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P-Stable Symmetric Super-Implicit Methods for Periodic Initial Value Problems

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Abstract—The idea of super-implicit methods (requiring not just past and present but also future values) was suggested by Fukushima recently. Here, we construct P-stable super-implicit methods for the solution of second-order initial value problems. The benefit of such methods is realized when using vector or parallel computers.

Keywords—Super-implicit, Initial value problems, P-stability, Cowell, Picard iteration.

1. INTRODUCTION

In this paper, we discuss the numerical solution of a special class (for which y' is missing) of the second-order IVPs,

$$y''(x) = f(x, y(x)), \qquad y(0) = y_0, \quad y'(0) = y_0'. \tag{1}$$

There is a vast literature for the numerical solution of these problems as well as for the general second-order IVPs,

$$y''(x) = f(x, y(x), y'(x)), \qquad y(0) = y_0, \quad y'(0) = y_0'. \tag{2}$$

See, for example, the excellent book by Hairer *et al.* [1] or Lambert [2].

For the multistep method to solve the second-order IVP (1),

$$\sum_{i=0}^{k} a_i y_{n+i} = h^2 \sum_{i=0}^{k'} b_i f_{n+i}, \tag{3}$$

we define the characteristic polynomials,

$$\rho(z) = \sum_{i=0}^{k} a_i z^i \tag{4}$$

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and

$$\sigma(z) = \sum_{i=0}^{k'} b_i z^i. \tag{5}$$

A method is called explicit if $k' = k - 1$ and implicit if $k' = k$. Later, we will consider other possibilities for k'. The order of the method is defined to be p, if for an adequately smooth arbitrary test function $\zeta(x)$,

$$\sum_{i=0}^{k} a_i \zeta(x + ih) - h^2 \sum_{i=0}^{k'} b_i \zeta''(x + ih) = C_{p+2} h^{p+2} \zeta^{(p+2)}(x) + O\left(h^{p+3}\right),$$

where the error constant, C_{p+2}, is given by

$$C_q = \frac{1}{q!} \sum_{j=0}^{k} j^{q-2} \left(j^2 a_j - q(q-1) b_j\right) - \sum_{k+1}^{k'} \frac{j^{q-2}}{(q-2)!} b_j, \qquad q > 2.$$

The method is assumed to satisfy the following,

1. $a_k = 1$, $|a_0| + |b_0| \neq 0$,
2. ρ and σ have no common factor (irreducibility),
3. $\rho(1) = \rho'(1) = 0$, $\rho''(1) = 2\sigma(1)$ (consistency),
4. the method is zero-stable.

The method is called symmetric if

$$a_i = a_{k-i}, \qquad \text{for } i = 0, 1, \dots, k, \text{ and similarly, for } b_j.$$

The definition of P-stability is based on the application of the method characterized by ρ, σ to the periodic IVP,

$$y'' + \omega^2 y = 0. \tag{6}$$

DEFINITION 1. *(See [3].) The method described by the characteristic polynomials ρ, σ is said to have interval of periodicity $(0, H_0^2)$ if, for all H^2 in the interval, the roots of*

$$P\left(z, H^2\right) = \rho(z) + H^2 \sigma(z) = 0, \qquad H = \omega h,$$

satisfy

$$z_1 = e^{i\theta(H)}, \qquad z_2 = e^{-i\theta(H)},$$
$$|z_s| \leq 1, \qquad s > 2,$$

where $\theta(H)$ is a real function.

DEFINITION 2. *(See [3].) The method described by the characteristic polynomials ρ, σ is said to be P-stable if its interval of periodicity is $(0, \infty)$.*

Lambert and Watson proved that a method described by ρ, σ has a nonvanishing interval of periodicity only if it is symmetric and, for P-stability, the order cannot exceed two. Fukushima [4] has proved that the condition is also sufficient. To be precise, we quote the result of [4].

