Available online at www.sciencedirect.com
ScienceDirect
Mathematics and Computers in Simulation 109 (2015) 74–91
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

Original articles

Basins of attraction for Zhou–Chen–Song fourth order family of methods for multiple roots

Changbum Chun[a,1], Beny Neta[b,*]

[a] Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Republic of Korea
[b] Naval Postgraduate School, Department of Applied Mathematics, Monterey, CA 93943, United States

Received 15 April 2014; received in revised form 10 July 2014; accepted 29 August 2014
Available online 16 September 2014

Abstract

There are very few optimal fourth order methods for solving nonlinear algebraic equations having roots of multiplicity m. In a previous paper we have compared 5 such methods, two of which require the evaluation of the $(m-1)$th root. We have used the basin of attraction idea to recommend the best optimal fourth order method. Here we suggest to improve on the best of those five, namely Zhou–Chen–Song method by showing how to choose the best weight function.
Published by Elsevier B.V. on behalf of International Association for Mathematics and Computers in Simulation (IMACS).

Keywords: Iterative methods; Order of convergence; Rational maps; Basin of attraction; Conjugacy classes

1. Introduction

There is a vast literature on the solution of nonlinear equations and nonlinear systems, see for example Ostrowski [19], Traub [24], Neta [12] and the recent book by Petković et al. [20] and references therein. "Calculating zeros of a scalar function f ranks among the most significant problems in the theory and practice not only of applied mathematics, but also of many branches of engineering sciences, physics, computer science, finance, to mention only some fields" (see [20]). One simple and well known example is finding the extremal points of a given function F. The candidates for extremum are those points for which $F'(x) = 0$. Most of the algorithms are for finding a simple root of a nonlinear equation $f(x) = 0$, i.e. for a root α we have $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. In this paper we are interested in the case that α is a root of multiplicity $m > 1$. Clearly, one can use the quotient $f(x)/f'(x)$ which has a simple root where $f(x)$ has a multiple root. Such an idea will not require a knowledge of the multiplicity, but on the other hand will require higher derivatives. Therefore the amount of information required to achieve a certain order of convergence is higher.

There are very few methods for multiple roots when the multiplicity is known, see e.g. [17,6–8,10,11,13,14,21,24,25,27]. Some of these methods are considered optimal in the sense of Kung and Traub [9], i.e. they have a maximal

* Corresponding author. Tel.: +1 831 656 2235; fax: +1 831 656 2355.
E-mail addresses: cbchun@skku.edu (C. Chun), bneta@nps.edu, byneta@gmail.com (B. Neta).
1 Tel.: +82 31 299 4523; fax: +82 31 290 7033.

http://dx.doi.org/10.1016/j.matcom.2014.08.005
0378-4754/Published by Elsevier B.V. on behalf of International Association for Mathematics and Computers in Simulation (IMACS).

order of 2^n when using $n+1$ function- (and derivative-) evaluation per iteration step. For methods containing a parameter, the authors did not discuss the question of choosing the parameter to get the best member. For example, Zhou et al. [28] have presented the family of optimal methods (denoted here by ZCS)

$$\begin{aligned} y_n &= x_n - \frac{2m}{m+2} u_n, \\ x_{n+1} &= x_n - \phi(t_n) \frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{1}$$

where $t_n = \frac{f'(y_n)}{f'(x_n)}$ and ϕ is at least twice differentiable function satisfying the following conditions

$$\begin{aligned} \phi(\lambda) &= m, \\ \phi'(\lambda) &= -\frac{1}{4} m^3 \left(\frac{m+2}{m}\right)^m, \\ \phi''(\lambda) &= \frac{1}{4} m^4 \left(\frac{m+2}{m}\right)^{2m}, \end{aligned} \tag{2}$$

and $\lambda = \left(\frac{m}{m+2}\right)^{m-1}$. They did not suggest how to choose the weight function $\phi(t)$ but gave several possibilities.

In a previous paper, Neta and Chun [15] have considered the special case of ϕ:

$$\phi(t) = \frac{B + Ct}{1 + At}, \tag{3}$$

where

$$\begin{aligned} A &= -\frac{m+2}{m\lambda}, \\ B &= -\frac{m^2}{2}, \\ C &= \frac{(m-2)(m+2)}{2\lambda}. \end{aligned}$$

It was shown in [15] that the method is equivalent to an optimal fourth-order method given by (75) in [10]. These two methods perform better than the other optimal fourth-order methods in [10,11]. Here we suggest a rational function ϕ having two parameters and examine the possibility of finding a better performer. The function is given by

$$\phi(t) = \frac{b + ct + dt^2}{1 + at + gt^2}. \tag{4}$$

In order to satisfy the conditions (2), we have the coefficients b, c, and d in terms of a and g as follows

$$\begin{aligned} b &= \frac{m}{8} ((m+2)^2 \lambda m a + (m+2)\lambda^2 m^2 g + m^3 + 6m^2 + 8m + 8), \\ c &= -\frac{m}{4\lambda} ((m^3 + 3m^2 + 2m - 4)\lambda a + (m^2 + m - 2)\lambda^2 m g + m(m+2)(m+3)), \\ d &= \frac{m}{8\lambda^2} (m^2(m+2)\lambda a + (m^3 - 4m + 8)\lambda^2 g + m(m+2)^2). \end{aligned} \tag{5}$$

We will also consider the following functions $\phi(t)$ that Zhou et al. [28] used:

- ZCSpoly

$$\phi(t) = At^2 + Bt + C$$

where

$$A = \frac{1}{8}\left(\frac{m(m+2)}{\lambda}\right)^2$$
$$B = -\frac{1}{4}\frac{m^2(m+2)(m+3)}{\lambda}$$
$$C = \frac{1}{8}m(m^3+6m^2+8m+8).$$

This is a special case of (4) with $a = g = 0$.

- ZCS21

$$\phi(t) = At + \frac{B}{t} + C \tag{6}$$

where

$$A = \frac{1}{8}\frac{m^3(m+2)}{\lambda}$$
$$B = \frac{1}{8}m^2(m+2)^2\lambda$$
$$C = -\frac{1}{4}m(m^3+3m^2+2m-4).$$

We prefer the form

$$\phi(t) = \frac{b+ct+dt^2}{t} \tag{7}$$

with $b = B,\ c = C,\ d = A.$

- ZCS22

$$\phi(t) = A + \frac{B}{t} + \frac{C}{t^2} \tag{8}$$

where

$$A = \frac{1}{8}m(m^3-4m+8)$$
$$B = -\frac{1}{4}m^2(m-1)(m+2)\lambda$$
$$C = \frac{1}{8}m^3(m+2)\lambda^2$$

- ZCS22a

$$\phi(t) = \frac{A}{t} + \frac{1}{B+Ct} \tag{9}$$

where

$$A = -\frac{m^2(m-2)(m+2)^2\lambda}{2(m^3-4m+8)}$$
$$B = -\frac{(m^3-4m+8)^2}{m(m^2+2m-4)^3}$$
$$C = \frac{m(m+2)(m^3-4m+8)}{(m^2+2m-4)^3\lambda}.$$

In the next 2 sections, we will analyze the basins of attraction to compare several cases of the pair of parameters a and g with (7)–(9). The idea of using basins of attraction was initiated by Stewart [23] and followed by the works of Amat et al. [1–4], Scott et al. [22] and Chun et al. [5]. The only papers comparing basins of attraction for methods to obtain multiple roots is due to Neta et al. [18] and Neta and Chun [16,15]. They have not considered the methods with such weight functions. In Section 4 we present the results of several numerical examples with six different combinations of the parameters a and g and the methods (7)-(9). We close with concluding remarks.

