Original article

Basins of attraction for several optimal fourth order methods for multiple roots

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Abstract

There are very few optimal fourth order methods for solving nonlinear algebraic equations having roots of multiplicity $m$. Here we compare five such methods, two of which require the evaluation of the $(m - 1)$st root. The methods are usually compared by evaluating the computational efficiency and the efficiency index. In this paper all the methods have the same efficiency, since they are of the same order and use the same information. Frequently, comparisons of the various schemes are based on the number of iterations required for convergence, number of function evaluations, and/or amount of CPU time. If a particular algorithm does not converge or if it converges to a different solution, then that particular algorithm is thought to be inferior to the others. The primary flaw in this type of comparison is that the starting point represents only one of an infinite number of other choices. Here we use the basin of attraction idea to recommend the best fourth order method. The basin of attraction is a method to visually comprehend how an algorithm behaves as a function of the various starting points.

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1. Introduction

There is a vast literature on the solution of nonlinear equations, see for example Ostrowski [21], Traub [27], Neta [13], Petković et al. [23] and references therein. In the recent book by Petković et al. [23], they have shown that some methods are a rediscovery of old ones and some are just special cases of other methods. Most of the algorithms are for finding a simple root or of a nonlinear equation $f(x) = 0$. In this paper we are interested in the case that the root is of a known multiplicity $m > 1$. Clearly, one can use the quotient $f(x)/f’(x)$ which has a simple root where $f(x)$ has a multiple

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root. Such an idea will not require a knowledge of the multiplicity, but on the other hand will require higher derivatives. For example, Newton’s method for the function \( F(x) = f(x) f'(x) \) will be

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.
\]

If we define the efficiency index of a method of order, \( p \) as

\[
I = p^{1/d},
\]

where \( d \) is the number of function- (and derivative-) evaluation per step then this method has an efficiency of \( 2^{1/3} = 1.2599 \) instead of \( \sqrt{2} = 1.4142 \) for Newton’s method for simple roots.

There are very few methods for multiple roots when the multiplicity is known. These method are based on the function \( G(x) = \sqrt[3]{f(x)} \) which obviously has a simple root at \( \alpha \), the multiple root with multiplicity \( m \) of \( f(x) \). The first one is due to Schröder [24] and it is also referred to as modified Newton,

\[
x_{n+1} = x_n - m u_n,
\]

where

\[
u_n = \frac{f(x_n)}{f'(x_n)}.
\]

Another method based on the same \( G \) is Laguerre’s-like method

\[
x_{n+1} = x_n - \frac{\lambda u_n}{1 + sgn(\lambda - m) \sqrt{(\lambda - m)/m} (\lambda - 1) - \lambda u_n f''(x_n)/f'(x_n)}
\]

where \( \lambda \ (\neq 0 , m) \) is a real parameter. When \( f(x) \) is a polynomial of degree \( n \), this method with \( \lambda = n \) is the ordinary Laguerre method for multiple roots, see Bodewig [5]. This method converges cubically. Some special cases are Euler–Cauchy, Halley, Ostrowski and Hansen–Patrick [9]. Petković et al. [22] have shown the equivalence between Laguerre family [5] and Hansen–Patrick family. When \( \lambda \to m \) the method becomes second order given by (3). Two other cubically convergent methods that sometimes mistaken as members of Laguerre’s family are: Euler–Chebyshev [27] and Osada’s method [20].

Other methods for multiple roots can be found in [30,14,28,7,8]. Li et al. [11] have developed 6 fourth order methods based on the results of Neta and Johnson [15] and Neta [16]. We will give just those two that are optimal. A method is called optimal if it attains the order \( 2^n \) and uses \( n + 1 \) function-evaluations. Thus a fourth-order optimal method is one that requires 3 function- and derivative-evaluation per step.

In the next section we will present the five optimal fourth-order methods to be analyzed. In the two sections following it, we will analyze the basins of attraction to compare all these fourth order optimal methods for multiple roots. The idea of using basins of attraction was initiated by Stewart [26] and followed by the works of Amat et al. [1–4], Scott et al. [25] and Chun et al. [6].

Neta et al. [19] and Neta and Chun [18] have compared several methods for multiple roots but they have not considered the methods appearing here.

2. Optimal fourth order methods for multiple roots

There are very few methods of optimal order for multiple roots. Li et al. [11] have developed six different methods but only two are optimal, in the sense of Kung and Traub [10]. These are denoted here by LCN5 and LCN6. Liu and Zhou [12] have developed two optimal fourth order methods, denoted here by LZ11 and LZ12. We also discuss a family of methods developed Zhou et al. [31].

