Galerkin spectral synthesis methods for diffusis equations with general boundary conditions

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Abstract

An existence and uniqueness theory is developed for the energy dependent, steady-state neutron diffusion equation with inhomogeneous oblique boundary conditions imposed. A convergence theory is developed for the Galerkin spectral synthesis approximations that arise when trial functions depending only on energy are utilized. The diffusion coefficient, total and scattering cross-sectional data are all assumed to be both spatially and energy dependent. Interior interfaces defined by spatial discontinuities in the cross-section data are assumed to be present. Our estimates are in a Sobolev-type norm, and our results show that the spectral synthesis approximations are optimal in the sense of being of the same order as the error generated at the best approximation to the actual solution from the subspace to which the spectral synthesis approximations belong. Published by Elsevier Science Ltd.

1. Introduction

As indicated in the survey article by Adams (1977), synthesis methods in general have been utilized quite satisfactorily to obtain computational results for the neutron diffusion approximation to nuclear reactor problems. Such methods consist of all techniques in which an approximate solution is sought in the form

$$\sum_{i=1}^{N} \omega_i(x) \psi_i(x, y)$$

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where $x$ is a subset of the independent variables, $y$ represents the remaining pendent variables; the $\psi_i$ are the known trial functions, and $\omega_i(x)$ are the common coefficients. The basic idea of synthesis methods is to substitute the approximant into the governing diffusion equation, take the inner product over $y$ of the result expressions with test functions $\xi_i(x, y)$, and then to determine the combining coefficients $\omega_i(x)$ by solving the resulting linear system. Galerkin synthesis method when the spaces of test and trial functions are the same.

Unfortunately, however, the theoretical underpinnings of synthesis methods is to be lacking. The research toward constructing a convergence theory for these methods seems to consist of the work by Meyer and Nelson (1979) and Nelsc Meyer (1977). They considered the special case of Galerkin spectral synthesis approximations in which $\xi_i = \psi_i = \psi_i(E)$, where $E$ represents energy. In Nelsc Meyer (1977), these authors considered the convergence question of Galerkin spectral synthesis approximations to the continuous energy diffusion equations, in the diffusion coefficient and the total cross-section are both spatially dependent, and the convergence of the spectral synthesis approximations was strictly in an $L_2$-setting of functions of both space and energy, and was obtained by using operator-theoretic results of Polskij (1962) concerning projection methods. In Meyer and Nelson (1979), on the other hand, spatially dependent cross-sections and diffusion were present, and the techniques employed to prove convergence were to those utilized in analyzing the convergence of finite element methods in approximating elliptic boundary-value problems (Aubin, 1972; Strang and Fix, 1978). Hilbert space for the analysis was chosen to be an $L_2$ space of Sobolev space functions. The elements of this Hilbert space are $L_2$ mappings of the energy $d$ $(0, \infty)$ into the Sobolev space $H^1_0(\Omega)$, where $\Omega$ is the spatial domain. In working a variational form of the diffusion equation, the authors imposed conditions total and scattering cross-sections which guarantee a submultiplying diffusivity in order to exploit the Lax–Milgram lemma (Friedman, 1969) to obtain unique weak solution to the diffusion problem. The boundary condition: homogeneous Dirichlet conditions (in the spatial variable). The convergence spectral synthesis approximations to the weak solution is optimal in the sense that the errors are of the same order as the errors predicted by the best approximator weak solution from the subspaces where the spectral synthesis approximants occur. Neta (1981) relaxed the stringent conditions imposed on the scattering section data by Meyer and Nelson to guarantee convergence of the spectral synthesis approximants, but, unfortunately, the convergence results in this more generalizing were not shown to be of the “optimal type” as those of Meyer and Nelson (1979, pp. 907–908). In all of the aforementioned works, the weak solution diffusion problem was not shown to be classical once smoothness assumptions the source and cross-section data were stipulated. It is appropriate to remark that a survey of theoretical and numerical questions in obtaining a rigorous convergence theory for synthesis methods can be found in Neta and Victory (1988).

The present work considers spectral schemes with Galerkin weighting, and techniques used in studying projection methods (Witsch, 1977) to examine convergence question. As in Meyer and Nelson (1979) we, too, allow for
discontinuities in the diffusion coefficient and underlying cross-section data. work, however, extends that of Meyer and Nelson in the sense that more gen boundary conditions (i.e. oblique boundary conditions) for the diffusion equa are treated. The coercivity assumption in [Meyer and Nelson, 1979, (2.13)], in ving the total and scattering cross-section data, will be weakened by observing the scattering operator will provide a compact mapping from its domain space t dual. We will apply the results of Witsch concerning projection methods for op tors which are compactly perturbed (Witsch, 1977, pp. 343–344).

