dot{y}^2 + gy) + constant$$

Note: The constant is arbitrary and dependent on the initial conditions.

To show the Euler Equation holds, recall

$$L = m(\frac{1}{2}\dot{y}^2 + gy) + constant$$

$$L_y = mg \qquad\qquad L_{y'} = m\dot{y} \qquad\qquad \frac{d}{dt}L_{y'} = m\ddot{y}$$

Thus,

$$L_y - \frac{d}{dt}L_{y'} = mg - m\ddot{y} = m(g - \ddot{y})$$

Since the particle falls under gravity (no initial velocity), $\ddot{y} = g$ and

$$L_y - \frac{d}{dt}L_{y'} = 0$$

The Euler Equation holds.
b. Let $p = m\dot{y}$. The Hamiltonian of the system is

$$\begin{aligned} H(t,x,p) &= -L(t,x,\phi(t,x,p)) + p\phi(t,x,p) \\ &= -\left(m(\frac{1}{2}\dot{y}^2 + gy) + constant\right) + my\phi(t,x,p) \end{aligned}$$

c. $\frac{\partial}{\partial p}H = \phi$

$$\begin{aligned} \frac{\partial}{\partial p}H &= \dot{y} \qquad \text{(by definition)} \\ \frac{\partial}{\partial y}H &= -mg = -m\ddot{y} = -\dot{p} \end{aligned}$$

4. Newton's second law: $m\ddot{R} - F = 0$ Note that $F = mg - kR$, so we have

$$\int_{t_1}^{t_2} \left(m\ddot{R}\delta R - mg\delta R + kR\delta R\right) dt = 0$$

This can also be written as

$$\delta \int_{t_1}^{t_2} \left(\frac{1}{2}m\dot{R}^2 + mgR - \frac{1}{2}kR^2\right) dt = 0$$

To obtain Euler's equation, we let

$$L = \frac{1}{2}m\dot{R}^2 + mgR - \frac{1}{2}kR^2$$

Therefore

$$L_R = mg - kR$$
$$L_{\dot{R}} = m\dot{R}$$
$$L_R - \frac{d}{dt}L_{\dot{R}} = mg - kR - m\ddot{R} = 0$$

5. The first two terms are as before (coming from ma and the gravity). The second integral gives the resistive force contribution which is proportional to $\dot{y}$ with a constant of proportionality c. Note that the same is negative because it acts opposite to other forces.

6. Here we notice that the first spring moves a distance of x_1 relative to rest. The second spring in the series moves a distance x_2 relative to its original position, but x_1 was the contribution of the first spring therefore, the total is $x_2 - x_1$. Similarly, the third moves $x_3 - x_2$ units.

CHAPTER 10

10 Degrees of Freedom - Generalized Coordinates

Problems

1. Consider the functional

$$I(y) = \int_a^b \left[r(t)\dot{y}^2 + q(t)y^2\right] dt.$$

Find the Hamiltonian and write the canonical equations for the problem.

2. Give Hamilton's equations for

$$I(y) = \int_a^b \sqrt{(t^2 + y^2)(1 + \dot{y}^2)}dt.$$

Solve these equations and plot the solution curves in the yp plane.

3. A particle of unit mass moves along the y axis under the influence of a potential

$$f(y) = -\omega^2 y + ay^2$$

where ω and a are positive constants.

a. What is the potential energy $V(y)$? Determine the Lagrangian and write down the equations of motion.

b. Find the Hamiltonian $H(y,p)$ and show it coincides with the total energy. Write down Hamilton's equations. Is energy conserved? Is momentum conserved?

c. If the total energy E is $\frac{\omega^2}{10}$, and $y(0) = 0$, what is the initial velocity?

d. Sketch the possible phase trajectories in phase space when the total energy in the system is given by $E = \frac{\omega^6}{12a^2}$.

Hint: Note that $p = \pm\sqrt{2}\sqrt{E - V(y)}$.

What is the value of E above which oscillatory solution is not possible?

4. A particle of mass m moves in one dimension under the influence of the force $F(y,t) = ky^{-2}e^t$, where $y(t)$ is the position at time t, and k is a constant. Formulate Hamilton's principle for this system, and derive the equations of motion. Determine the Hamiltonian and compare it with the total energy.

5. A Lagrangian has the form

$$L(x, y, y') = \frac{a^2}{12}(y')^4 + a(y')^2 G(y) - G(y)^2,$$

where G is a given differentaible function. Find Euler's equation and a first integral.

6. If the Lagrangian L does not depend explicitly on time t, prove that $H =$ constant, and if L doesn't depend explicitly on a generalized coordinate y, prove that $p =$ constant.