- LCN5 (Li et al. [11])

\[
y_{n} = x_{n} - \frac{2m}{m+2} u_{n},
\]

\[
x_{n+1} = x_{n} - a_{3} \frac{f(x_{n})}{f'(y_{n})} - \frac{f(x_{n})}{b_{1} f'(x_{n}) + b_{2} f'(y_{n})},
\]

\[
\text{where } \lambda = m.
\]

The efficiency index of this method is

\[
I = 2^{1/d}.
\]
\[ a_3 = -\frac{1}{2} \frac{(m/(m + 2))m(m - 2)(m + 2)^3}{m^3 - 4m + 8}, \]
\[ b_1 = -\frac{(m^3 - 4m + 8)^2}{m(m^4 + 4m^3 - 4m^2 - 16m + 16)(m^2 + 2m - 4)}, \]
\[ b_2 = \frac{m^2(m^3 - 4m + 8)}{(m/(m + 2))^m(m^4 + 4m^3 - 4m^2 - 16m + 16)(m^2 + 2m - 4)}. \]

- LCN6 (Li et al. [11])
\[
\begin{align*}
y_n &= x_n - \frac{2m}{m + 2} u_n, \\
x_{n+1} &= x_n - a_3 \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n)}{b_1 f'(x_n) + b_2 f'(y_n)},
\end{align*}
\]
where
\[
\begin{align*}
a_3 &= -\frac{1}{2} m(m - 2), \\
b_1 &= -\frac{1}{m}, \quad b_2 = \frac{1}{m(m/(m + 2))^m}.
\end{align*}
\]
There are two other optimal fourth order methods from the family developed by Liu and Zhou [12]
\[
\begin{align*}
y_n &= x_n - m u_n, \\
x_{n+1} &= x_n - m H(w_n) \frac{f(x_n)}{f'(x_n)},
\end{align*}
\]
where
\[
\begin{align*}
w_n &= \left(\frac{f'(y_n)}{f'(x_n)}\right)^{m-1}, \\
and & \quad H(0) = 0, \quad H'(0) = 1, \quad H''(0) = 4m/(m - 1).
\end{align*}
\]
The two members given there are
- LZ11 (Liu and Zhou [12])
\[
\begin{align*}
y_n &= x_n - m \frac{f(x_n)}{f'(x_n)}, \\
x_{n+1} &= y_n - m \left( w_n + \frac{2m}{m - 1} w_n^2 \right) \frac{f(x_n)}{f'(x_n)},
\end{align*}
\]
- LZ12 (Liu and Zhou [12])
\[
\begin{align*}
y_n &= x_n - m \frac{f(x_n)}{f'(x_n)}, \\
x_{n+1} &= y_n + \frac{(m - 1)m w_n}{1 - m + 2m w_n} \frac{f(x_n)}{f'(x_n)}.
\end{align*}
\]
Zhou et al. have presented the optimal method [31]
• ZCS3 Zhou et al. [31]

\[
y_n = x_n - \frac{2m}{m+2}u_n, \\
x_{n+1} = x_n - \phi(t_n) \frac{f(x_n)}{f'(x_n)},
\]

where \( t_n = f(y_n)/f'(x_n) \) and \( \phi \) is at least twice differentiable function satisfying the condition \( \phi(\lambda) = m, \phi'(\lambda) = -1/4m^{2-m}(2 + m)^m, \phi''(\lambda) = 1/4m^4(m/(m + 2))^{-2m} \) and \( \lambda = (m/(m + 2))^{m-1} \). Here we have considered the special case of \( \phi \):

\[
\phi(t) = \frac{B + Ct}{1 + At},
\]

where

\[
A = -\left( \frac{m + 2}{m} \right)^m, \\
B = -\frac{m^2}{2}, \\
C = \frac{1}{2}m(m - 2) \left( \frac{m + 2}{m} \right)^m.
\]

The two methods (denoted LZ11 and LZ12) are the only ones known to the authors where the root of the function is required at each step. We have shown [17] that the cost did not increase by having to evaluate \( (m-1)/\sqrt{f'(y_n)/f'(x_n)} \). We compare these two to other optimal fourth order methods listed above (namely, LCN5, LCN6 and ZCS3).