In Section 2, assumptions about the cross-section data and the diffusion coeffic are formally stated, and the existence and uniqueness questions for both clas and weak solutions to interface diffusion problems are treated. The spectral sy synthesis approximations are formulated in Section 3, and the existence of these app imations is settled. In Section 4, we present our convergence results, which ind that our spectral synthesis approximations to the diffusion solution are of the s order as the best approximation to the diffusion solution from the subspace which the spectral synthesis solutions belong. Section 5 treats the convergence of multigroup approximations to the diffusion equation.

2. The energy-dependent neutron diffusion problem

Let $D(x, E)$ be the diffusion coefficient, $\Sigma(x, E)$ the total cross-section, $S(x, E)$ spontaneous source, and $k(x, E, E')$ the scattering kernel with $x$ an element of spatial domain $\Omega \subset \mathcal{R}^n$ and the energy variable $E$ an element of the interval $(0, \infty)$. We also assume that $D$ and $\Sigma$ can be viewed as piecewise continuous mappings (to $L_{\infty}((0, \infty))$ which is the Banach space of functions essentially bounded in ene with norm $\|f\|_{\infty} = \text{ess sup}_{E \in (0, \infty)} |f(E)|$; moreover, as a function of two variables, $D(x, E)$ is bounded below by a positive constant $c_1$. Our scattering kernel will be viewed piecewise continuous mapping from $\Omega$ to $L_2((0, \infty) \times (0, \infty))$. The region $\Omega$ is ta to be bounded and open. Further, we assume that $\Omega$ is possibly divided into a number of subdomains $\Omega_i$, each with boundary $\partial \Omega_i$. Also, we assume these bot aries are sufficiently well behaved so that integration by parts may be perfr. Clearly, portions of $\partial \Omega_i$ will make up the interfaces and the rest will make up boundary $\partial \Omega$ of $\Omega$.

The underlying diffusion problem we consider is

$$A\phi = K\phi + S(x, E), \quad x \in \Omega, \quad E \in (0, \infty)$$

$$\alpha \phi(x, E) + D(x, E) \frac{\partial \phi(x, E)}{\partial n} = \gamma, \quad \alpha \geq 0, \quad x \in \Omega, \quad E \in (0, \infty)$$

where both $\phi(x, E)$ and $D(x, E) \frac{\partial \phi(x, E)}{\partial n}$ are continuous across interfaces, and w the operator $K$ is defined by
\[ K \phi = \int_0^\infty k(x, E, E') \Sigma(x, E') \phi(x, E') dE', \]

and the operator \( A \) is defined by

\[ A \phi = -\nabla \cdot (D(x, E) \nabla \phi(x, E)) + \Sigma(x, E) \phi(x, E). \]

Note that if the term containing \( \frac{\partial \phi}{\partial n} \) in (2.2) is not present, i.e. the boundary conditions are of Dirichlet type, the integrals over the boundary in (2.7) and (2.8) will not be present, and the test functions must vanish on the boundary. If the boundary conditions are of Neumann type \((\alpha = 0)\), then we have a boundary integral of (2.7).

Let us introduce inner products and norms:

\[ (u, v) = \int_0^\infty \int_\Omega u(x, E) v(x, E) dx dE \]

\[ < u, v >= (u, v) + \int_0^\infty \int_\Omega \nabla u(x, E) \cdot \nabla v(x, E) dx dE \]

\[ (u, v)_{\partial \Omega} = \int_0^\infty \int_{\partial \Omega} u(x, E) v(x, E) dx dE \]

\[ ||u||_0 = (u, u)^{1/2}, \]

\[ ||u||_{\partial \Omega} = (u, u)^{1/2}, \]

\[ ||u||_1 = < u, u >^{1/2}. \]

Our analysis will primarily utilize the space \( L_2((0, \infty); H_0^1(\Omega)), \) equipped with norm (2.4d). It is easily seen that this particular space is the function space of primary importance in Meyer and Nelson (1979) and Neta (1981). Also \( L_2(\Omega \times \Omega) \) will denote the Hilbert space of functions square-integrable with respect to position and energy variables equipped with the norm described in (2.4d).

