Several New Methods for Solving Equations

BENY NETA

Naval Postgraduate School, Department of Mathematics, Code N3Nd, Monterey, CA 93943, USA

(Received November 1986; in final form June 1987)

Several new methods for solving one nonlinear equation are developed. Most of the methods are of order three and they require the knowledge of $f$, $f'$ and $f''$. The methods will be compared to others in the literature. An extensive bibliography is given.

KEY WORDS: Nonlinear equations, order of convergence, iteration, computational efficiency, efficiency index.

C.R. CATEGORIES: 5.1, 5.15.

1. INTRODUCTION

Many iterative procedures were developed to obtain a simple zero $\xi$ of a nonlinear function $f(x)$. There are also methods for obtaining multiple zeros. The algorithms can be classified as bracketing techniques, fixed point methods and hybrid ones. The bracketing methods include the well-known bisection, Regula Falsi and modified Regula Falsi. Other algorithms of this class are: method F (King [66]), modified method F (Popovski [108]), Illinois algorithm (Snyder [116]), Pegasus (Dowell and Jarratt [24]), and improved Pegasus method (King [64]), algorithms A, M and R (Dekker [21]; Bus and Dekker [11]), algorithm B (Brent [9]), algorithm C
(Anderson and Bjorck [5]), Cox method (Cox [18]), and Stone's method (Stone [120]). In all these methods one assumes that an interval \( [a, b] \) is given on which \( f(x) \) changes its sign. The methods successively produce smaller and smaller intervals containing the zero. Thus one guarantees the convergence of the iterative process. On the other hand such methods cannot find zeros of even multiplicity. In order to overcome such difficulty one can use fixed point type method. The list of such methods is long and includes among others Cauchy, Chebyshev, Euler, Halley, Hansen and Patrick, Jarratt, King, Laguerre, Muller, Murakami, Neta, Newton, Ostrowski, Popovski, Steffensen, secant, Traub, Wegstein and Werner.

The last class uses a combination of two methods, one from each class to guarantee the convergence of the iterative process, see for example Nedsore [79] and Popovski [86–108].

In this article, we develop several new fixed-point type methods based on the idea of Popovski [107]. All such methods will require the evaluation of \( f, f' \) and \( f'' \) at each step. These methods are all of order three and thus the informational efficiency (see e.g., Ostrowski) is 1. The efficiency index is 1.442. A fourth-order method based on Nourein's algorithm [83] will be given. A special case of a fifth-order method developed by Murakami [77] will also be discussed. These methods will be compared numerically. Tables comparing the informational efficiency and the efficiency index of all methods (known to the author) will be given.

2. THIRD-ORDER METHODS

In 1982 Popovski has suggested the construction of third-order methods by what he called the method of replacement. Let

\[
h = x_{n+1} - x_n, \quad f_n = f(x_n), \quad u_n = f_n/f'_n, \quad A_j = \frac{f^{(j)}(x_n)}{j! f'(x_n)^j},
\]

then

\[
0 = u + h + A_3 h^2.
\]  

This equation can be solved for \( h \) directly, which yields Cauchy's
NONLINEAR EQUATIONS

method [13]. If (1) is written as

\[ 0 = u + h + A_2(h) \cdot \{h\} \]  

(2)

one can replace \( (h) \) and \( \{h\} \) by various iteration functions and then solve for \( h \). This is the method of replacement. Popovski obtained the following 9 algorithms this way.

1. \[ h = - \frac{u(A_2 - 1)}{2uA_2 - 1}, \]  

(3)

2. \[ h = \frac{u}{(uA_2 + 1)uA_2 - 1}, \]  

(4)

3. \[ h = \frac{u(2uA_2 - 1)}{(uA_2 - 3)uA_2 + 1}, \]  

(5)

4. \[ h = - \frac{u[(uA_2 - 3)uA_2 + 1]}{3(uA_2 - 4uA_2 + 1)}, \]  

(6)

5. \[ h = - \frac{u[(uA_2 + 1)uA_2 - 1]}{2uA_2 - 1}, \]  

(7)

6. \[ h = u \left( \frac{uA_2}{(uA_2 + 1)uA_2 - 1} - 1 \right), \]  

(8)