Frequently, authors will pick a collection of sample equations, initial points and a collection of algorithms for comparison. The comparisons of the various schemes are based on the number of iterations required for convergence, number of function evaluations, and/or amount of CPU time. If a particular algorithm does not converge or if it converges to a different solution, then that particular algorithm is thought to be inferior to the others. The primary flaw in this type of comparison is that the starting point represents only one of an infinite number of other choices. To overcome that, the idea of basin of attraction was introduced. In an ideal case, if a function has \( n \) distinct zeroes, then the plane is divided to \( n \) basins. For example, if we have the polynomial \( z^2 - 1 \), then the roots are \( z = 1 \) and \( z = -1 \). Ideally the basins boundaries are straight lines. Actually, depending on the numerical method, we find the basin boundaries are much more complex, see examples later. The methods are compared by using initial points in a square containing the roots and coloring each point by the root to which it converged. The intensity of the shade depends on the number of iterations required for convergence. If the iterative process did not converge in a given number of iterations, the point assigned a black color. It will be shown by the following examples that the basin boundaries are complex and that from some starting points a method does not converge in a fixed number of iterations or does not converge to the closest root. The best method is picked by comparing the basins qualitatively and quantitatively as we will see later.

3. Corresponding conjugacy maps for quadratic polynomials

Given two maps \( f \) and \( g \) from the Riemann sphere into itself, an analytic conjugacy between the two maps is a diffeomorphism \( h \) from the Riemann sphere onto itself such that \( h \circ f = g \circ h \). Here we consider only quadratic polynomials raised to \( m \)th power.

**Theorem 1** (LCN5 method (6)). For a rational map \( R_a(z) \) arising from LCN5 method applied to \( p(z) = ((z - a)(z - b))^m \), \( a \neq b \), \( R_a(z) \) is conjugate via the Möbius transformation given by \( M(z) = (z - a)/(z - b) \) to

\[
S(z) = z \frac{\rho_1(m, z) + \psi(m, z)\rho_2(m, z) - \rho_3(m, z)}{\rho_1(m, z) + \zeta(m, z)\rho_2(m, z) - z\rho_3(m, z)}.
\]
where
\[ \mu = \left( \frac{m}{m + 2} \right)^m \] (16)
and
\[ \alpha(m, z) = zm + m + 2 \]
\[ \beta(m, z) = \alpha(m, z) + z\phi(m, z) \]
\[ \phi(m, z) = mz + 2z + m \]
\[ \gamma(m, z) = m^4 - 8m^2 - 8m \]
\[ \delta(m, z) = -2m^3 + 8m - 16 \]
\[ \psi(m, z) = \delta(m, z) - z\gamma(m, z) \]
\[ \xi(m, z) = z\delta(m, z) - \gamma(m, z) \]
\[ \eta(m, z) = (m + 2)(z + 1) \]
\[ \rho_1(m, z) = 2m^3\beta(m, z)^2(\alpha(m, z)\phi(m, z))^{2m-2} \]
\[ \rho_2(m, z) = \mu\beta(\alpha(m, z)\phi(m, z))^{m-1}\eta^{2m-1} \]
\[ \rho_3(m, z) = \mu^2(m - 2)(m + 2)^2\eta^{4m-1} \]

**Proof.** Let \( p(z) = ((z - a)(z - b))^m, a \neq b \) and let \( M \) be the Möbius transformation given by \( M(z) = (z - a)(z - b) \) with its inverse \( M^{-1}(u) = (ub - a)/(u - 1) \), which may be considered as a map from \( C \cup \{ \infty \} \). We then have with the help of Maple
\[
S(u) = M \circ R_p \circ M^{-1}(u) = M \circ R_p \left( \frac{ub - a}{u - 1} \right) = u \frac{\rho_1(m, u) + \psi(m, u)\rho_2(m, u) - \rho_3(m, u)}{\rho_1(m, u) + \xi(m, u)\rho_2(m, u) - u\rho_3(m, u)}.
\]

\[ \square \]

**Theorem 2 (LCN6 method (8)).** For a rational map \( R_p(z) \) arising from LCN6 method applied to \( p(z) = ((z - a)(z - b))^m, a \neq b \), \( R_p(z) \) is conjugate via the Möbius transformation given by \( M(z) = (z - a)(z - b) \) to
\[
S(z) = z \frac{(m + 2z)K_1(m, z) - (m + 2 + 2z)K_2(m, z)}{(mz + 2)K_1(m, z) - ((m + 2)z + 2)K_2(m, z)}.
\]

where
\[ K_1(m, z) = \beta(m, z)[\alpha(m, z)\phi(m, z)]^{m-1} \]
\[ K_2(m, z) = m(z + 1)^{2m}(m + 2)^{2m-1} \]
and \( \alpha(m, z), \beta(m, z), \) and \( \phi(m, z) \) are as in the previous theorem.