From our initial assumptions, we see that

\[ (Du_x, u_x) \geq c_1 ||u_x||_0^2, \]

and

\[ (Du_x, v_x) + (\Sigma u, v) \leq c_2 ||u||_1 ||v||_1. \]

Concerning the scattering kernel \( k \), we assume further that
\[
\begin{align*}
\sup_{\Omega} \int_0^\infty \int_0^\infty k^2(x, E, E') \Sigma^2(x, E') dE'dE &< \infty, \\
\sup_{\Omega} \int_0^\infty \int_0^\infty |k(x + h, E + t, E') \Sigma(x + h, E') - k(x, E, E') \Sigma(x, E')|^2 dE'dE &\rightarrow 0,
\end{align*}
\]

as \(|h| + |t| \rightarrow 0\) where \(k \equiv 0\) outside \(\Omega \times (0, \infty) \times (0, \infty)\);

\[
\sup_{\Omega} \int_s^\infty \int_0^\infty k^2(x, E, E') \Sigma^2(x, E') dE'dE \rightarrow 0, \quad \text{as} \ s \rightarrow \infty.
\]

The mapping properties of the operator \(K\) defined by (2.3a) are described in following lemma the proof of which appears in Appendix A.

**Lemma 1.** The operator \(K\) is a compact mapping of \(L_2((0, \infty); H^1(\Omega))\) into \((\Omega \times (0, \infty))\).

We now turn to the formulation of a weak version of the problem (2.1)–(2.2). In this end we need the following trace properties to hold, see Lions (private communication) and Adams (1975, pp. 184–191).

**Lemma 2.** Assuming \(\partial \Omega\) is smooth enough there is a trace operator \(\gamma_0\)

\[
\gamma_0 v = v|_{\partial \Omega \times (0, \infty)}
\]

which is linear and continuous from \(L_2((0, \infty); H^1(\Omega))\) into \(L_2((0, \infty); H^{1/2}(\partial \Omega))\).

**Remark.** We will use \(v\) instead of \(\gamma_0 v\) on \(\partial \Omega\).

**Weak Formulation:** find \(u \in L_2((0, \infty); H^1(\Omega))\) such that

\[
B(u, v) = (S, v) + (\gamma, v)_{\partial \Omega}, \quad \text{for all} \ v \in L_2((0, \infty); H^1(\Omega)),
\]

where

\[
B(u, v) = a(u, v) - (Ku, v),
\]

\[
a(u, v) = \int_0^\infty \int_{\Omega} \left\{ D(x, E) \nabla u(x, E) \cdot \nabla v(x, E) + \Sigma(x, E) u(x, E) v(x, E) \right\} dx dE
\]

\[
+ \alpha \int_0^\infty \int_{\partial \Omega} u(x, E) v(x, E) dx dE,
\]
and

\[ \gamma \in L_2((0, \infty); H^{-1/2}(\partial \Omega)), \quad S \in L_2(\Omega \times (0, \infty)). \]

Clearly the last integral in (2.8b) is on the boundary \( \partial \Omega \) and thus \( u, v \) a restrictions of these functions to \( \partial \Omega \) (see Lemma 2).

Since \( a(u, v) \) is a continuous bilinear form (the boundary term is continuous result of Lemma 2), we can consider \( A \) a continuous linear mapping of \( L_2((0, \infty); H^1(\Omega)) \) into its dual, by virtue of the duality mapping

\[ a(u, v) = \langle Au, v \rangle. \]

By the continuity of the bilinear form, the operator \( A \) can be extended to elements of \( L_2((0, \infty); H^1(\Omega)) \) and have range in the dual of \( L_2((0, \infty); H^1(\Omega)) \).

**Theorem 3.** Under subcritical conditions (in which 1 is not in the spectrum of \( K \) to \( A \)), the weak formulation of the classical diffusion problem described by (2.7), unique solution \( u \in L_2((0, \infty); H^1(\Omega)) \).

**Proof.** Since \( \Sigma (x, E) \) is positive, one obtains using (2.8b), (2.4) and Lemma 2

\[ a(u, u) \geq c_1\|u_x\|_0^2 + \sigma\|u\|_0^2 + \alpha\|u\|_1^2 \geq c_3\|u\|_1^2, \]

\[ a(u, v) \leq c_2\|u\|_1\|v\|_1 + \alpha\|u\|_1\|v\| \leq c_4\|u\|_1\|v\|_1. \]

Since \( K \) is compact, one obtains by the Riesz–Fredholm alternative (Aubin, Theorem 1-15, p. 62) that since 1 is not in the spectrum of \( K \) relative to \( A \) exists a unique solution \( u \) of (2.7).