7. Consider the differential equations

$$r^2\dot{\theta} = C, \qquad \ddot{r} - r\dot{\theta}^2 + \frac{k}{m}r^{-2} = 0$$

governing the motion of a mass in an inversely square central force field.

a. Show by the chain rule that

$$\dot{r} = Cr^{-2}\frac{dr}{d\theta}, \qquad \ddot{r} = C^2r^{-4}\frac{d^2r}{d\theta^2} - 2C^2r^{-5}\left(\frac{dr}{d\theta}\right)^2$$

and therefore the differential equations may be written

$$\frac{d^2r}{d\theta^2} - 2r^{-1}\left(\frac{dr}{d\theta}\right)^2 - r + \frac{k}{C^2m}r^2 = 0$$

b. Let $r = u^{-1}$ and show that

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{C^2m}.$$

c. Solve the differential equation in part b to obtain

$$u = r^{-1} = \frac{k}{C^2m}\left(1 + \epsilon\cos(\theta - \theta_0)\right)$$

where ϵ and θ_0 are constants of integration.

d. Show that elliptical orbits are obtained when $\epsilon < 1$.

CHAPTER 11

11 Integrals Involving Higher Derivatives

Problems

1. Derive the Euler equation of the problem

$$\delta \int_{x_1}^{x_2} F(x, y, y', y'')\, dx = 0$$

in the form

$$\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y} = 0,$$

and show that the associated natural boundary conditions are

$$\left[\left(\frac{d}{dx}\frac{\partial F}{\partial y''} - \frac{\partial F}{\partial y'}\right) \delta y\right]\Bigg|_{x_1}^{x_2} = 0$$

and

$$\left[\frac{\partial F}{\partial y''}\, \delta y'\right]\Bigg|_{x_1}^{x_2} = 0.$$

2. Derive the Euler equation of the problem

$$\delta \int_{x_1}^{x_2} \int_{y_1}^{y_2} F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})\, dxdy = 0,$$

where $x_1, x_2, y_1,$ and y_2 are constants, in the form

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial u_{xx}}\right) + \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial F}{\partial u_{xy}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial u_{yy}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) + \frac{\partial F}{\partial u} = 0,$$

and show that the associated natural boundary conditions are then

$$\left[\left(\frac{\partial}{\partial x}\frac{\partial F}{\partial u_{xx}} + \frac{\partial}{\partial y}\frac{\partial F}{\partial u_{xy}} - \frac{\partial F}{\partial u_x}\right) \delta u\right]\Bigg|_{x_1}^{x_2} = 0$$

$$\left[\frac{\partial F}{\partial u_{xx}} \delta u_x\right]\Bigg|_{x_1}^{x_2} = 0,$$

and

$$\left[\left(\frac{\partial}{\partial y}\frac{\partial F}{\partial u_{yy}} + \frac{\partial}{\partial x}\frac{\partial F}{\partial u_{xy}} - \frac{\partial F}{\partial u_y}\right) \delta u\right]\Bigg|_{y_1}^{y_2} = 0$$

$$\left[\frac{\partial F}{\partial u_{yy}} \delta u_y\right]\Bigg|_{y_1}^{y_2} = 0.$$

3. Specialize the results of problem 2 in the case of the problem

$$\delta \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left[\frac{1}{2}u_{xx}^2 \,+\, \frac{1}{2}u_{yy}^2 \,+\, \alpha u_{xx}u_{yy} \,+\, (1-\alpha)u_{xy}^2\right] dxdy \,=\, 0,$$

where α is a constant.

Hint: Show that the Euler equation is $\nabla^4 u \,=\, 0$, regardless of the value of α, but the natural boundary conditions depend on α.

4. Specialize the results of problem 1 in the case

$$F \,=\, a(x)(y'')^2 \,-\, b(x)(y')^2 \,+\, c(x)y^2.$$

5. Find the extremals

a. $I(y) \,=\, \int_0^1 (yy' + (y'')^2)dx, \qquad y(0) = 0,\, y'(0) = 1,\, y(1) = 2,\, y'(1) = 4$

b. $I(y) \,=\, \int_0^\infty (y^2 + (y')^2 + (y'' + y')^2)dx, \qquad y(0) = 1,\, y'(0) = 2,\, y(\infty) = 0,\, y'(\infty) = 0.$

6. Find the extremals for the functional

$$I(y) \,=\, \int_a^b (y^2 + 2\dot{y}^2 + \ddot{y}^2)dt.$$

7. Solve the following variational problem by finding extremals satisfying the given conditions

$$I(y) \,=\, \int_0^1 (1 + (y'')^2)dx, \qquad y(0) = 0,\, y'(0) = 1,\, y(1) = 1,\, y'(1) = 1.$$