2. Corresponding conjugacy maps for quadratic polynomials

Given two maps f and g from the Riemann sphere into itself, an analytic conjugacy between the two maps is a diffeomorphism h from the Riemann sphere onto itself such that $h \circ f = g \circ h$. Here we consider only quadratic polynomials raised to mth power.

Theorem 1 (*ZCS Method* (1), (3)). *For a rational map $R_p(z)$ arising from method* (1), (3) *applied to $p(z) = ((z-a)(z-b))^m$, $a \neq b$, $R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ to*

$$S(z) = \frac{-m\lambda\alpha(m,z)^{m-1}\beta(m,z)(\psi(m,z)-2z)\phi(m,z)^{m-1} + \psi(m,z)\eta(m,z)^{2m}}{m\lambda\alpha(m,z)^{m-1}\beta(m,z)(\psi(m,z)+2z)\phi(m,z)^{m-1} - \psi(m,z)\eta(m,z)^{2m}}$$

where

$$\begin{aligned}
&\alpha(m,z) = zm + m + 2\\
&\beta(m,z) = \alpha(m,z) + z\phi(m,z)\\
&\phi(m,z) = (m+2)z + m\\
&\psi(m,z) = z^2 + zm - 1\\
&\eta(m,z) = (m+2)(z+1).
\end{aligned}$$

The proof can be found in [15].

Theorem 2 (*ZCS Method* (1), (4)). *For a rational map $R_p(z)$ arising from method* (1), (4) *applied to $p(z) = ((z-a)(z-b))^m$, $a \neq b$, $R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ to*

$$S(z) = \frac{N_1(m,z)V^2T^2 + N_2(m,z)z^{2m} - N_3(m,z)VTz^m}{D_1(m,z)V^2T^2 + D_2(m,z)z^{2m} - D_3(m,z)VTz^m}$$

where α, β, ϕ as before and

$$\begin{aligned}
&V = \left[\frac{z}{1+z}\left(z + \frac{m}{m+2}\right)\right]^m\\
&T = \left[\frac{z}{1+z}\left(1 + \frac{m}{m+2}z\right)\right]^m\\
&N(m,z) = 2\lambda\alpha(m,z)\phi(m,z)\beta(m,z)\\
&\nu_1(m,z) = (m+2)N(m,z)\\
&\nu_2(m,z) = (m+2)^2\beta(m,z)^2\\
&\nu_3(m,z) = \lambda^2\phi(m,z)^2\alpha(m,z)^2\\
&\nu_4(m,z) = m(m+1)(m+2) + 4z\\
&\nu_5(m,z) = m(m+1)(m+2)z + 4\\
&\nu_6(m,z) = m(m+2)(m+4) - 8z\\
&\nu_7(m,z) = m(m+2)(m+4)z - 8\\
&N_1(m,z) = \nu_2(m,z)\left[m(m+2)^2 + \lambda^2 g(m^3 - 8z - 4m) + \lambda a m^2(m+2)\right]z\\
&N_2(m,z) = \nu_3(m,z)\left[\nu_6(m,z) + \lambda^2 g m^2(m+2) + \lambda a m(m+2)^2\right]z\\
&N_3(m,z) = \nu_1(m,z)\left[m(m+2)(m+3) + \lambda^2 g m(m+2)(m-1) + \lambda a \nu_4(m,z)\right]z\\
&D_1(m,z) = \nu_2(m,z)\left[m(m+2)^2 z + \lambda^2 g(m^3 z - 4mz - 8) + \lambda a m^2(m+2)z\right]\\
&D_2(m,z) = \nu_3(m,z)\left[\nu_7(m,z) + \lambda^2 g m^2(m+2)z + \lambda a m(m+2)^2 z\right]\\
&D_3(m,z) = \nu_1(m,z)\left[m(m+2)(m+3)z + \lambda^2 g m(m+2)(m-1)z + \lambda a \nu_5(m,z)\right].
\end{aligned}$$

Proof. Let $p(z) = ((z-a)(z-b))^m,\ a \neq b$ and let M be the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ with its inverse $M^{-1}(u) = \frac{ub-a}{u-1}$, which may be considered as a map from $C \cup \{\infty\}$. We then have with the help of Maple

$$\begin{aligned} S(u) &= M \circ R_p \circ M^{-1}(u) = M \circ R_p\left(\frac{ub-a}{u-1}\right) \\ &= \frac{N_1(m,u)V^2T^2 + N_2(m,u)z^{2m} - N_3(m,u)VTz^m}{D_1(m,u)V^2T^2 + D_2(m,u)z^{2m} - D_3(m,u)VTz^m}. \end{aligned}$$

Theorem 3 (*ZCS21 Method* (1), (7)). *For a rational map $R_p(z)$ arising from method (1), (7) applied to $p(z) = ((z-a)(z-b))^m$, $a \neq b$, $R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ to*

$$S(z) = \frac{N_1(m,z)V^2T^2 + N_2(m,z)z^{2m} - N_3(m,z)VTz^m}{N_1(m,z)V^2T^2 + N_2(m,z)z^{2m} - N_4(m,z)VTz^m}$$

where $V,\ T,\ \nu_2,\ \nu_3,\ \nu_4,\ \nu_5$ as before and

$$\begin{aligned} N_1(m,z) &= m^2\nu_2(m,z)z \\ N_2(m,z) &= m(m+2)\nu_3(m,z)z \\ N_3(m,z) &= N(m,z)\nu_4(m,z)z \\ N_4(m,z) &= N(m,z)\nu_5(m,z). \end{aligned}$$

Theorem 4 (*ZCS22 Method* (1), (8)). *For a rational map $R_p(z)$ arising from method (1), (8) applied to $p(z) = ((z-a)(z-b))^m$, $a \neq b$, $R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ to*

$$S(z) = \frac{N_1(m,z)V^2T^2 + N_2(m,z)z^{2m} - N_3(m,z)VTz^m}{N_4(m,z)V^2T^2 + N_2(m,z)z^{2m} - N_3(m,z)VTz^m}$$

where $V,\ T,\ N,\ \beta,\ \nu_3$ as before and

$$\begin{aligned} N_1(m,z) &= (m+2)(m^3-4m-8z)\beta(m,z)^2z \\ N_2(m,z) &= m^2\nu_3(m,z)z \\ N_3(m,z) &= m(m-1)(m+2)N(m,z)z \\ N_4(m,z) &= (m+2)(m^3z-4mz-8)\beta(m,z)^2. \end{aligned}$$