In our next theorem we show the "equivalence" between the weak and classical solution which will yield, using the Lax–Milgram theorem, the existence and uniqueness to the classical problem.

**Theorem 4.** The interface problem (2.1)–(2.2) is equivalent to the weak form (2.7) in the sense that every solution of (2.1)–(2.2) is a solution of (2.7) and sufficiently differentiable solution of (2.7) is a solution of (2.1)–(2.2).

**Proof.** Multiply (2.1) by \( \psi(x, E) \) and integrate over \( \Omega \times (0, \infty) \) one ha: applying Green's formula

\[ \int_0^\infty \int_\Omega D(x, E) \nabla \phi(x, E) \cdot \nabla \psi(x, E) dxdE + \int_\Omega \int_0^\infty \Sigma(x, E) \phi(x, E) \psi(x, E) dxdE \]

\[ - \int_0^\infty \int_{\partial \Omega} D(x, E) \nabla \phi(x, E) \cdot n \psi(x, E) dxdE \]

\[ = \int_0^\infty \int_\Omega K \phi(x, E) \psi(x, E) dxdE + \int_0^\infty \int_\Omega S(x, E) \psi(x, E) dxdE. \]
Using (2.2) one obtains (2.7):

\[ B(\phi, \psi) = (S, \psi) + \int_0^\infty \int_{\partial \Omega} \gamma \psi(x, E) dx dE. \]

Conversely, let \( \phi \) be a solution of (2.7): Then

\[
\int_0^\infty \int_{\Omega} D(x, E) \nabla \phi(x, E) \cdot \nabla \psi(x, E) dx dE + \int_0^\infty \int_{\Omega} \Sigma(x, E) \phi(x, E) \psi(x, E) dx dE \\
+ \alpha \int_0^\infty \int_{\partial \Omega} \phi(x, E) \psi(x, E) dx dE - \int_0^\infty \int_{\Omega} K \phi \psi(x, E) dx dE \\
= \int_0^\infty \int_{\Omega} S(x, E) \psi(x, E) dx dE + \int_0^\infty \int_{\partial \Omega} \gamma \psi(x, E) dx dE, \quad \forall \psi \in L_2((0, \infty); H^1(\Omega)).
\]

Integration by parts yields

\[
\int_0^\infty \int_{\Omega} \{-\nabla \cdot [D(x, E) \nabla \phi(x, E)] + \Sigma(x, E) \phi(x, E) - K \phi - S(x, E)\} \psi(x, E) dx dE \\
+ \int_0^\infty \int_{\partial \Omega} \{-\gamma + \alpha \phi(x, E) + D(x, E) \nabla \phi(x, E) \cdot n\} \psi(x, E) dx dE = 0.
\] \hspace{1cm} (2)

If \( \psi \) vanishes everywhere on \( \partial \Omega \), then the boundary integral is zero and one (2.1). Returning to (2.12) one now has

\[
\int_0^\infty \int_{\partial \Omega} \{\alpha \phi(x, E) + D(x, E) \nabla \phi(x, E) \cdot n - \gamma\} \psi(x, E) dx dE = 0
\]

which yields (2.2).

As a result of the equivalence theorem, we can use the Lax–Milgram theorem prove existence and uniqueness of the solutions, assuming

\[
\left(D \frac{\partial \phi}{\partial n}, \phi\right)_{\partial \Omega} \geq 0 \quad \text{for any } \phi \in L_2((0, \infty); H^{1/2}(\partial \Omega)),
\]

see Aubin (1972, p. 172).

3. Spectral synthesis approximation

To specify the synthesis method, we first prescribe a set of functions

\[
\{\xi_i(E) : E \in (0, \infty), i = 1, 2, \ldots, n\}
\]
in $L_2((0, \infty))$. These functions define a subspace $W_n \subset L_2((0, \infty); H^1(\Omega))$ given

$$W_n = \left\{ \Psi \in L_2((0, \infty); H^1(\Omega)) : \Psi(x, E) = \sum_{i=1}^{n} \omega_i(x) \xi_i(E), \omega_i \in H^1(\Omega), \ i = 1, 2, \ldots, n \right\}.$$

Clearly, with respect to the topology of $L_2((0, \infty); H^1(\Omega))$, $W_n$ is closed in $\infty); H^1(\Omega))$, and is thus a Hilbert space.