**Proof.** Let \( p(z) = ((z - a)(z - b))^m, a \neq b \) and let \( M \) be the Möbius transformation given by \( M(z) = (z - a)(z - b) \) with its inverse \( M^{-1}(u) = (ub - a)/(u - 1) \), which may be considered as a map from \( C \cup \{ \infty \} \). We then have
\[
S(u) = M \circ R_p \circ M^{-1}(u) = M \circ R_p \left( \frac{ub - a}{u - 1} \right) = u \frac{(m + 2m)K_1(m, u) - (m + 2 + 2m)K_2(m, u)}{(mu + 2)K_1(m, u) - ((m + 2)u + 2)K_2(m, u)}.
\]

\[ \square \]

**Theorem 3 (LZ11 method (11)).** For a rational map \( R_p(z) \) arising from LZ11 method applied to \( p(z) = ((z - a)(z - b))^m, a \neq b \), \( R_p(z) \) is conjugate via the Möbius transformation given by \( M(z) = (z - a)(z - b) \) to
\[
S(z) = z \frac{(K_3(m, z) - z)(m - 1) + 2mK_3(m, z)^2}{(zK_3(m, z) - 1)(m - 1) + 2zmK_3(m, z)^2}.
\]
Proof. Let \( p(z) = ((z - a)(z - b))^{m} \), \( a \neq b \) and let \( M \) be the Möbius transformation given by \( M(z) = (z - a)/(z - b) \) with its inverse \( M^{-1}(u) = (ub - a)/(u - 1) \), which may be considered as a map from \( C \cup \{ \infty \} \). We then have
\[
S(u) = M \circ R_p \circ M^{-1}(u) = M \circ R_p \left( \frac{ub - a}{u - 1} \right) = u \left( \frac{K_3(m, u) - u(m - 1) + 2mK_3(m, u)^2}{uK_3(m, u) - (m - 1)(m - 2)K_3(m, u)} \right) .
\]
\[\square\]

Theorem 4 (LZ12 method (12)). For a rational map \( R_p(z) \) arising from LZ12 method applied to \( p(z) = ((z - a)/(z - b))^{m} \), \( a \neq b \), \( R_p(z) \) is conjugate via the Möbius transformation given by \( M(z) = (z - a)/(z - b) \) to
\[
S(z) = \frac{z (m - 1)(K_3(m, z) - z) + 2zmK_3(m, z)}{zK_3(m, z) - (m - 1) + 2mK_3(m, z)},
\]
where \( K_3(m, z) \) is as in the previous theorem.
\[\square\]

Theorem 5 (ZCS3 method (13)). For a rational map \( R_p(z) \) arising from ZCS3 method applied to \( p(z) = ((z - a)/(z - b))^m \), \( a \neq b \), \( R_p(z) \) is conjugate via the Möbius transformation given by \( M(z) = (z - a)/(z - b) \) to
\[
S(z) = \frac{-\mu(\alpha(m, z), \beta(m, z), \psi(m, z), \eta(m, z))}{\mu(\alpha(m, z), \beta(m, z), \psi(m, z), \eta(m, z))} \phi(m, z)^m + \psi(m, z)\eta(m, z)^{2m}
\]
where \( \mu \) is given by (16), \( \alpha(m, z), \beta(m, z), \phi(m, z), \eta(m, z) \) are as in LCN5 and
\[
\psi(m, z) = z^2 + zm - 1
\]
\[\square\]

4. Extraneous fixed points

In solving a nonlinear equation iteratively we are looking for fixed points which are zeros of the given nonlinear function. Many iterative methods have fixed points that are not zeros of the function of interest. These points are called extraneous fixed points (see Vrcsay and Gilbert [29]). Those points could be attractive which will trap an iteration sequence and give erroneous results. Even if those extraneous fixed points are repulsive or indifferent they can complicate the situation by converging to a root not close to the initial guess.
Table 1

$H_f(x_n)$ for our fourth order methods. Note that $H_f$ for LCN5 and LCN6 seem similar but the parameters $a_3$, $b_1$, and $b_2$ are different.