7. \[ h = \frac{u[(uA_2 + 2)uA_2 - 1]}{(uA_2 - 3)uA_2 + 1}, \]  

(9)

8. \[ h = - u \frac{uA_2}{(uA_2 - 1)^2 + 1}, \]  

(10)

9. \[ h = u \frac{(uA_2)^2 + 1}{uA_2 - 1}, \]  

(11)

These methods based on combining \( (h) = h \), \( -u \) or one of the following methods

\[ h = \frac{u}{uA_2 - 1} \]  

(Halley),  

(12)

\[ h = -u(uA_2 + 1) \]  

(Euler),  

(13)

\[ h = -u[(2uA_2 + 1)uA_2 + 1], \]  

(14)

and (3)-(5).
In a similar fashion, one can obtain the following 21 new methods.

\[ h = -\frac{u}{1 - uA_2(1 + uA_2(1 + 2uA_2))}, \quad (h), \quad (14) \quad (15) \]

\[ h = -\frac{u}{1 + uA_2((1 + uA_2)uA_2 - 1)}, \quad (h), \quad (4) \quad (16) \]

\[ h = -u - u^2A_2[(2uA_2 + 1)uA_2 + 1], \quad (-u), \quad (12) \quad (17) \]

\[ h = -u - u^2A_2(uA_2 + 1), \quad (-u), \quad (13) \quad (18) \]

\[ h = -u - u^2A_2[(2uA_2 + 1)uA_2 + 1], \quad (-u), \quad (14) \quad (19) \]

\[ h = -u + u^2A_2 \frac{(2uA_2 + 1)uA_2 + 1}{uA_2 - 1}, \quad (12), \quad (14) \quad (20) \]

\[ h = -u - \frac{u^2A_2}{(uA_2 - 1)((uA_2 + 1)uA_2 - 1)}, \quad (12), \quad (4) \quad (21) \]

\[ h = -u - \frac{u^2A_2}{uA_2 - 1} \frac{2uA_2 - 1}{(uA_2 - 3)uA_2 + 1}, \quad (12), \quad (5) \quad (22) \]

\[ h = -u - u^2A_2(uA_2 + 1)^2, \quad (13), \quad (13) \quad (23) \]

\[ h = -u - A_2u^2(uA_2 + 1)[(2uA_2 + 1)uA_2 + 1], \quad (13), \quad (14) \quad (24) \]

\[ h = -u - A_2 \frac{u^2(uA_2 + 1)(uA_2 - 1)}{2uA_2 - 1}, \quad (13), \quad (3) \quad (25) \]

\[ h = -u + A_2 \frac{u^2(uA_2 + 1)(2uA_2 - 1)}{(uA_2 - 3)uA_2 + 1}, \quad (13), \quad (5) \quad (26) \]

\[ h = -u - A_2u^2[(2uA_2 + 1)uA_2 + 1]^2, \quad (14), \quad (14) \quad (27) \]

\[ h = -u - A_2 \frac{u^2(uA_2 - 1)[(2uA_2 + 1)uA_2 + 1]}{2uA_2 - 1}, \quad (14), \quad (3) \quad (28) \]
NONLINEAR EQUATIONS

\begin{align*}
  h &= -u + A_2 \frac{u^2[(2uA_2 + 1)uA_2 + 1]}{(uA_2 + 1)uA_2 - 1}, \quad (14), (4) (29) \\
  h &= -u + A_2 \frac{u^2(2uA_2 - 1)[(2uA_2 + 1)uA_2 + 1]}{(uA_2 - 3)uA_2 + 1}, \quad (14), (5) (30) \\
  h &= -u - A_2 \left[ \frac{u(uA_2 - 1)}{(2uA_2 - 1)} \right]^2, \quad (3), (3) (31) \\
  h &= -u - A_2 \frac{u^2(uA_2 - 1)}{(2uA_2 - 1)[(uA_2 + 1)uA_2 - 1]}, \quad (3), (4) (32) \\
  h &= -u - A_2 \left[ \frac{u}{(uA_2 + 1)uA_2 - 1} \right]^2, \quad (4), (4) (33) \\
  h &= -u - A_2 \frac{u^2(2uA_2 - 1)}{[(uA_2 + 1)uA_2 - 1][(uA_2 - 3)uA_2 + 1]}, \quad (4), (5) (34) \\
  h &= -u - A_2 \left[ \frac{u(2uA_2 - 1)}{(uA_2 - 3)uA_2 + 1} \right]^2. \quad (5), (5) (35)
\end{align*}

Remark: The two quantities in parentheses to the left of the equation number indicate how the new method was developed.