We seek a solution $\Psi$ in $W_n$ of the form

$$\Psi(x, E) = \sum_{i=1}^{n} \omega_i(x) \xi_i(E).$$

The weights $\omega_i(x)$ are determined by substituting (3.2) into the following analog of (2.7). Find a $\Psi \in W_n$, such that

$$B(\Psi, \xi) = (S, \xi) + (\gamma, \xi)_{\partial \Omega}, \text{ for all } \xi \in W_n.$$

**Theorem 5.** The approximate problem (3.3) has a unique solution $\Psi \in W_n$.

**Proof.** Since $W_n$ is a subspace of $L_2((0, \infty); H^1(\Omega))$, we can follow the steps proof of Theorem 3 to get the existence and uniqueness of solutions of the approximate problem.

To get another proof of uniqueness, we reformulate (3.3) as

$$((A - K)\Psi, \xi) = (S, \xi) + (\gamma, \xi)_{\partial \Omega}, \text{ for all } \xi \in W_n,$$

where the operator $A$ is given by (2.9). It is easy to see, from Lemma 1, that operator $A^{-1}K$ will indeed be a compact operator on $L_2((0, \infty); H^1(\Omega))$.

We now show that the spectral synthesis approximations for $B = A - K$ is a projection method in the sense of Witsch (1977). Indeed, from (2.10), we have is $L_2((0, \infty); H^1(\Omega))$-elliptic, i.e., there is a positive $C_3$ such that

$$(Au, u) \geq c_3\|u\|_1^2.$$

Hence, not only $A$ is a bounded mapping from $L_2((0, \infty); H^1(\Omega))$ to its dual to each $u \in W_n$, there corresponds a $v \in W_n$, namely $v = u$, such that

$$(Au, u) \geq c_3\|u\|_1^2.$$

Therefore the projection method for $A$ satisfies the criterion
\( u \in W_n, \quad (Au, v) = 0 \quad \text{for all} \quad v \in W_n \quad \text{implies} \quad u = 0. \)

Since \( K \) is a compact operator and \( 1 \) is not in the spectrum of \( K \) relative to \( A \), we have a similar result for \( B = A - K \), i.e.,

\( u \in W_n, \quad (Bu, v) = 0 \quad \text{for all} \quad v \in W_n \quad \text{implies} \quad u = 0. \)

We can use this as an alternative proof of uniqueness. Indeed suppose the problem does not have a unique solution, i.e., we have two solutions \( w_1 \) and \( w_2 \). Let \( u = w_2 \). Then (3.5) applied to the operator \( B \) implies that \( u = 0 \) or \( w_1 = w_2 \), and thus solution is unique.

### 4. Convergence results

In this section we show that under the assumptions in Section 2, the spectra synthesis approximations to the diffusion problem are of the same order as the approximation to the diffusion solution from the subspaces \( W_n \).

**Theorem 6.** Let \( u \) be the solution of (2.7) and \( \phi_n \) be the solution of (3.3). Let \( \hat{\phi}_n \) be the best approximation of the solution \( u \) from the subspaces \( W_n \). Then there is a constant \( M \) such that

\[
\|u - \phi_n\|_1 \leq M \|u - \hat{\phi}_n\|_1 \quad \text{for all} \quad n.
\]

**Proof.** Using (3.5) we can conclude (see Witsch, 1977) that there is a projection operator associated with \( W_n \), which we shall denote by \( \Pi_n \). So our problem becomes equivalent to

\( u_n \in W_n, \quad \Pi_n(Bu_n - S) = 0, \)

and

\( u_n = A^{-1}(P_n S) = Q_n(A^{-1}S) = Q_n u \)

with

\[
P_n = (\Pi_n|_{A(W_n)})^{-1} \Pi_n, \quad Q_n = (\Pi_n A|_{W_n})^{-1} \Pi_n A.
\]

See the diagram in Section 2.1 of Witsch (1977). The projections \( P_n \) are especially adapted for investigations concerning the convergence of the residual \( Au_n - S \) (stability) and the \( Q_n \) are for the convergence of \( u_n \) to \( u \). We apply Theorem 2 Witsch (1977) to conclude that we have consistency and convergence.
The question now is, can we prove the same for the operator $\mathbf{B}$. From The 2.6 of Witsch (1977), we know that the $Q_n$ for (A–K) are defined for sufficiently $n$, and are uniformly bounded. Hence we get that $\phi_n$, given by the spectral syn method, will converge to $u$ in the topology of $L_2((0, \infty); H^1(\Omega))$, and moreover $= Q_n u$. Now to the error estimate:

$$\|u - \phi_n\|_1 \leq \|u - \hat{\phi}_n\|_1 + \|\hat{\phi}_n - \phi_n\|_1 \leq \|u - \hat{\phi}_n\|_1 + \|Q_n(\hat{\phi}_n - u)\|_1 \leq (1 + \|Q_n\|_1)\|u - \hat{\phi}_n\|_1.$$  