<table>
<thead>
<tr>
<th>Method</th>
<th>$H_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LCN5</td>
<td>$a_3 \frac{f'(x_n)}{f''(x_n)} + \frac{1}{b_1 + b_2 f'(x_n)/f''(x_n)}$</td>
</tr>
<tr>
<td>LCN6</td>
<td>$a_3 + \frac{1}{b_1 + b_2 f'(x_n)/f''(x_n)}$</td>
</tr>
<tr>
<td>LZ11</td>
<td>$m \left( 1 + w_n + \frac{2m}{m-1} w_n^2 \right)$</td>
</tr>
<tr>
<td>LZ12</td>
<td>$m - \frac{(m-1)w_n}{1-m+2m w_n}$</td>
</tr>
<tr>
<td>ZCS3</td>
<td>$\frac{B + C f'(x_n)/f''(x_n)}{1 + Af'(x_n)/f''(x_n)}$</td>
</tr>
</tbody>
</table>

All of the methods discussed here can be written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} H_f(x_n).$$

Clearly the root $\alpha$ of $f(x)$ is a fixed point of the method. The points $\xi \neq \alpha$ at which $H_f(\xi) = 0$ are also fixed points of the family, since the second term on the right vanishes.

It is easy to see that $H_f(x_n)$ for our methods is given in Table 1.

**Theorem 6.** The extraneous fixed points for LCN5 can be found by solving

$$\frac{f'(y(\xi))}{f'(\xi)} = \frac{b_1}{b_2 + 1/a_3}. \quad (17)$$

In the case $a_3 = 0$, there are no extraneous fixed points. From (7) we can see that $a_3 = 0$ when $m = 2$.

**Proof.** The extraneous fixed points can be found by solving (17). For the polynomial $(z^2 - 1)^m$ this leads to a very complicated equation. We will give the solution for several values of the multiplicity.

![Fig. 1. LCN5 for the roots of the polynomial $(z^2 - 1)^2$.](image)
For $m = 2$, we have $a_3 = 0$ and therefore $H_f = 1/[b_1 + b_2 f(y_n) f'(x_n)]$ and thus no extraneous fixed points. For other values of $m$ the solution of (17) is complicated. Using Maple we find that for $m = 3$

$$
\frac{(4z + 1)^2 (4z - 1)^2 (4z^2 + 1)^2}{15625z^7} = \frac{27}{31}
$$
Fig. 4. LZ12 for the roots of the polynomial \((z^2 - 1)^2\).

which has roots 
\[ \xi = -0.211240797758941 \pm 0.0512093740287757i, \]
\[ \xi = -0.0699905764436473 \pm 0.287956581131709i, \]
\[ \xi = -0.239769043335697 \pm 0.1851822151220i, \]
\[ \xi = 0.20016603699392, \] and \[ \xi = 3.20538632164326. \]
The fixed point \[ \xi = 3.20538632164326 \] is attractive and all the other fixed points are repulsive.

For \( m = 4 \) the equation becomes
\[
\frac{(5z + 1)^3(5z - 1)^3(5z^2 + 1)^2}{1679616z^9} = \frac{16}{39}
\]
whose roots are
\[ \xi = -0.187518815617671 \pm 0.0612652938601174i, \]
\[ \xi = -0.0604588615401252 \pm 0.289530856026817i, \]
\[ \xi = -0.166776678644070 \pm 0.0264760552329123i, \]
\[ \xi = 0.249701917765884 \pm 0.0264760552329123i, \]
\[ \xi = 1.58888530847133, \] and \[ \xi = -1.61859676206409. \]

The fixed point \[ \xi = 1.58888530847133 \] is attractive and all the other fixed points are repulsive.

For \( m = 5 \) we have
\[
\frac{(6z + 1)^4(6z - 1)^4(6z^2 + 1)^2}{282475249z^{11}} = \frac{9375}{29939}
\]
whose roots are
\[ \xi = -0.167129063349016 \pm 0.0657131609870331i, \]
\[ \xi = -0.135628601644570 \pm 0.0172476674102900i, \]
\[ \xi = -0.0530754369414943 \pm 0.273332371459065i, \]
\[ \xi = 0.144279717830285 \pm -0.0370333131209247i, \]
\[ \xi = 0.229538783536978 \pm 0.199595622775183i, \]
\[ \xi = 1.29364934863688, \] and \[ \xi = 1.33236880189436. \]