THEOREM. *Consider an irreducible, convergent, symmetric multistep method. Define a function*

$$g(\theta) = -\frac{\rho\left(e^{i\theta}\right)}{\sigma\left(e^{i\theta}\right)}.$$

Then, the method has a nonvanishing interval of periodicity if and only if

1. *$g(\theta)$ has no nonzero double roots in the interval $[0, \pi]$, or*
2. *$g''(\theta)$ is positive on all the nonzero double roots of $g(\theta)$ in the interval $[0, \pi]$.*

Quinlan and Tremaine [5] have extended the work of Lambert and Watson [3] to derive high-order symmetric methods for planetary integrations. The essential difference is that without symmetry a linear growth in the error can cause a quadratic growth in position error. Higher-order P-stable methods were developed by introducing off-step points or higher derivatives of $f(x, y)$. (See [6–8].)

We now recall the super-implicit methods developed by Fukushima [9],

$$\sum_{i=0}^{k} a_i y_{n+1-i} = h^2 \sum_{i=0}^{k'} b_i f_{n+1+m-i}. \tag{7}$$

The method is explicit for $m < 0$, implicit for $m = 0$, and super-implicit for $m > 0$. Fukushima [9] has developed a Störmer-Cowell type formulas, namely $k = 2$ and $a_1 = -2$, $a_2 = a_0 = 1$. In that case, one can get methods of order up to $k' + 2$. Clearly, the methods require additional formulas to treat the additional starting and final values. Solving the nonlinear system one obtains the solution for a block of points. Fukushima [10] suggested the use of Picard iteration. In any event the methods are recommended for parallel or vector machines [11].

2. CONSTRUCTION OF P-STABLE SUPER-IMPLICIT METHODS

We start by writing the super-implicit symmetric k-step methods in the form

$$\sum_{j=0}^{k/2} a_j \left(y_{n+j} + y_{n-j}\right) = h^2 \sum_{j=0}^{k'/2} b_j \left(f_{n+j} + f_{n-j}\right), \tag{8}$$

In our previous notation, we have $a_{k/2} = 1$, and $|a_0| + |b_0| \neq 0$. It is better to choose $a_0 = 1$, since it appears only once. Clearly, k and k' are even and $k' \geq k$ for a super-implicit. Upon applying the method (8) to (6), we have

$$\sum_{j=0}^{k/2} \left(a_j + h^2\omega^2 b_j\right)\left(y_{n+j} + y_{n-j}\right) + \sum_{j=k/2+1}^{k'/2} h^2\omega^2 b_j \left(y_{n+j} + y_{n-j}\right) = 0. \tag{9}$$

Now, substitute $y_n = e^{i\omega x_n}$ to have

$$\sum_{j=0}^{k/2} \left(a_j + H^2 b_j\right)\cos(jH) + \sum_{j=k/2+1}^{k'/2} H^2 b_j \cos(jH) = 0, \tag{10}$$

where $H = h\omega$. We can use one of the parameters to ensure P-stability. For example, suppose we take $k = 4$, $k' = 8$, then the method becomes

$$\begin{aligned} &y_{n+2} - 2y_{n+1} + 2y_n - 2y_{n-1} + y_{n-2} \\ =&h^2 \left[b_4\left(f_{n+4} + f_{n-4}\right) + b_3\left(f_{n+3} + f_{n-3}\right) + b_2\left(f_{n+2} + f_{n-2}\right) + b_1\left(f_{n+1} + f_{n-1}\right) + 2b_0 f_n\right]. \end{aligned} \tag{11}$$

Choose b_0 to satisfy the P-stability condition (10), i.e.,

$$1 + H^2 b_0 + \left(-2 + H^2 b_1\right)\cos(H) + \left(1 + H^2 b_2\right)\cos(2H) + H^3 b_3 \cos(3H) + H^4 b_4 \cos(4H) = 0. \tag{12}$$

Using MAPLE [12], we found the following values

$$\begin{aligned} b_1 &= \frac{362771}{453600}, & b_2 &= \frac{47057}{453600}, \\ b_3 &= -\frac{2707}{453600}, & b_4 &= \frac{641}{1814400}, \end{aligned} \tag{13}$$

and the method is of order ten with an error constant $-4139/79833600$. The choice for b_0 should be $(7411/72576) + O(h^{10})$. The same method is obtained even if we don't restrict the coefficients of the first characteristic polynomial ρ.