The norm of $Q_n$ can be estimated by

$$\|Q_n\| \leq \|Q_n\|_{B^{-1}} \|\mathbf{B}\| \leq \sup_{x \in W_n} \inf_{y \in W_n} \frac{\|y^*\|}{\|y^*\|} \|\mathbf{B}\| \|x\| \leq \sup_{x \in W_n} \frac{\|x\|^2}{c_3 \|x\|^2} \|\mathbf{B}\| \leq \frac{\|\mathbf{B}\|}{c_3}.$$  

Using this estimate in (4.2), we have

$$\|u - \phi_n\|_1 \leq \left(1 + \frac{\|\mathbf{B}\|}{c_3}\right)\|u - \hat{\phi}_n\|_1.$$  

5. The multigroup method

The multigroup method is perhaps the most widely used technique for approximating the energy distribution of particles in a system modeled by transport (Victory, 1985). It is usually assumed that the solutions of the multigroup equations approximate the corresponding solution of the exact transport equations which the energy variable is not discretized. In spite of its wide usage, there seem to be a dearth of work concerned with studies of how the multigroup solution approximate the exact solution and with an evaluation of the sources of the inaccuracy incurred in using the method (Victory, 1985; Victory and Allen, 1985). One of the authors (Victory, 1985) has discussed the convergence of the multigroup approximation for multidimensional media. Here we are only interested in the synthesis approximation.

In the multigroup method, the flux is assumed to be separable into a product of known function dependent on energy and position and a function dependent on position and direction over each of the specified energy intervals. This approach leads to a set of coupled multigroup transport equations that describe the behavior of the multigroup fluxes as functions of position and direction (see Allen et al 1989). In this case the operator $\mathbf{K}$ is
\[ K \phi = \int_{E_0}^{E_{\text{max}}} k(x, E, E') \Sigma(x, E') \phi(x, E') dE'. \]

Let \( \Pi_G \) be a partition of \([E_0, E_{\text{max}}] \subset (0, \infty) \) specified by
\[ 0 = E_0 < E_1 < \ldots < E_G = E_{\text{max}}, \]
with \( E_0 \) and \( E_{\text{max}} \), respectively, denoting the minimum and maximum energy attainable by a particle, and \( G \) is the number of energy groups. Let \( \{\psi_g, g = 1, 2, \ldots, G\} \) be a set of functions in \( L_2((0, \infty)) \) satisfying
\[
\psi_g(E) = \begin{cases} 
1 & E \in [E_{g-1}, E_g), \\
0 & \text{otherwise}, 
\end{cases}
\quad g = 1, 2, \ldots, G. \tag{5} \]

Let \( W_G \) be the space of functions defined by
\[ W_G = \{ \psi \in L_2((0, \infty); H^1(\Omega)); \psi(x, E) = \psi_g(x, E)\psi_g(E), \quad 1 \leq g \leq G \}. \tag{5} \]

We now define our multigroup solution \( \eta(x, E) \in W_G \) and require that \( \eta(x, E) \) satisfies the following system of \( G \) equations for all \( \psi \in W_G \):
\[
\int_{E_0}^{E_G} \int_{\Omega} \left\{ -\nabla \cdot [D(x, E)\nabla \eta(x, E)] + \Sigma(x, E)\eta(x, E) - K\eta \right\} w_g(x, E)\psi_g(E) dxdE \]
\[ = \int_{E_0}^{E_G} \int_{\Omega} S(x, E)w_g(x, E)\psi_g(E) dxdE, \quad 1 \leq g \leq G. \tag{5} \]

As a result of (5.1), we can rewrite the above system as follows:
\[
\int_{E_{g-1}}^{E_G} \int_{\Omega} \left\{ -\nabla \cdot [D(x, E)\nabla \eta(x, E)] + \Sigma(x, E)\eta(x, E) - K\eta \right\} w_g(x, E)\psi_g(E) dxdE \]
\[ = \int_{E_{g-1}}^{E_G} \int_{\Omega} S(x, E)w_g(x, E)\psi_g(E) dxdE, \quad 1 \leq g \leq G. \tag{5} \]