The fixed point \[ \xi = 1.29364934863688 \] is attractive and all the other fixed points are repulsive.

\[ \Box \]

**Theorem 7.** The extraneous fixed points for LCN6 can be found by solving
\[
\frac{f'(y(\xi))}{f'(\xi)} = -\frac{1/a_3 + b_1}{b_2}. \tag{18}
\]

If \( a_3 = 0 \), then there are no extraneous fixed points. This happens when \( m = 2 \).
**Proof.** The extraneous fixed points can be found by solving (18). The case $m = 2$ gives $a_3 = 0$ and therefore $H_f$ will not vanish. For higher values of the multiplicity, we have to solve (18) which is very complicated even for the polynomial $(z^2 - 1)^m$. In the case $m = 3$ we have the equation

$$\frac{(4z^2 + 1)((4z^2 + 1)^2/(25z^2) - 1)^2}{z^2(z^2 - 1)^2} = \frac{81}{125},$$

Fig. 5. ZCS3 for the roots of the polynomial $(z^2 - 1)^2$.

Fig. 6. LCN5 for the roots of the polynomial $(z^3 + 4z^2 - 10)^3$. 
for which the roots are $\xi = \pm .2022849824$, and $\xi = \pm .3160112363 \pm .2374587218i$. For $m = 4$ we have

$$\frac{(5z^2 + 1)((5z^2 + 1)^2/(36z^2) - 1)^3}{6z^2(z^2 - 1)^3} = \frac{32}{81},$$

for which the roots are $\xi = \pm .166501731563599 \pm .0278871903068296i$, and $\xi = \pm .363341035970950 \pm .250420169165790i$.

For $m = 5$

$$\frac{(6z^2 + 1)((6z^2 + 1)^2/(49z^2) - 1)^4}{7z^2(z^2 - 1)^4} = \frac{15625}{50421},$$

for which the roots are $\xi = \pm .132338087814173$, $\xi = \pm .144764426655851 \pm .0387259411374153i$, and $\xi = \pm .38927788485548 \pm .253836300351146\,i$.

All the roots are repulsive for $m = 3, 4, 5$. □

**Theorem 8.** The extraneous fixed points for LZ11 can be found by solving

$$1 + w_n(\xi) + \frac{2m}{m - 1}w_n(\xi)^2 = 0. \quad \text{(19)}$$

**Proof.** The extraneous fixed points can be found by solving (19)

$$w_n(\xi) = \frac{-1 \pm i\sqrt{(7m + 1)/(m - 1)}}{4m/(m - 1)}.$$

For the polynomial $(z^2 - 1)^m$ this leads to the equation

$$\frac{((z - (z^2 - 1)/(2z))^2 - 1)^{(m-1/2+1)/(2z^2)}}{z^2 - 1} = \frac{-1 \pm i\sqrt{(7m + 1)/(m - 1)}}{4m/(m - 1)}.$$

In the case $m = 2$ we have the roots $\xi = \pm .6663501590 \pm .1869521088i$, and $\xi = \pm .1869521087 \pm .6663501590i$.

For $m = 3$ we have the roots $\xi = \pm .4948728557 \pm .3835352469i$, $\xi = \pm .5591405393 \pm .1463393609i$, and
\[ \xi = \pm .1329918966 \pm .6235127764i. \] For \( m = 4 \) we have the roots \( \xi = \pm .526875760404095 \pm .0676890178076593i, \)
\( \xi = \pm .487374303113312 \pm .290345428838413i, \quad \xi = \pm .392476333852103 \pm .460699070648885i, \quad \) and
\( \xi = \pm .10316374744143 \pm .609063365078721. \)

For \( m = 5 \) the roots are \( \xi = \pm .506451385403059 \pm .0783152038942477i, \)
\( \xi = \pm .492672405068604 \pm .187015072214805i, \quad \xi = \pm .427577857095005 \pm .373771355527361i, \)
\( \xi = \pm .325541986702564 \pm .499109673996577i, \) and \( \xi = \pm .0843250537544179 \pm .601875671477542i. \)

All the roots are repulsive for \( m = 2, 3, 4, 5. \)

**Theorem 9.** The extraneous fixed points for LZ12 can be found by solving
\[ w_m(\xi) = \frac{m(m - 1)}{2m^2 - m + 1}. \] (20)