Theorem 5 (*ZCS22a Method* (1), (9)). *For a rational map $R_p(z)$ arising from method (1), (9) applied to $p(z) = ((z-a)(z-b))^m$, $a \neq b$, $R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \frac{z-a}{z-b}$ to*

$$S(z) = z\frac{N_1(m,z)V^2T^2 + N_2(m,z)z^{2m} - N_3(m,z)VTz^m}{N_1(m,z)V^2T^2 + z^{2m+1}N_2(m,z) - N_4(m,z)VTz^m}$$

where $V,\ T,\ N,\ \nu_2,\ \nu_3$ as before and

$$\begin{aligned} N_1(m,z) &= -2m^2\nu_2(m,z)(z+1) \\ N_2(m,z) &= m(m-2)(m+2)\nu_3(m,z) \\ N_3(m,z) &= N(m,z)(m^4-2m^3z-8m^2+8mz-8m-16z)/2 \\ N_4(m,z) &= N(m,z)(m^4z-2m^3-8m^2z+8m-8mz-16)/2. \end{aligned}$$

3. Extraneous fixed points

In solving a nonlinear equation iteratively we are looking for fixed points which are zeros of the given nonlinear function. Many iterative methods have fixed points that are not zeros of the function of interest. Those points are called extraneous fixed points (see Vrscay and Gilbert [26]). Those points could be attractive which will trap an iteration sequence and give erroneous results. Even if those extraneous fixed points are repulsive or indifferent they can complicate the situation by converging to a root not close to the initial guess.

Table 1
$H_f(x_n)$ for our fourth order methods.

Method	H_f
(1), (3)	$\frac{B+C\frac{f'(y_n)}{f'(x_n)}}{1+A\frac{f'(y_n)}{f'(x_n)}}$
(1), (4)	$\frac{b+c\frac{f'(y_n)}{f'(x_n)}+d\left(\frac{f'(y_n)}{f'(x_n)}\right)^2}{1+a\frac{f'(y_n)}{f'(x_n)}+g\left(\frac{f'(y_n)}{f'(x_n)}\right)^2}$,
(1), (7)	$\frac{B+C\frac{f'(y_n)}{f'(x_n)}+A\left(\frac{f'(y_n)}{f'(x_n)}\right)^2}{\frac{f'(y_n)}{f'(x_n)}}$,
(1), (8)	$\frac{C+B\frac{f'(y_n)}{f'(x_n)}+A\left(\frac{f'(y_n)}{f'(x_n)}\right)^2}{\left(\frac{f'(y_n)}{f'(x_n)}\right)^2}$,
(1), (9)	$\frac{AB+(AC+1)\frac{f'(y_n)}{f'(x_n)}}{B\frac{f'(y_n)}{f'(x_n)}+C\left(\frac{f'(y_n)}{f'(x_n)}\right)^2}$.

The method discussed here can be written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} H_f(x_n).$$

Clearly the root α of $f(x)$ is a fixed point of the method. The points $\xi \neq \alpha$ at which $H_f(\xi) = 0$ are also fixed points of the family, since the second term on the right vanishes.

It is easy to see that $H_f(x_n)$ for our methods is given in Table 1.

Theorem 6. *The extraneous fixed points for (1), (3) can be found by solving*

$$B + C\frac{f'(y(\xi))}{f'(\xi)} = 0. \tag{10}$$

The proof can be found in [15].

Theorem 7. *The extraneous fixed points for (1), (4) can be found by solving*

$$b + c\frac{f'(y(\xi))}{f'(\xi)} + d\left(\frac{f'(y(\xi))}{f'(\xi)}\right)^2 = 0. \tag{11}$$

Proof. The extraneous fixed points can be found by solving (11). For the polynomial $(z^2-1)^m$ this leads to a very complicated equation. We will give the solution for several values of the multiplicity. All the computations were done using Maple.

For $m = 2$ Eq. (11) becomes

$$\frac{N_{2,a,g}(z)}{D_{2,a,g}(z)} = 0 \tag{12}$$

where

$$\begin{aligned} N_{2,a,g}(z) &= (5776 + 1049g + 2468a)z^8 - (2496 + 60g + 624a)z^6 \\ &\quad + (992 + 46g + 248a)z^4 - (192 + 12g + 48a)z^2 + 16 + g + 4a \end{aligned}$$

$$D_{2,a,g}(z) = (4096 + 729g + 1728a)z^8 + (324g + 384a)z^6 - (18g + 64a)z^4 - 12gz^2 + g.$$

The roots are as follows:
If $a = 0$, $g = 0$ then

$$\xi = \pm 0.444075071235753 \pm 0.351364125343937i,$$
$$\xi = \pm 0.391443492288385 \pm 0.104438718753136i.$$

If $a = -4$, $g = 0$ then $\xi = 0$.
If $a = -4$, $g = 4$ then there are no extraneous fixed points.
If $a = -3.83$, $g = -0.68$ then
$\xi = \pm 0.2403584225i$, $\xi = \pm 0.2071260458$.
If $a = -6.01$, $g = 8.04$ then
$\xi = \pm 0.4454613602$, $\xi = \pm 1.020308653$.
If $a = -1.51$, $g = -9.96$ then
$\xi = \pm 0.303516853705238$, $\xi = \pm 0.453837415669222i$.
For $m = 3$ Eq. (11) becomes

$$\frac{N_{3,a,g}(z)}{D_{3,a,g}(z)} = 0 \tag{13}$$

where

$$\begin{aligned} N_{3,a,g}(z) \; = \; & (2978921875 + 322660071g + 981284625a)z^{12} \\ & - (70488576g + 1304000000 + 324864000a)z^{10} \\ & + (25593840g + 541250000 + 126360000a)z^{8} \\ & - (122187500 + 5002020g + 26730000a)z^{6} \\ & + (3510000a + 16250000 + 645840g)z^{4} \\ & - (875000 + 189000a + 34776g)z^{2} + 3375a + 621g + 15625 \\ D_{3,a,g}(z) \; = \; & (703125000 + 75497472g + 230400000a)z^{12} \\ & + (28800000a + 18874368g)z^{10} - (6300000a + 2949120g)z^{8} \\ & + (225000a - 368640g)z^{6} + 74880gz^{4} - 4032gz^{2} + 72g. \end{aligned}$$

The roots are as follows:
If $a = 0$, $g = 0$ then

$$\xi = \pm 0.207973827611037 \pm 0.003003140014000793i,$$
$$\xi = \pm 0.422506765232095 \pm 0.362486535539093i,$$
$$\xi = \pm 0.386866717970422 \pm 0.145445174529573i.$$

If $a = -4$, $g = 0$ then

$$\xi = \pm 0.361571220508882 \pm 0.238768269303465i,$$
$$\xi = \pm 0.578549967743772i, \qquad \xi = \pm 0.204907492042522,$$
$$\xi = \pm 0.224126362952842, \qquad \xi = \pm 0.300418297005595.$$

If $a = -4$, $g = 4$ then

$$\xi = \pm 0.193720627416559 \pm 0.202742293909324i,$$
$$\xi = \pm 0.445931691446149 \pm 0.134769278028657i,$$
$$\xi = \pm 0.537187376035398 \pm 0.311544575924712i.$$