It is clear that (5.4) is equivalent to the system
\[ B(\eta, \psi_g) = (S, \psi_g) + (\gamma, \psi_g)_{\partial \Omega}, \quad g = 1, 2, \ldots, G, \tag{5} \]
where now the integration is on a finite energy interval. For our spectral synthesis method we seek a solution \( \Psi(x, E) \in W_{n,G} \subset W_G \) of the form
\[
\sum_{i=1}^{n} \sum_{g=1}^{G} w_{ig}(x)\psi_{ig}(E), \tag{5} \]
where we let

\[ W_{n,G} = \left\{ \psi \in L_2((0, \infty); H^1(\Omega)); \psi(x, E) = \sum_{i=1}^n \sum_{g=1}^G w_{ig}(x) \psi_{ig}(E) \right\}. \]

The \( w_{ig}(x) \) are determined by substituting (5.6) into (2.1) and then projecting the space spanned by the functions \( \psi_{ig}(E) \). Thus \( \Psi \) must satisfy

\[ B(\psi, \psi_g) = (S, \psi_g) + (\gamma, \psi_g)_{\partial\Omega}, \quad \text{for all } \psi_g \in W_{n,G} \text{ and } g = 1, 2, \ldots, G. \]

Note the similarity between (5.5), (5.8) and (2.7), (3.3) respectively. The observation is that the energy integration is on a finite interval. Therefore one obtains results similar to Theorems 5 and 6.

**Theorem 7.** The approximate system (5.8) has a unique solution in \( W_{n,G} \).

**Theorem 8.** Let \( \eta \) be the solution of (5.5) and \( \phi \) be the solution of (5.8). Let \( \hat{\phi} \) be the best approximation of the solution \( \eta \) from the subspaces \( W_{n,G} \). Then there is a constant \( M \) such that

\[ \| \eta - \phi \|_1 \leq M \| \eta - \hat{\phi} \|_1 \text{ for all } n. \]

**Appendix A**

**Proof of Lemma 1.** Condition (2.5c) assures us that the mapping \( K \) will be a bounded mapping of \( L_2((0, \infty); H^1(\Omega)) \) to \( L_2(\Omega \times (0, \infty)) \). From the Fréchet–Kolmogorov theorem (Krasnosel'skii and Rutickii, 1961, p. 97), we see that we must show, \( u \in B \equiv \{ u \in L_2((0, \infty); H^1(\Omega)) : \| u \|_1 \leq 1 \}, \)

(a) \( \sup_{B} \int_0^\infty \int_\Omega |Ku(y, E)|^2 dE dy < \infty; \)

(b) \( \int_\Omega \int_0^\infty |K\tilde{u}(y + x, E + t) - K\tilde{u}(y, E)|^2 dE dy \to 0, \)

uniformly for \( u \in B \) as \( |x| + |t| \to 0 \), where \( \tilde{u} \) is the extension of \( u \) to all of which has the value zero outside the set \( \Omega \times (0, \infty) \), and

(c) \( \lim_{a \to \infty} \int_\Omega \int_a^\infty |Ku(y, E)|^2 dE dy = 0, \)
uniformly for \( u \in \mathcal{B} \).

To prove (a), we use the Schwarz inequality, and obtain

\[
\int_{\Omega} \int_{0}^{\infty} \int_{0}^{\infty} k(y, E, E') \Sigma(y, E') u(y, E') dE' dE dy \leq \sup_{\Omega} \left( \int_{0}^{\infty} k^2(y, E, E') \Sigma^2(y, E') dE' \left( \int_{0}^{\infty} u^2(y, E') dE' \right) \right) dE dy \leq \sup_{\Omega} \left( \int_{0}^{\infty} k^2(y, E, E') \Sigma^2(y, E') dE' dE' dE \right) \| u \|_{1}^2.
\]

In order to prove (b), we must extend \( u \in L_2((0, \infty); H^1(\Omega)) \) to a function belonging to \( L_2((0, \infty); H^1(\mathcal{R}^n)) \). We say that \( \Gamma \) is an extension operator with respect to \( \Omega \) provided there exists a constant \( \nu \) such that for every \( u \in L_2((0, \infty); H^1(\Omega)) \),

(i) \( \Gamma u(x, E) = u(x, E), \quad x \in \Omega; \)