Since Popovski [107] recommended the use of method (3) we compare that method with the newly developed ones in Section 5.

3. FOURTH-ORDER METHOD

This method is based on Nourein’s algorithm [83]. The method of neglecting discussed by Popovski [107] is used here. Setting $f^{(4)}(\xi_i) = 0$, in Nourein’s method one obtains

\begin{equation}
  h = \frac{(2uA_2 - 1 - 6u^2A_3)u}{(uA_2 - 3)uA_2 + 1 + 6u^2A_3}. \quad (36)
\end{equation}

The method is of order four and requires four new pieces of
information. Therefore the informational efficiency is 1 as for the previous methods. On the other hand the efficiency index is only 1.414.

4. FIFTH-ORDER METHOD

Murakami [77] has developed the following family of methods of order five,

\[ h = -a_1 u_n - a_2 w_2(x_n) - a_3 w_3(x_n) - \psi(x_n), \]  

(37)

where

\[ w_2(x_n) = \frac{f_n}{f'(x_n - u_n)}, \]  

(38)

\[ w_3(x_n) = \frac{f_n}{f'(x_n + \beta u_n + \gamma w_2(x_n))}, \]  

(39)

\[ \psi(x_n) = \frac{f_n}{b_1 f''_n + b_2 f'''(x_n - u_n)}. \]  

(40)

The asymptotic error constant is

\[ C = -\left[ \frac{3\gamma^2}{2} + \frac{5\gamma}{3} - \frac{2}{3} + \frac{1}{6(4\gamma + 1)} \right] A_2^4 + (8\gamma^2 - 4\gamma) A_2^2 A_3, \]

\[ + \frac{1367}{3} A_2 A_4 - \frac{1}{6} A_3^2 + \frac{1}{24} A_6. \]  

(41)

Murakami has suggested that \( \gamma = 0 \) or \( \gamma = -\frac{1}{2} \). These two choices annihilate one term of the asymptotic error constant. Another possibility is \( \gamma = 17795/131072 \) which annihilates the first term. This choice leads to the values

\[ \beta = -\frac{1}{2} - \gamma, \quad a_1 = 0.3879870, \quad a_2 = -1.420700, \]

\[ a_3 = \frac{2}{3}, \quad b_1 = -0.1186015, \quad b_2 = 0.8506410. \]  

(42)
and the asymptotic error constant is then
\[ C = 0.6905251 A_1^2 A_3 + 5.792695 A_2 A_4 - \frac{3}{8} A_3^2 + \frac{1}{2} A_3. \] (43)

5. NUMERICAL EXPERIMENTS

We have used the following six examples to compare the performance of the methods (3), (15)–(37).

\[ f_1(x) = \sin x - \frac{1}{2} x, \quad x_0 = 2 \]
\[ f_2(x) = x^5 + x - 10000, \quad x_0 = 4 \]
\[ f_3(x) = x^{1/2} - \frac{1}{x} - 3, \quad x_0 = 1 \]
\[ f_4(x) = e^x + x - 20, \quad x_0 = 0 \]
\[ f_5(x) = \ln x + x^{1/2} - 5, \quad x_0 = 1 \]
\[ f_6(x) = x^3 - x^2 - 1, \quad x_0 = 0.5. \]

The first example is simple and it’s taken from Gerald and Wheatley [35]. All methods performed very well in this case. The other examples are taken from Popovski [86] and thus we added the method of that article to our comparison. In our notations, the method is
\[ h = -\frac{e^{2u A_2} - 1}{2 A_2}. \] (44)

The following table (Table 1) gives the number of iterations required to obtain the zero with tolerance of $10^{-14}$. All computations were performed in double precision on an IBM 3033. The letter D stands for divergence (within 30 iterations) and * denotes computational difficulties (overflow, etc.). Note that the method recommended in Popovski [107] didn’t perform as well as the new
methods developed here. The method taken from Popovski [86] did not converge for one of the examples. The following new methods converge for all the examples (17)–(19), (21), (23), (31), (33), (35)–(37). Counting the total number of iterations one can say that methods (21), (31), (33) (35) are the best third-order ones of those considered. Next comes (23), and then (17)–(19). Certainly method (37) which is fifth-order performed better than the others. Surprisingly though the fourth-order method (36) didn't perform better than the third-order methods.