If $a = -3.83$, $g = -0.68$ then

$$\xi = \pm 0.3602196283 \pm 0.2388217529i,$$
$$\xi = \pm 0.5836396200i, \qquad \xi = \pm 0.2048441800,$$
$$\xi = \pm 0.2236709356, \qquad \xi = \pm 0.3022381361.$$

If $a = -6.01$, $g = 8.04$ then

$$\xi = \pm 0.4290202944 \pm 0.2283073945i,$$
$$\xi = \pm 0.2256118365 \pm 0.04680875069i,$$
$$\xi = \pm 0.3903942642i, \qquad \xi = \pm 0.2073026506.$$

If $a = -1.51$, $g = -9.96$ then

$$\xi = \pm 0.2043856550, \qquad \xi = \pm 0.2210548492, \qquad \xi = \pm 0.3142783605, \qquad \xi = \pm 0.6215038328i$$
$$\xi = \pm 0.3508841438 \pm 0.2390342012i.$$

For $m = 4$ and $m = 5$ the results are messy and we will only give the extraneous fixed points for the 2 methods that we found to be the best. As we can see later, the choices are $a = -4$, $g = 0$ and $a = -6.01$, $g = 8.04$.

If $m = 4$, $a = -4$, $g = 0$ then

$$\xi = \pm 0.167254183868000 \pm 0.0275766173167561i,$$
$$\xi = \pm 0.171276201276364 \pm 0.0257659847307305i,$$
$$\xi = \pm 0.404656022666154 \pm 0.246635543973501i,$$
$$\xi = \pm 0.822428106896555i, \qquad \xi = \pm 0.358951084238680.$$

If $m = 4$, $a = -6.01$, $g = 8.04$ then

$$\xi = \pm 0.1716490433 \pm 0.02558466458i,$$
$$\xi = \pm 0.1675417107 \pm 0.02745567013i,$$
$$\xi = \pm 0.4253366951 \pm 0.2427062392i,$$
$$\xi = \pm 0.7629373002i, \qquad \xi = \pm 0.3492384896.$$

If $m = 5$, $a = -4$, $g = 0$ then

$$\xi = \pm 0.145161247808070 \pm 0.0384620191849354i,$$
$$\xi = \pm 0.146325370628400 \pm 0.0376584498253377i,$$
$$\xi = \pm 0.436560989278621 \pm 0.246477741429716i,$$
$$\xi = \pm 10.4625350048809i, \qquad \xi = \pm 0.132642382665641,$$
$$\xi = \pm 0.133546301109759, \qquad \xi = \pm 0.437860827955717.$$

If $m = 5$, $a = -6.01$, $g = 8.04$ then

$$\xi = \pm 0.1463289573 \pm 0.03765739425i,$$
$$\xi = \pm 0.1452411500 \pm 0.03840682169i,$$
$$\xi = \pm 0.4489276217 \pm 0.2433403312i,$$
$$\xi = \pm 7.576277121i, \qquad \xi = \pm 0.1326999971,$$
$$\xi = \pm 0.1335531628, \qquad \xi = \pm 0.4373850651.$$

All the fixed points are repulsive except the fixed point $\xi = 0$, for $m = 2$, $a = -4$, $g = 4$, which is indifferent and the fixed point $\xi = \pm 0.453837415669222i$ for $m = 2$, $a = -1.51$, $g = -9.96$ which is super-attractive.

Remark. The above results include the case ZCSpoly upon taking $a = g = 0$. For the other weight functions, the numerator is the same quadratic and so we just have to check that the denominator does not vanish at those points.

Theorem 8. *The extraneous fixed points for (1), (7) can be found by solving*

$$B + C\frac{f'(y(\xi))}{f'(\xi)} + A\left(\frac{f'(y(\xi))}{f'(\xi)}\right)^2 = 0. \tag{14}$$

Proof. We will give the solution for several values of the multiplicity. All the computations were done using Maple.

For $m = 2$ Eq. (14) becomes

$$\frac{617z^8 - 156z^6 + 62z^4 - 12z^2 + 1}{z^4(3z^2+1)(9z^2-1)} = 0. \tag{15}$$

The roots are as follows:

$$\xi = \pm 0.376053998399425 \pm 0.372021694971105i,$$
$$\xi = \pm 0.365565414545151 \pm 0.101175037160889i.$$

For $m = 3$ Eq. (14) becomes

$$\frac{290751z^{12} - 96256z^{10} + 37440z^8 - 7920z^6 + 1040z^4 - 56z^2 + 1}{z^6(4z^2+1)(4z+1)^2(4z-1)^2} = 0. \tag{16}$$

The roots are as follows:

$$\xi = \pm 0.379418914953478 \pm 0.364917660486372i,$$
$$\xi = \pm 0.207293547759891 \pm 0.00385580624857242i,$$
$$\xi = \pm 0.365694148699927 \pm 0.148161455848097i.$$

For $m = 4$ Eq. (14) becomes

$$\frac{N_{16}(z)}{z^8(5z^2+1)(5z+1)^3(5z-1)^3} = 0 \tag{17}$$

where

$$N_{16}(z) = 763076329z^{16} - 276237500z^{14} + 105359500z^{12} - 21843860z^{10} + 3169198z^8 - 222500z^6 + 7900z^4 - 140z^2 + 1.$$

The roots are as follows:

$$\xi = \pm 0.377152762339161 \pm 0.348349168808825i,$$
$$\xi = \pm 0.166682596277652 \pm 0.0260250253831761i,$$
$$\xi = \pm 0.167973107288453 \pm 0.0291026495068239i,$$
$$\xi = \pm 0.368560920564186 \pm 0.173797198907396i.$$

For $m = 5$ Eq. (14) becomes

$$\frac{N_{20}(z)}{z^{10}(6z^2+1)(6z+1)^4(6z-1)^4} = 0 \tag{18}$$

where

$$N_{20}(z) = 6814163076491z^{20} - 2561362891776z^{18} + 953797379328z^{16} - 188350092288z^{14} + 27962048856z^{12} - 2094017212z^{10} + 87526656z^8 - 2187648z^6 + 32868z^4 - 276z^2 + 1.$$

The roots are as follows:

$$\xi = \pm 0.1323752253 \pm 0.0005714857100i,$$
$$\xi = \pm 0.144323895765696 \pm 0.0379527963619449i,$$
$$\xi = \pm 0.145310514565242 \pm 0.0394468622361660i,$$
$$\xi = \pm 0.374148893224887 \pm 0.328772261860497i,$$
$$\xi = \pm 0.370213286163991 \pm 0.193599979476163i.$$

All the fixed points are repulsive, no exception.

Theorem 9. *The extraneous fixed points for (1), (8) can be found by solving*

$$C + B\frac{f'(y(\xi))}{f'(\xi)} + A\left(\frac{f'(y(\xi))}{f'(\xi)}\right)^2 = 0. \tag{19}$$

Proof. We will give the solution for several values of the multiplicity. All the computations were done using Maple.