**Proof.** The extraneous fixed points can be found by solving (20). For the polynomial \((z^2 - 1)^m\) this leads to the equation
\[ \left( \frac{z^2 - 1}{4z^2} \right)^{(m-1)/2} \left( \frac{z^2 + 1}{2z^2} \right)^{m(m-1)/(2m^2 - m + 1)} = \]
for which the solution is complicated. Using Maple we find that for the case \( m = 2 \) we have the roots \( \xi = \pm .6640471009 \pm .6640471009i. \) For \( m = 3 \) we have the roots \( \xi = \pm .6760352238, \)
\( \xi = \pm .578774836 \pm .7622471679i. \) For \( m = 4 \) we have the roots \( \xi = \pm .650850372513114 \pm .18390536629925i, \)
\( \xi = \pm .357835243719505 \pm .801286135404517i. \)

For \( m = 5 \) the roots are \( \xi = \pm .631081540347638, \ \xi = \pm .633106510826521 \pm .307859301123081i, \)
\( \xi = \pm .297394807289425 \pm .82483141656687i. \)

All the roots are repulsive for \( m = 2, 3, 4, 5. \)

**Theorem 10.** The extraneous fixed points for ZCS3 can be found by solving
\[ B + C \left( \frac{f'(y(\xi))}{f'(\xi)} \right) = 0. \] (21)

**Proof.** The extraneous fixed points can be found by solving (21). For the polynomial \((z^2 - 1)^m\) this leads to a very complicated equation. We will give the solution for several values of the multiplicity.

For \( m = 2 \), we have \( C = 0 \) and thus no extraneous fixed points. For other values of \( m \) the solution of (21) is complicated. Using Maple we find that for \( m = 3 \)
\[ \frac{(4z + 1)^2(4z - 1)^2(4z^2 + 1)^2}{15625z^7} = \]
which has roots \( \xi = -0.216994739857028 \pm .0486258386387605i, \)
\( \xi = -0.0712147935919297 \pm .302196154630797i, \)
\( \xi = .267373626805780 \pm .192552841858781i, \) \( \xi = .204286724043434, \) and \( \xi = .230930891736792. \) All the fixed points are repulsive.

For \( m = 4 \) the equation becomes
\[ \frac{(5z + 1)^3(5z - 1)^3(5z^2 + 1)^2}{1679616z^9} = \]
whose roots are \( \xi = -0.188066600158518 \pm .0606825978478731i, \)
\( \xi = -0.0604973983572096 \pm .0291102666144582i, \)
\( \xi = .167118505473424 \pm .0263484265250046i, \)
\( \xi = .254287017873980 \pm .201824092821040i, \) \( \xi = 1.51517564848636, \) and \( \xi = -0.162165578149714. \) All the fixed points are repulsive.

For \( m = 5 \) we have
\[ \frac{(6z + 1)^4(6z - 1)^4(6z^2 + 1)^2}{282475249z^{11}} = \]
whose roots are \( \xi = -0.167259719413823 \pm .0655872574002358i, \)
\( \xi = -0.135688232865044 \pm .0172310884130640i, \)
\( \xi = -0.0530792120139345 \pm .273713596709721i, \) \( \xi = .14435395735977. \)
Fig. 8. LZ11 for the roots of the polynomial \((z^3 + 4z^2 - 10)^3\).

\[
\pm 0.0369890290391367i, \quad \xi = 0.230741400615992 \pm 0.200052526722947i, \quad \xi = 1.27626370323085, \quad \text{and} \quad \xi = 0.133292881519958.
\]

All the fixed points are repulsive. Vrcsay and Gilbert [29] show that if the points are attractive then the method will give erroneous results. If the points are repulsive then the method may not converge to a root near the initial guess. \(\square\)

5. Numerical experiments

We have used the above methods for 6 different polynomials having multiple roots with multiplicity \(m = 2, 3, 4\) and 5.

Example 1. In our first example, we have taken the polynomial

\[
p_1(z) = (z^2 - 1)^2 \tag{22}
\]

whose roots \(z = \pm 1\) are both real and of multiplicity \(m = 2\). The results are presented in Figs. 1–5. Notice that the darker the shade in each basin, the slower the convergence to the root. At black points the method did not converge in 40 iterations. Method LZ12 (Fig. 4) is best since there are no black points. Notice also that LCN5 (Fig. 1), LCN6 (Fig. 2) and ZCS3 (Fig. 5) have regions of no convergence (black) near the imaginary axis. The worst one is LZ11 (Fig. 3).