Table 1

<table>
<thead>
<tr>
<th>Method</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3)</td>
<td>3</td>
<td>6</td>
<td>D</td>
<td>*</td>
<td>D</td>
<td>6</td>
</tr>
<tr>
<td>(15)</td>
<td>3</td>
<td>10</td>
<td>D</td>
<td>8</td>
<td>D</td>
<td>D</td>
</tr>
<tr>
<td>(16)</td>
<td>3</td>
<td>D</td>
<td>*</td>
<td>D</td>
<td>*</td>
<td>13</td>
</tr>
<tr>
<td>(17)</td>
<td>3</td>
<td>29</td>
<td>4</td>
<td>14</td>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>(18)</td>
<td>3</td>
<td>12</td>
<td>4</td>
<td>26</td>
<td>4</td>
<td>17</td>
</tr>
<tr>
<td>(19)</td>
<td>3</td>
<td>29</td>
<td>4</td>
<td>14</td>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>(20)</td>
<td>3</td>
<td>16</td>
<td>*</td>
<td>10</td>
<td>*</td>
<td>11</td>
</tr>
<tr>
<td>(21)</td>
<td>3</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>5</td>
<td>12</td>
</tr>
<tr>
<td>(22)</td>
<td>3</td>
<td>7</td>
<td>*</td>
<td>D</td>
<td>*</td>
<td>7</td>
</tr>
<tr>
<td>(23)</td>
<td>3</td>
<td>14</td>
<td>5</td>
<td>13</td>
<td>5</td>
<td>14</td>
</tr>
<tr>
<td>(24)</td>
<td>3</td>
<td>16</td>
<td>7</td>
<td>*</td>
<td>6</td>
<td>19</td>
</tr>
<tr>
<td>(25)</td>
<td>3</td>
<td>8</td>
<td>5</td>
<td>*</td>
<td>5</td>
<td>23</td>
</tr>
<tr>
<td>(26)</td>
<td>3</td>
<td>D</td>
<td>4</td>
<td>D</td>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>(27)</td>
<td>3</td>
<td>20</td>
<td>*</td>
<td>11</td>
<td>5</td>
<td>19</td>
</tr>
<tr>
<td>(28)</td>
<td>3</td>
<td>D</td>
<td>4</td>
<td>*</td>
<td>4</td>
<td>11</td>
</tr>
<tr>
<td>(29)</td>
<td>3</td>
<td>9</td>
<td>*</td>
<td>25</td>
<td>*</td>
<td>13</td>
</tr>
<tr>
<td>(30)</td>
<td>3</td>
<td>D</td>
<td>5</td>
<td>D</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>(31)</td>
<td>3</td>
<td>11</td>
<td>6</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>(32)</td>
<td>3</td>
<td>15</td>
<td>5</td>
<td>*</td>
<td>5</td>
<td>11</td>
</tr>
<tr>
<td>(33)</td>
<td>3</td>
<td>6</td>
<td>7</td>
<td>9</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>(34)</td>
<td>3</td>
<td>D</td>
<td>D</td>
<td>D</td>
<td>D</td>
<td>D</td>
</tr>
<tr>
<td>(35)</td>
<td>3</td>
<td>6</td>
<td>4</td>
<td>7</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>(36)</td>
<td>3</td>
<td>6</td>
<td>5</td>
<td>8</td>
<td>5</td>
<td>15</td>
</tr>
<tr>
<td>(37)</td>
<td>2</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>(44)</td>
<td>3</td>
<td>6</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>D</td>
</tr>
</tbody>
</table>
Remark More experiments were performed with functions suggested by Nerinckx and Haegemans [78] and the results were consistently better than those reported there.