For $m = 2$ Eq. (19) becomes

$$\frac{2098z^8 - 120z^6 + 92z^4 - 24z^2 + 2}{(3z^2+1)^2(3z+1)^2(3z-1)^2} = 0. \tag{20}$$

The roots are as follows:

$$\xi = \pm 0.293495779172449 \pm 0.407670123526465i,$$
$$\xi = \pm 0.339862208343043 \pm 0.0827798166231053i.$$

For $m = 3$ Eq. (19) becomes

$$\frac{11950373z^{12} - 2610688z^{10} + 947920z^8 - 185260z^6 + 23 - 1288z^2 + 23920z^4}{(4z^2+1)^2(4z+1)^4(4z-1)^4} = 0. \tag{21}$$

The roots are as follows:

$$\xi = \pm 0.326520807335722 \pm 0.364069002072458i,$$
$$\xi = \pm 0.206224285167792 \pm 0.00537434434264770i,$$
$$\xi = \pm 0.338352284754117 \pm 0.147719470913145i.$$

For $m = 4$ Eq. (19) becomes

$$\frac{N_{16}(z)}{(5z^2+1)^2(5z+1)^6(5z-1)^6} = 0 \tag{22}$$

where

$$N_{16}(z) = 8641095007z^{16} - 2495262500z^{14} + 872300500z^{12} - 159196940z^{10} + 22274242z^8 - 1557500z^6 + 55300z^4 - 980z^2 + 7.$$

The roots are as follows:

$$\xi = \pm 0.340902770554521 \pm 0.335070944394960i,$$
$$\xi = \pm 0.165964395216490 \pm 0.0260221960139433i,$$
$$\xi = \pm 0.167445157903423 \pm 0.0296351165220827i,$$
$$\xi = \pm 0.345731486146877 \pm 0.181976981910545i.$$

For $m = 5$ Eq. (19) becomes

$$\frac{N_{20}(z)}{(6z^2+1)^2(6z+1)^8(6z-1)^8} = 0 \tag{23}$$

where

$$N_{20}(z) = 1145335160540383z^{20} - 371567845029888z^{18} + 128312563428864z^{16} - 22297558431744z^{14} + 3179956272528z^{12} - 236770646056z^{10} + 9890512128z^8 - 247204224z^6 + 3714084z^4 - 31188z^2 + 113.$$

The roots are as follows:

$$\xi = \pm 0.1320896898 \pm 0.0005336039749i,$$
$$\xi = \pm 0.143987253161952 \pm 0.0382426862700286i,$$
$$\xi = \pm 0.144897387809369 \pm 0.0396454823984612i,$$
$$\xi = \pm 0.347909746557530 \pm 0.307393208388302i,$$
$$\xi = \pm 0.350911261316159 \pm 0.208855972149650i.$$

All the fixed points are repulsive.

Theorem 10. *The extraneous fixed points for (1), (9) can be found by solving*

$$AB + (AC + 1)\frac{f'(y(\xi))}{f'(\xi)} = 0. \tag{24}$$

Proof. We will give the solution for several values of the multiplicity. All the computations were done using Maple.

For $m = 2$ Eq. (24) becomes

$$\frac{32z^4}{11z^4 + 6z^2 - 1} = 0. \tag{25}$$

The roots are all zero.

For $m = 3$ Eq. (24) becomes

$$\frac{z^6(52631z^6 - 3968z^4 + 868z^2 - 31)}{(4z^2 + 1)(4z + 1)^2(4z - 1)^2(449z^6 + 128z^4 - 28z^2 + 1)} = 0. \tag{26}$$

The roots are as follows:

$\xi = \pm 0.265452876482484 \pm 0.228707652173347i$,
$\xi = \pm 0.197678401401779$.

For $m = 4$ Eq. (24) becomes

$$\frac{27648z^8(477367z^8 - 81250z^6 + 19500z^4 - 910z^2 + 13)}{(5z^2 + 1)(5z + 1)^3(5z - 1)^3(29741z^8 + 6250z^6 - 1500z^4 + 70z^2 - 1)} = 0. \tag{27}$$

The roots are as follows:

$\xi = \pm 0.166140901646348 \pm 0.0280330990141669i$,
$\xi = \pm 0.347928616940534 \pm 0.250537480519098i$,
$\xi = \pm 0.347928616940534$.

For $m = 5$ Eq. (24) becomes

$$\frac{z^{10}N_{10}(z)}{(6z^2 + 1)(6z + 1)^4(6z - 1)^4 D_{10}(z)} = 0 \tag{28}$$

where

$$N_{10}(z) = 1563243369z^{10} - 342081792z^8 + 85520448z^6 - 4223232z^4 + 84318z^2 - 611$$
$$D_{10}(z) = 3294871z^{10} + 559872z^8 - 139968z^6 + 6912z^4 - 138z^2 + 1.$$

The roots are as follows:

$\xi = \pm 0.132279278794500$,
$\xi = \pm 0.144687458801151 \pm 0.0387765609887301i$,
$\xi = \pm 0.382013739639782 \pm 0.254362184937334i$.

All the fixed points are repulsive. Vrscay and Gilbert [26] show that if the points are attractive then the method will give erroneous results. If the points are repulsive then the method may not converge to a root near the initial guess.

4. Numerical experiments

We have used the above method (1) with weight function (4) with 6 different combinations of parameters in (5). We have also used the 3 methods ZCS21, ZCS22, and ZCS22a for comparison. We ran the 9 cases on 6 different polynomials having multiple roots with multiplicity $m = 2, 3, 4$ and 5. In several cases we have included the basins of attractions to show the best and worst of the nine cases. In general we prefer to have a more qualitative comparison, by computing the average number of iterations required for convergence per initial point. In each case we have taken a 6 by 6 square centered at the origin. The total number of initial points in the square is 360,000 uniformly spaced.

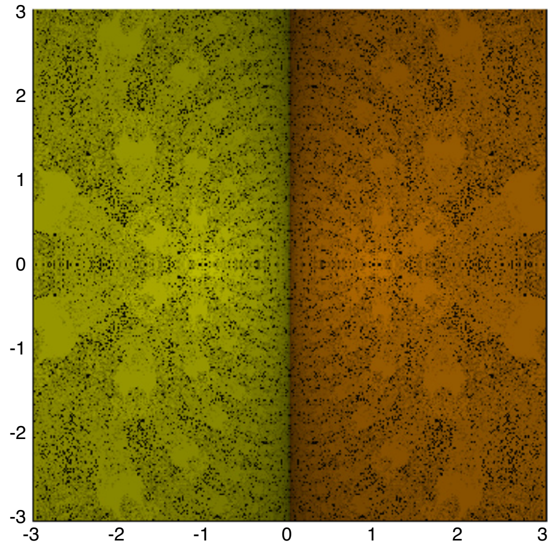

Fig. 1. ZCS with $a = -4$, $g = 4$ for the roots of the polynomial $(z^2 - 1)^2$.