Example 2. The second example is a polynomial whose roots are all of multiplicity three. The roots are \(-2.68261500670705 \pm 0.358259359924043i, 1.36523001341410, i.e.

\[
p_4(z) = (z^3 + 4z^2 - 10)^3 \tag{23}
\]

The results are presented in Figs. 6–10. Based on these figures, we find that LCN6 (Fig. 7) and ZCS3 (Fig. 10) perform best. LZ12 (Fig. 9) has very complex basin boundaries.

Example 3. The third example is a polynomial whose roots are all of multiplicity four. The roots are the three roots of unity, i.e.

\[
p_3(z) = (z^3 - 1)^4 \tag{24}
\]
The results are presented in Figs. 11–15. Methods LCN6 (Fig. 12), ZCS3 (Fig. 15) and LZ12 (Fig. 14) are now the only ones with no black regions. Although LZ12 basins are not as separated as the other two.

Example 4. In our next example we took the polynomial

$$p_5(z) = (z^4 - 1)^5$$

(25)
where the roots are symmetrically located on the axes. In some sense this is similar to the first example, since in both cases we have an even number of roots. The results are shown in Figs. 16–20. Again we can see the best methods are LCN6 (Fig. 17) and ZCS3 (Fig. 20) and the worst is method LZ11 (Fig. 18). Method LZ12 (Fig. 19) came third as before.
In order to have a more quantitative comparison of the methods, we have computed the average number of iterations per point for each method and each example. In this part we have included two more examples, the fifth is for the 5 roots of unity with multiplicity 3 and the sixth is for the 3 roots of unity with multiplicity 2. These values are given in Table 2. The total averages for each method are given in the last column. It can be seen that ZCS3 and LCN6 are the best with ZCS3 having a minute advantage. The next one is LZ12, followed by LCN5. The worst performer is LZ11. In our pervious work [17] we have found that LZ11 performed better than LZ12 on 31 examples. There we did not use the basin of attraction idea.

Fig. 13. LZ11 for the roots of the polynomial $(z^3 - 1)^4$.

Fig. 14. LZ12 for the roots of the polynomial $(z^3 - 1)^4$. 
**Remark.** We have noticed that LCN6 and ZCS3 gave essentially the same results. We have dug deeper to find that basically for that choice of $\phi(t)$ the two methods are identical. Take $H_f$ for LCN6,

$$H_f(\text{LCN6}) = a_3 + \frac{1}{b_1 + b_2 t_n} = \frac{(1/b_1 + a_3) + (a_3 b_2/b_1) t_n}{1 + (b_2/b_1) t_n}$$
Fig. 17. LCN6 for the roots of the polynomial \((z^4 - 1)^5\).

Fig. 18. LZ11 for the roots of the polynomial \((z^4 - 1)^5\).
It is easy to see that \(1/b_1 + a_3 = B, a_3 b_2/b_1 = C, \) and \(b_2/b_1 = A.\) Using the values of \(a_3, b_1, \) and \(b_2\) from (9) and simplifying the algebra, we get the parameters of ZCS3 given in (15). We should emphasize that that form of \(\phi\) is only one of the special cases considered by Zhou et al. [31].
6. Conclusions

In this paper we have considered 5th order optimal methods for finding multiple roots of a nonlinear equation. Note that the conjugacy map does not tell the whole story as can be seen when comparing the maps for LZ11 and LZ12. Even though the maps are identical, the results are different. LZ11 required the highest average number of iterations per point than any other method. We have studied all of the extraneous fixed points for low multiplicity ($2 \leq m \leq 5$) and they are repulsive except one point for LCN5 which is attractive. The only methods having no extraneous fixed points are LCN5 and LCN6 and this is only for $m = 2$. Nevertheless, LCN5 and LCN6 still have black regions in the first example with $m = 2$ (see Figs. 1 and 2).

Note the similarity of the basins of attraction for LCN5, LCN6 and ZCS3. In example 2 ($m = 3$), example 3 ($m = 4$) and example 4 ($m = 5$) we note that LCN5 has black regions but not in LCN6 and ZCS3. We can see that LCN6 and ZCS3 were better than the others, as is also evident from the table. We have proved that ZCS3 for the current choice of $\phi$ is the same as LCN6 developed a year earlier. Method LZ12 came always third. Neta et al. [17] demonstrated that on average LZ12 requires more CPU time. Therefore we conclude that LCN6 and ZCS3 are best overall. Can we improve on ZCS3 by taking a different function $\phi$ in (14)? This will be the subject of a follow-on paper.

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