6. EFFICIENCY COMPARISON

In this section we collected information concerning the order, informational usage, informational efficiency and efficiency index (for definitions, see e.g. Neta [81]) of all methods known to the author. The first table (Table 2) consists of the information for bracketing methods and the others will give fixed point methods (Tables 3–7). For the fixed point methods we gave separate tables for derivative free methods, for methods using $f'$ at only one point, for those using $f''$ at more than one point, for those using $f''$ and $f'''$ and those using derivatives of order 3 or higher.

| Method       | $P$ | $d$ | $E$ | $l$ | $N = \log_2 \frac{|b-a|}{l}$ |
|--------------|-----|-----|-----|-----|-----------------------------|
| Bisection    | 1   | 1   | 1   | 1   | $N = \log_2 \frac{|b-a|}{l}$ |
| Regula Falsi | 1   | 1   | 1   | 1   |                             |
| Modified R.F.| 1.618 | 1 | 1.618 | 1.618 | $2^N$                      |
| Algorithm A  | 1.618 | 1 | 1.618 | 1.618 |                             |
| Algorithm B  | 1.618 | 1 | 1.618 | 1.618 | $(N+1)^2 - 2$               |
| Algorithm M  | 1.618 | 1 | 1.618 | 1.618 | $4N$                        |
| Algorithm R  | 1.839 | 1 | 1.839 | 1.839 | $5N$                        |
| Method F     | 1.839 | 1 | 1.839 | 1.839 |                             |
| Modified     | 1.839 | 1 | 1.839 | 1.839 |                             |
| Method F     | 1.839 | 1 | 1.839 | 1.839 |                             |
| Illinois     | 3    | 1   | 1.442 |     |                             |
| Pegasus      | 7.275 | 4 | 1.818 | 1.622 |                             |
| Improved     | 3    | 2   | 1.5   | 1.732 |                             |
| Pegasus      | 5    | 3   | 1.667 | 1.710 |                             |
| Algorithm C  | 5    | 3   | 1.667 | 1.710 |                             |
|              | 8    | 4   | 2     | 1.682 |                             |
| Cox          | 2    | 2   | 1.414 |     |                             |
| Stone        | 3    | 2   | 1.5   | 1.732 |                             |

In the last two algorithms we assumed exact complements in evaluating $f, f'$ The efficiency may be higher in other cases.
### B. NETA

#### Table 3

<table>
<thead>
<tr>
<th></th>
<th>$p$</th>
<th>$d$</th>
<th>$E$</th>
<th>$l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed point (Picard)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Wegstein</td>
<td>1.618</td>
<td>1.618</td>
<td>1.618</td>
<td>1.618</td>
</tr>
<tr>
<td>Secant</td>
<td>1.618</td>
<td>1.618</td>
<td>1.618</td>
<td>1.618</td>
</tr>
<tr>
<td>Popovski [98]</td>
<td>1.839</td>
<td>1.839</td>
<td>1.839</td>
<td>1.839</td>
</tr>
<tr>
<td>Muller</td>
<td>1.839</td>
<td>1.839</td>
<td>1.839</td>
<td>1.839</td>
</tr>
<tr>
<td>Jarratt and Nudds</td>
<td>1.839</td>
<td>1.839</td>
<td>1.839</td>
<td>1.839</td>
</tr>
<tr>
<td>Popovski [100]</td>
<td>1.839</td>
<td>1.839</td>
<td>1.839</td>
<td>1.839</td>
</tr>
<tr>
<td>Traub (3 methods)</td>
<td>1.839</td>
<td>1.839</td>
<td>1.839</td>
<td>1.839</td>
</tr>
<tr>
<td>Steffensen</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1.414</td>
</tr>
<tr>
<td>Chambers</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1.414</td>
</tr>
<tr>
<td>Chambers</td>
<td>2.732</td>
<td>2.1366</td>
<td>1.653</td>
<td></td>
</tr>
</tbody>
</table>