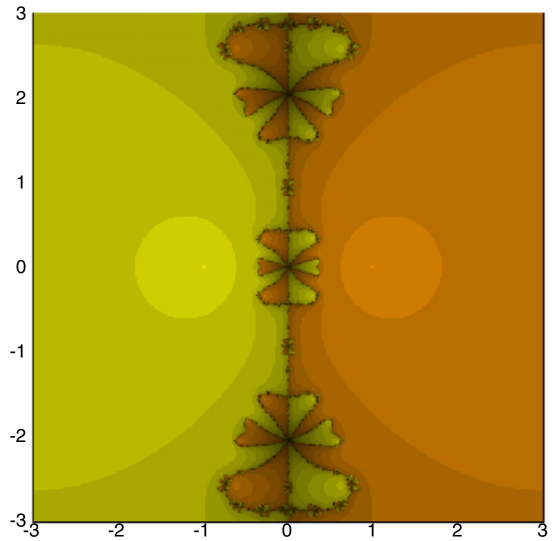

Fig. 2. ZCS with $\phi(t) = At + B/t + C$ for the roots of the polynomial $(z^2 - 1)^2$.

The code will assign a color to each point based on the root it converged to. If the method did not converge after 40 iterations the code will assign a black color to the point. We have also used the intensity of the color to indicate the number of iterations, i.e. the lighter the shade the faster the method converged to that root.

Example 1. In our first example, we have taken the polynomial

$$p_1(z) = (z^2 - 1)^2 \tag{29}$$

whose roots $z = \pm 1$ are both real and of multiplicity $m = 2$. The results are presented in Figs. 1–3. Notice that the darker the shade in each basin, the slower the convergence to the root. Case 8 (Fig. 3) is best because there are no black points and in most cases the method converged to the closest root. The next best is case 7 (Fig. 2). Here we have some black points and the lobes along the vertical line are larger, which means that at those points the method did not converge to the closest root. The worst performer is case 4 with $a = -4,\ g = 4$ (Fig. 1). It is clear that we have many

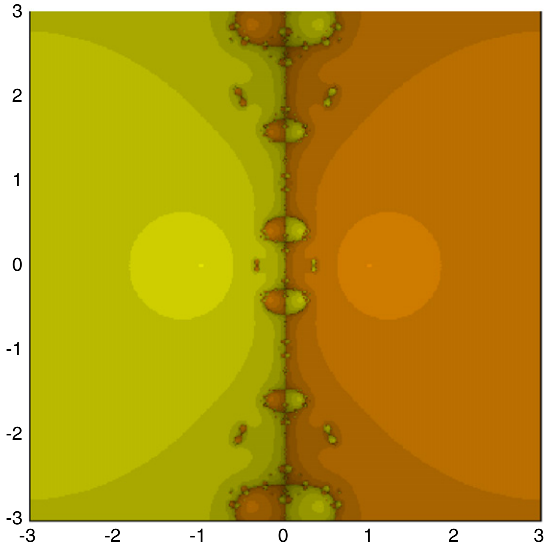

Fig. 3. ZCS with $\phi(t) = A + B/t + C/t^2$ for the roots of the polynomial $(z^2 - 1)^2$.

Table 2
Average number of iterations per point for Examples 1–6 and each pair of parameters.

Case	a	g	Ex1	Ex2	Ex3	Ex4	Ex5	Ex6	Total
1	−4	0	4.4206	4.4134	4.5920	6.1554	6.6395	5.9335	32.1544
2	0	0	4.2093	6.1057	7.1300	10.5871	7.3328	10.6584	46.0233
3	−3.83	−0.68	4.4206	4.3286	4.3817	8.7711	6.6396	6.1922	34.7338
4	−4	4	8.6656	5.5184	5.7198	8.2915	9.0724	8.7456	46.0133
5	−6.01	8.04	4.4208	4.0609	4.0634	8.0337	6.6396	5.0637	32.2821
6	−1.51	−9.96	4.4206	4.2554	4.2897	7.9462	6.6396	5.6391	33.1906
7	(7)		3.8958	5.3691	6.4282	9.9231	5.9295	9.7258	41.2715
8	(8)		3.5829	4.7609	5.1452	7.6153	4.5407	7.1463	32.7913
9	(9)		4.4206	7.1740	7.0934	9.5781	6.6395	10.1501	45.0557

black points which means that the method did not converge within the 40 iterations allowed. For multiplicity higher than 2 all the extraneous fixed points are repulsive. Note that case 8 (Fig. 3) is the only example with no black points. We have not shown the figures for the other cases but collected the average number of iterations per point for each case and each example in Table 2.

Example 2. The second example is a polynomial whose roots are all of multiplicity three. The roots are $-2.68261500670705 \pm .358259359924043i$, 1.36523001341410, i.e.

$$p_2(z) = (z^3 + 4z^2 - 10)^3. \tag{30}$$

The results for only 3 cases are presented in Figs. 4–6. Now that $m \neq 2$ we find that case 5 (Fig. 5) is best (no black points) and cases 9 (Fig. 6) and 2 (Fig. 4) are worst. Both of these latter cases have black points. Again, we can see the rest of the results tabulated in Table 2. The best and worst performers in the last example are somewhere between best and worst performers in this example.

Example 3. The third example is a polynomial whose roots are all of multiplicity four. The roots are the three roots of unity, i.e.

$$p_3(z) = (z^3 - 1)^4. \tag{31}$$

The results are presented in Figs. 7–9. Now case 5 (Fig. 8) is best and the worst performers are cases 9 (Fig. 9) and 2 (Fig. 7). This is the exact same situation as in Example 2, even though the multiplicity is different.

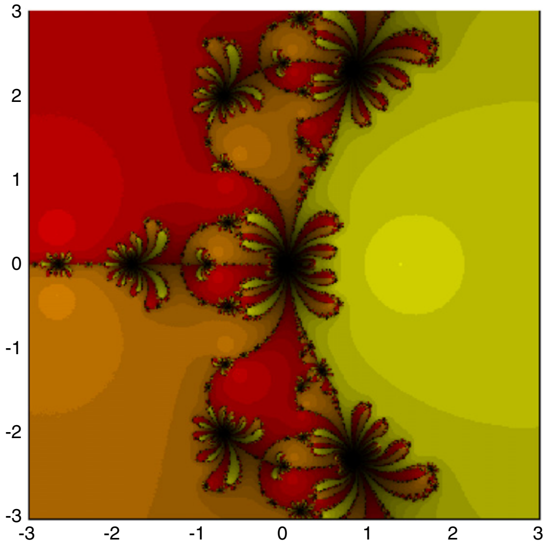

Fig. 4. ZCS with $a = 0$, $g = 0$ for the roots of the polynomial $(z^3 + 4z^2 - 10)^3$.

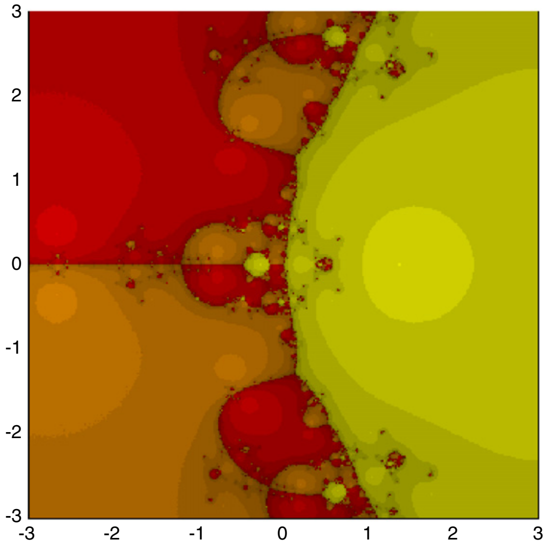

Fig. 5. ZCS with $a = -6.01$, $g = 8.04$ for the roots of the polynomial $(z^3 + 4z^2 - 10)^3$.