To increase the order, we have to add two terms on the right, namely $b_5(f_{n+5} + f_{n-5})$. The method is now of order 12 and the coefficients as computed by MAPLE are

$$\begin{gathered} b_1 = \frac{31489253}{39916800}, \qquad b_2 = \frac{1097339}{9979200}, \qquad b_3 = -\frac{662687}{79833600}, \\ b_4 = \frac{1657}{1900800}, \qquad b_5 = -\frac{4139}{79833600}. \end{gathered} \tag{14}$$

The error constant is $-11370133/1307674368000$ and $b_0 = (4336807/39916800) + O(h^{12})$.

2.1. Störmer-Cowell Type P-Stable Methods

Störmer-Cowell type methods have left-hand side of the form $y_{n+1} - 2y_n + y_{n-1}$. For example, the following method will be of order ten,

$$\begin{aligned} &y_{n+1} - 2y_n + y_{n-1} \\ &= h^2 \left[b_4 (f_{n+4}+f_{n-4})+b_3 (f_{n+3}+f_{n-3})+b_2 (f_{n+2}+f_{n-2})+b_1 (f_{n+1}+f_{n-1})+2b_0 f_n\right]. \end{aligned} \tag{15}$$

The coefficients can be found using MAPLE,

$$\begin{gathered} b_0 = \frac{57517}{145152} + O\left(h^{10}\right), \qquad b_1 = \frac{101741}{907200}, \qquad b_2 = -\frac{8593}{907200}, \\ b_3 = \frac{149}{129600}, \qquad b_4 = -\frac{289}{3628800}. \end{gathered} \tag{16}$$

The error constant is $317/22809600$.

If we allow more points, we can get a 12^{th}-order method,

$$\begin{aligned} y_{n+1} - 2y_n + y_{n-1} = h^2 \big[&b_5 (f_{n+5} + f_{n-5}) + b_4 (f_{n+4} + f_{n-4}) \\ &+b_3 (f_{n+3} + f_{n-3}) + b_2 (f_{n+2} + f_{n-2}) \\ &+b_1 (f_{n+1} + f_{n-1}) + 2b_0 f_n\big]. \end{aligned} \tag{17}$$

The coefficients can be found using MAPLE

$$\begin{aligned} b_0 &= \frac{31494553}{79833600} + O\left(h^{12}\right), & b_1 &= \frac{9186203}{79833600}, \\ b_2 &= -\frac{222331}{19958400}, & b_3 &= \frac{40489}{22809600}, \\ b_4 &= -\frac{17453}{79833600}, & b_5 &= \frac{317}{22809600}. \end{aligned} \tag{18}$$

The error constant is $-6803477/2615348736000$.

Notice that the order depends only on the number of parameters on the right hand side. These tenth and twelfth order Störmer-Cowell type methods are identical to the methods in Table 1 of [9] with $m = 3$, $n = 4$ matching the tenth-order method and $m = 4$, $n = 5$ matching the twelfth-order one. We included here the error constant that was not given in [9]. Thus, the super-implicit methods of [9] are P-stable.

3. IMPLEMENTATION ISSUES

Consider the numerical solution of (1) for $0 \le x \le x_f$. We first subdivide the whole interval into N blocks, say $0 = \xi_0 < \xi_1 < \cdots < \xi_N = x_f$. In each subinterval, $[\xi_i, \xi_{i+1}]$, $i = 0, 1, \ldots, N-1$, we take M equally spaced (h) points $\xi_0 = x_0 < x_1 < \cdots < x_M = \xi_{i+1}$. Now, we create a set of equations relating y_n to past, present and future values using, for example, (11), (see [9]). Thus, we have to solve a system of M equations,

$$y_n = F_n\left(x_0, x_1, \ldots, x_M; f_0, f_1, \ldots, f_M\right), \qquad n = 1, 2, \ldots, M. \tag{19}$$

Fukushima [9] suggested to solve this nonlinear system using Picard iteration. Clearly the number of unknowns in each equation of the system will increase when increasing the order of the numerical method.

Once we get the solution of the first block of points, we have to move on to the next block where the first point is now the same as the last point of the previous block.

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