*Only $l$ values are required.*

#### Table 4

<table>
<thead>
<tr>
<th></th>
<th>$p$</th>
<th>$d$</th>
<th>$E$</th>
<th>$l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Newton</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1.414</td>
</tr>
<tr>
<td>Dordjevic</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1.414</td>
</tr>
<tr>
<td>Ostrowski</td>
<td>2.414</td>
<td>2.1207</td>
<td>1.554</td>
<td></td>
</tr>
<tr>
<td>Popovski [99]</td>
<td>2.414</td>
<td>2.1207</td>
<td>1.554</td>
<td></td>
</tr>
<tr>
<td>Popovski [105] 3 methods</td>
<td>2.414</td>
<td>2.1207</td>
<td>1.554</td>
<td></td>
</tr>
<tr>
<td>Werner [131]</td>
<td>2.414</td>
<td>2.1207</td>
<td>1.554</td>
<td></td>
</tr>
<tr>
<td>Chambers [15]</td>
<td>2.414</td>
<td>2.1207</td>
<td>1.554</td>
<td></td>
</tr>
<tr>
<td>Jarratt [53]</td>
<td>2.732</td>
<td>2.1366</td>
<td>1.653</td>
<td></td>
</tr>
<tr>
<td>Jain (implicit)</td>
<td>3</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Werner [134]</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1.442</td>
</tr>
<tr>
<td>King [65]</td>
<td>4</td>
<td>3</td>
<td>1.333</td>
<td>1.587</td>
</tr>
<tr>
<td>Popovski [105] 10 methods</td>
<td>4.562</td>
<td>3.1.520</td>
<td>1.66</td>
<td></td>
</tr>
<tr>
<td>Murakami</td>
<td>5</td>
<td>4</td>
<td>1.25</td>
<td>1.495</td>
</tr>
<tr>
<td>Neta (this article)</td>
<td>5</td>
<td>4</td>
<td>1.25</td>
<td>1.495</td>
</tr>
<tr>
<td>Neta [80]</td>
<td>6</td>
<td>4</td>
<td>1.5</td>
<td>1.565</td>
</tr>
<tr>
<td>Popovski [104]</td>
<td>7</td>
<td>4</td>
<td>1.75</td>
<td>1.626</td>
</tr>
<tr>
<td>Neta [82]</td>
<td>10.815</td>
<td>4.2.704</td>
<td>1.813</td>
<td></td>
</tr>
<tr>
<td>Neta [81]</td>
<td>14</td>
<td>5</td>
<td>2.8</td>
<td>1.685</td>
</tr>
<tr>
<td>Neta [81]</td>
<td>16</td>
<td>5</td>
<td>3.2</td>
<td>1.741</td>
</tr>
<tr>
<td>Werner [133]</td>
<td>2k</td>
<td>2</td>
<td>k+2</td>
<td>-</td>
</tr>
<tr>
<td>Werner [133]</td>
<td>k+</td>
<td>k+2</td>
<td>k+2</td>
<td>-</td>
</tr>
<tr>
<td>Werner [133]</td>
<td>2k+1</td>
<td>k+2</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Werner [133]</td>
<td>2k+2</td>
<td>k+2</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>

* $l$ is required at one point only.*
### Table 5

<table>
<thead>
<tr>
<th></th>
<th>$P$</th>
<th>$d$</th>
<th>$E$</th>
<th>$I$</th>
<th>$f'$ at</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jarratt [52]</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1.442</td>
<td>2</td>
</tr>
<tr>
<td>Jarratt [55]</td>
<td>4</td>
<td>3</td>
<td>1.333</td>
<td>1.587</td>
<td>2</td>
</tr>
<tr>
<td>Jarratt [51] 4 methods</td>
<td>4</td>
<td>3</td>
<td>1.333</td>
<td>1.587</td>
<td>2</td>
</tr>
<tr>
<td>Jarratt [52] 2 methods</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>1.414</td>
<td>3</td>
</tr>
<tr>
<td>Jain (semi-explicit)</td>
<td>4</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>2</td>
</tr>
<tr>
<td>Jarratt [52]</td>
<td>5</td>
<td>4</td>
<td>1.25</td>
<td>1.495</td>
<td>3</td>
</tr>
<tr>
<td>Jain (implicit)</td>
<td>5</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>2</td>
</tr>
<tr>
<td>King [52]</td>
<td>5</td>
<td>4</td>
<td>1.25</td>
<td>1.495</td>
<td>2</td>
</tr>
<tr>
<td>Popovski [101]</td>
<td>7.864</td>
<td>$m$</td>
<td>$\sqrt{\frac{m^2}{4} + 1}$</td>
<td>1.866</td>
<td>1.653</td>
</tr>
<tr>
<td>Werner</td>
<td>$\frac{m^2}{2} + \sqrt{\frac{m^2}{4} + 1}$</td>
<td>$m$</td>
<td>$m - 1$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Methods require $f$ and $f'$ at more than one point.