Example 4. In our next example we took the polynomial

$$p_4(z) = (z^4 - 1)^5 \tag{32}$$

where the roots are symmetrically located on the axes. The results are shown in Figs. 10 and 11. Now we can see that the best case is 1 (Fig. 10) and the worst is case 2 (Fig. 11). This is the only example when case 1 was best performer but in all other examples it was close to best.

Example 5. Our next example is having double roots. The polynomial has the three roots of unity,

$$p_5(z) = (z^3 - 1)^2. \tag{33}$$

The results are presented only in Table 2. The results are different than those in Example 3 even though we have polynomials of the same degree. Based on Table 2 the best performer is case 8 and the worst is case 4.

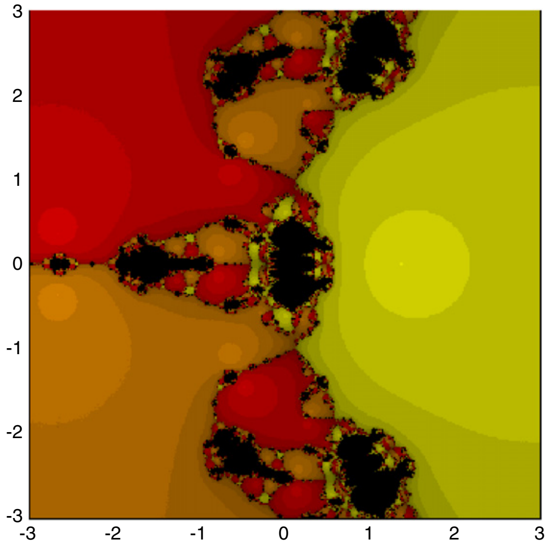

Fig. 6. ZCS with $\phi(t) = A/t + 1/(B + Ct)$ for the roots of the polynomial $(z^3 + 4z^2 - 10)^3$.

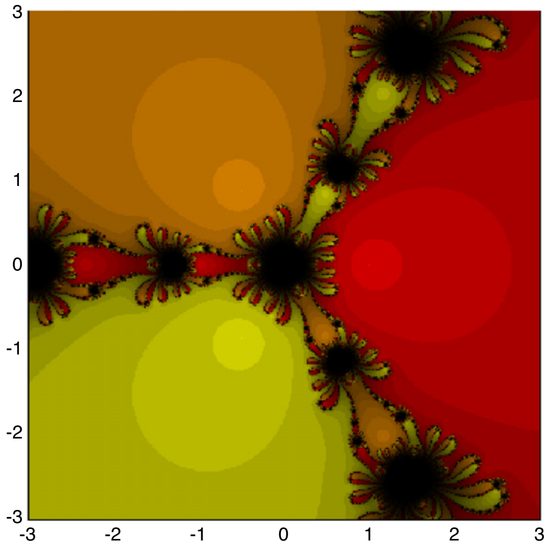

Fig. 7. ZCS with $a = 0$, $g = 0$ for the roots of the polynomial $(z^3 - 1)^4$.

Example 6. In our last example we have the 5 roots of unity all with multiplicity three

$$p_6(z) = (z^5 - 1)^3. \tag{34}$$

The results are given in Table 2. Based on this table, the best performer is case 5 and the worst is case 2.

We can now summarize that case 1 was best in one out of the 6 examples. Case 2 was worst performer in 4 examples and was never best. Cases 3, 6 and 7 were always somewhere in between, even though case 7 was closer to the bottom than the others. Case 4 was never best performer but was worst in 2 examples. Case 5 was best in 3 examples and was never worst performer. Case 8 was best in 2 examples and case 9 was worst or close to that in 3 examples.

The total averages for each method are given in the last column of Table 2. It can be seen that overall cases 1 and 5 are best followed closely by cases 8, 6 and 3. The worst performers are cases 2 and 4. Case 2 corresponds to a choice of a quadratic polynomial as a weight function. Therefore it is not recommended to take polynomial weight functions.

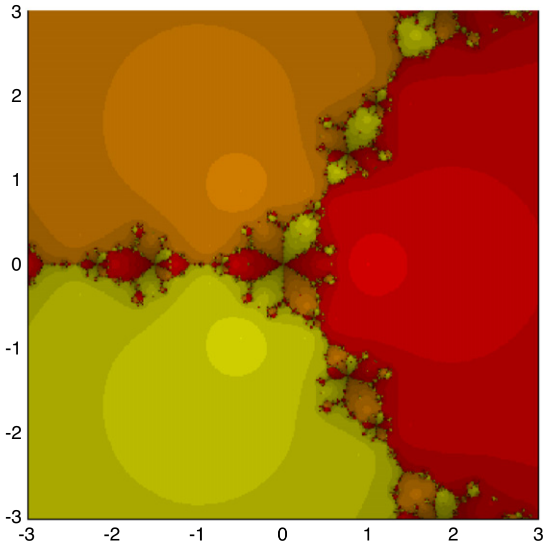

Fig. 8. ZCS with $a = -6.01$, $g = 8.04$ for the roots of the polynomial $(z^3 - 1)^4$.

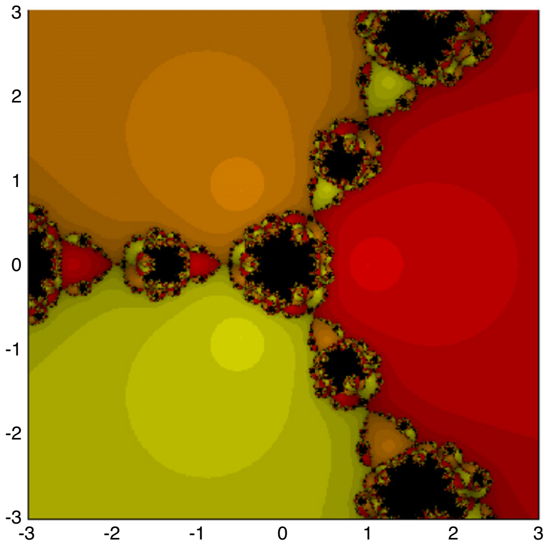

Fig. 9. ZCS with $\phi(t) = A/t + 1/(B + Ct)$ for the roots of the polynomial $(z^3 - 1)^4$.

Note that for $m = 2$ case 1 is basically the case analyzed in the previous paper [15]. If $m \neq 2$, then the best performers are cases 1 ($a = -4$, $g = 0$) and 5 ($a = -6.01$, $g = 8.04$). Therefore we have found better performers (always in top 4) by using a quotient of two quadratic polynomials.