### Table 6

<table>
<thead>
<tr>
<th></th>
<th>$P$</th>
<th>$d$</th>
<th>$E$</th>
<th>$I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hansen and Patrick</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1.442</td>
</tr>
<tr>
<td>Popovski [3]</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1.442</td>
</tr>
<tr>
<td>Halley</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1.442</td>
</tr>
<tr>
<td>Laguerre</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1.442</td>
</tr>
<tr>
<td>Chebyshev</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1.442</td>
</tr>
<tr>
<td>Cauchy</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1.442</td>
</tr>
<tr>
<td>Euler</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1.442</td>
</tr>
<tr>
<td>Ostrowski</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1.442</td>
</tr>
<tr>
<td>Popovski [89]</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1.442</td>
</tr>
<tr>
<td>Milovanovic et al.</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1.442</td>
</tr>
<tr>
<td>Popovski [90]</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1.442</td>
</tr>
<tr>
<td>Popovski [94]</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1.442</td>
</tr>
<tr>
<td>Neto (this article) 21 methods</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1.442</td>
</tr>
<tr>
<td>Popovski [107] 9 methods</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1.442</td>
</tr>
<tr>
<td>Werner [132]</td>
<td>4</td>
<td>3</td>
<td>1.333</td>
<td>1.587</td>
</tr>
<tr>
<td>Werner [133] 4 methods</td>
<td>4</td>
<td>3</td>
<td>1.333</td>
<td>1.587</td>
</tr>
<tr>
<td>Popovski [106]</td>
<td>6</td>
<td>4</td>
<td>1.5</td>
<td>1.565</td>
</tr>
</tbody>
</table>

Methods require $f$ and $f'$.
Table 7

<table>
<thead>
<tr>
<th></th>
<th>$P$</th>
<th>$d$</th>
<th>$E$</th>
<th>$I$</th>
<th>$f'$</th>
<th>$f''$</th>
<th>$f'''$</th>
<th>$f^{(4)}$</th>
<th>$f^{(5)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kiss/Lika</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>1.414</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Neta (this article)</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>1.414</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nounen</td>
<td>5</td>
<td>5</td>
<td>1</td>
<td>1.380</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Werner</td>
<td>$k + 2$</td>
<td>$k + 1$</td>
<td>--</td>
<td>--</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>...</td>
</tr>
<tr>
<td>Varyukhin and Kas'yanyuk</td>
<td>$k + 2$</td>
<td>$k + 2$</td>
<td>--</td>
<td>--</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>...</td>
</tr>
</tbody>
</table>

Methods requiring derivatives of $f$ of order three or higher.

Acknowledgement

The author would like to thank the NPS Foundation Research Program for its support of this research.

References

NONLINEAR EQUATIONS


[33] L. Galbene. Un metodo iterativo non stazionario per la risoluzione di
95-96.
Nauk SSSR 78 (1951), 193-196.
generally and that without any previous reduction. Phil. Trans. Roy. Soc.
London 18 (1694), 136-148.
(1946), 113-121.
Monthly 57 (1950), 517-522.
(1977), 257-269.
Helv. 21 (1948), 321-326.
(1949), 230-236.
[47] D. J. Hofmann. Note on the computation of the zeros of functions satisfying
a second order differential equation, Math. Table & Other Aids Comp. 12
(1958), 58-60.
Amer. Math. Soc. 2 (1951), 718-719.
[49] L. C. Hsu. A few useful modifications of Newton's approximation method of
Comp. 20 (1966), 434-437.
[54] P. Jarratt. A note on the asymptotic error constant of a certain method for
NONLINEAR EQUATIONS

[59] L. V. Kantorovich, On some further applications of the Newton approximation method (Russian), Vestnik Leningrad Univ. 7 (1957), 68–103.


NONLINEAR EQUATIONS


[116] J. N. Snyder, Inverse interpolation, a real root of $f(x) = 0$, University of Illinois Digital Computer Laboratory, ILLIAC I Library Routine H1-71 (1953), 4 pp.


[140] P. Wynn, On a cubically convergent process for determining the zeros of certain functions, Math. Table & Other Aids Comp. 10 (1956), 97-100.