5. Conclusion

In a previous paper [15], we have shown how to choose the coefficients of a weight function in the form of a quotient of two linear polynomials for the method due to Zhou et al. [28]. Here we have analyzed the possibility of using a rational function being a quotient of two quadratic polynomials. We have also included the other cases originally suggested by Zhou et al. [28]. We have ran 9 cases with the first 6 being new with a certain choice for the parameters a and g and the last 3 are those suggested by Zhou et al. [28]. In case 2 the extraneous fixed points are all complex and in case 4 there are no extraneous fixed points. These cases were the worst. Cases 1 and 5 performed best

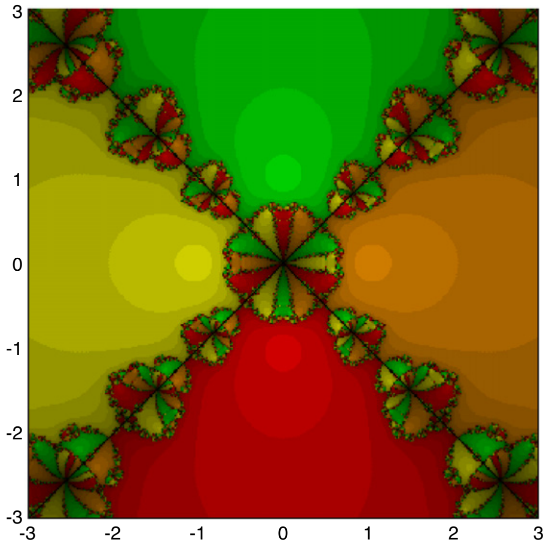

Fig. 10. ZCS with $a = -4$, $g = 0$ for the roots of the polynomial $(z^4 - 1)^5$.

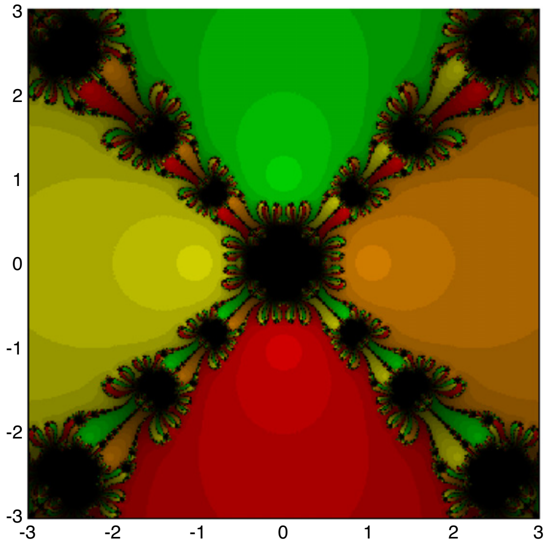

Fig. 11. ZCS with $a = 0$, $g = 0$ for the roots of the polynomial $(z^4 - 1)^5$.

overall. Therefore we have found better performers than the ones originally suggested by Zhou et al. [28] and the best performer in the previous paper [15]. We can also conclude that one should not choose a polynomial weight function (case 2).

Acknowledgment

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2013R1A1A2005012).

References

[1] S. Amat, S. Busquier, S. Plaza, Iterative root-finding methods, 2004, unpublished manuscript.
[2] S. Amat, S. Busquier, S. Plaza, Review of some iterative root-finding methods from a dynamical point of view, Sci. Ser. A Math. Sci. 10 (2004) 3–35.

[3] S. Amat, S. Busquier, S. Plaza, Dynamics of a family of third-order iterative methods that do not require using second derivatives, Appl. Math. Comput. 154 (2004) 735–746.
[4] S. Amat, S. Busquier, S. Plaza, Dynamics of the King and Jarratt iterations, Aeq. Math. 69 (2005) 212–223.
[5] C. Chun, M.Y. Lee, B. Neta, J. Džunić, On optimal fourth-order iterative methods free from second derivative and their dynamics, Appl. Math. Comput. 218 (2012) 6427–6438.
[6] C. Dong, A basic theorem of constructing an iterative formula of the higher order for computing multiple roots of an equation, Math. Numer. Sinica 11 (1982) 445–450.
[7] C. Dong, A family of multipoint iterative functions for finding multiple roots of equations, Int. J. Comput. Math. 21 (1987) 363–367.
[8] E. Hansen, M. Patrick, A family of root finding methods, Numer. Math. 27 (1977) 257–269.
[9] H.T. Kung, J.F. Traub, Optimal order of one-point and multipoint iteration, J. ACM 21 (1974) 643–651.
[10] S. Li, L.Z. Cheng, B. Neta, Some fourth-order nonlinear solvers with closed formulae for multiple roots, Comput. Math. Appl. 59 (2010) 126–135.
[11] B. Liu, X. Zhou, A new family of fourth-order methods for multiple roots of nonlinear equations, Nonlinear Anal. Model. Control 18 (2013) 143–152.
[12] B. Neta, Numerical methods for the solution of equations, Net-A-Sof, 1983.
[13] B. Neta, New third order nonlinear solvers for multiple roots, Appl. Math. Comput. 202 (2008) 162–170.
[14] B. Neta, Extension of Murakami's high order nonlinear solver to multiple roots, Int. J. Comput. Math. 8 (2010) 1023–1031.
[15] B. Neta, C. Chun, Basins of attraction for several optimal fourth order methods for multiple roots, Math. Comput. Simul. 103 (2014) 39–59.
[16] B. Neta, C. Chun, On a family of Laguerre methods to find multiple roots of nonlinear equations, Appl. Math. Comput. 219 (2013) 10987–11004.
[17] B. Neta, A.N. Johnson, High order nonlinear solver for multiple roots, Comput. Math. Appl. 55 (2008) 2012–2017.
[18] B. Neta, M. Scott, C. Chun, Basin attractors for various methods for multiple roots, Appl. Math. Comput. 218 (2012) 5043–5066.
[19] A. Ostrowski, Solution of Equations in Euclidean and Banach Space, Academic Press, 1973.
[20] M.S. Petković, B. Neta, L.D. Petković, J. Džunić, Multipoint Methods for Solving Nonlinear Equations, Elsevier Science, 2013.
[21] E. Schröder, Über unendlich viele Algorithmen zur auflösung der Gleichungen, Math. Ann. 2 (1870) 317–365.
[22] M. Scott, B. Neta, C. Chun, Basin attractors for various methods, Appl. Math. Comput. 218 (2011) 2584–2599.
[23] B.D. Stewart, Attractor basins of various root-finding methods (Master's thesis), Naval Postgraduate School, Dept. of Applied Mathematics, Monterey CA 93943, 2001.
[24] J. Traub, Iterative Methods for the Solution of Equations, Chelsea Publishing Company, 1997.
[25] H.D. Victory, B. Neta, A higher order method for multiple zeros of nonlinear functions, Int. J. Comput. Math. 12 (1983) 329–335.
[26] E.R. Vrscay, W.J. Gilbert, Extraneous fixed points, basin boundaries and chaotic dynamics for Schröder and König rational iteration functions, Numer. Math. 52 (1988) 1–16.
[27] W. Werner, Iterationsverfahren höherer ordnung zur lösung nicht linearer Gleichungen, Z. Angew. Math. Mech. 61 (1981) T322–T324.
[28] X. Zhou, X. Chen, Y. Song, Constructing higher-order methods for obtaining the multiple roots of nonlinear equations, J. Comput. Appl. Math. 235 (2011) 4199–4206.