(ii) \( \| \Gamma u \|_{1, \mathcal{R}^n} \leq \nu \| u \|_{1} \)

where \( \| \|_{1, \mathcal{R}^n} \) is the norm in \( L_2((0, \infty); H^1(\mathcal{R}^n)) \) given by (2.4f). Also, we note that results in (Adams, 1975, pp. 84–94) concerning the extension operators for Sobolev spaces of real- or complex-valued functions, with domain \( \Omega \subset \mathcal{R}^n \) possessing uniform cone property (Adams, 1975, pp. 65–70), can be modified to produce existence of extension operators for Sobolev spaces of Banach space- or Hilbert space-valued functions defined on certain domains. In particular, the Calderón Extension theorem (Adams, 1975, pp. 91–94) can be generalized in a straightforward manner to prove the existence of extension operators with respect to domains \( \mathcal{R}^n \) for functions in \( L^2((0, \infty); H^{p,m}(\Omega)) \), where

\[
H^{p,m}(\Omega) = \{ f \in L_p(\Omega) : f^{(i)} \in L_p(\Omega), \quad i = 0, 1, \ldots, m \}.
\]

More precisely, we have

The Calderón Extension theorem. let \( \Omega \) be a domain in \( \mathcal{R}^n \) having the uniform cone property (Adams, 1975, p. 66) modified as follows:

1. The open cover \( \{ U_j \} \) of the boundary of \( \Omega \) is required to be finite, and
2. The sets \( U_j \) are not required to be bounded.

Then for any \( m \), and any \( p \), \( 1 < p < \infty \), there exists an extension operator \( \Gamma = \Gamma_p \) with respect to \( \Omega \) for functions in \( L_2((0, \infty); H^{p,m}(\Omega)) \).

Henceforth we shall let \( \tilde{u}(x, E) = \Gamma u(x, E), \quad (x, E) \in \mathcal{R}^n \times (0, \infty) \). To show (b), have with \( k_1(x, E, E') = k(x, E, E') \Sigma(x, E') \),
\[ \int_{\Omega} \int_{0}^{\infty} \left| K_u(x + y, E + t) - K_u(y, E) \right|^2 dE dy \leq \\
2 \int_{\Omega} \int_{0}^{\infty} \left\{ \int_{0}^{\infty} \left| k_1(x + y, E + t, E') u(y, E') dE' \right|^2 dE dy \right\} \\
2 \int_{\Omega} \left\{ \left\{ \int_{0}^{\infty} \left| k_1(x + y, E + t, E') dE' \right|^2 dE \right\} \times \\
\left\{ \int_{0}^{\infty} \left| \tilde{u}(x + y, E') - u(u, E') \right|^2 dE' \right\} dy + \\
2 \int_{\Omega} \left\{ \left\{ \int_{0}^{\infty} \left| k_1(x + y, E + t, E') - k_1(y, E, E') \right|^2 dE' \right\} \times \\
\left\{ \int_{0}^{\infty} \left| \tilde{u}(y, E') \right|^2 dE' \right\} dE dy. \]

To ascertain that

\[ \int_{\Omega} \int_{0}^{\infty} \left| \tilde{u}(x + y, E') - u(u, E') \right|^2 dE' dy \to 0 \]

uniformly in \( u \in B \), as \( |x| \to 0 \), we estimate this integral. If \( u \in L_2((0, \infty); C_0^\infty \mathcal{R}^n) \) can deduce (Adams, 1975, p. 186), by virtue of the absolute continuity of \( \mathcal{I} \) mapping of \( \mathcal{R}^n \) to \( L^2((0, \infty)) \):

\[ \int_{\Omega} \int_{0}^{\infty} \left| u(x + y, E') - u(y, E') \right|^2 dE' dy = \\
\int_{\Omega} \int_{0}^{\infty} \left[ \int_{0}^{1} \frac{\partial}{\partial t} u(y + tx, E') dt \right]^2 dE' dy = \\
\int_{\Omega} \int_{0}^{\infty} \left[ \int_{0}^{1} \frac{\partial}{\partial y} u(y + tx, E') dt \right]^2 dE' dy \leq \\
\int_{\Omega} \int_{0}^{\infty} \left[ \int_{0}^{1} \frac{\partial}{\partial y} u(y + tx, E') dt \right]^2 dE' dy \leq \\
|\mathcal{I}| \int_{0}^{1} dt \int_{\mathcal{R}^n} dy \int_{0}^{\infty} \left| \frac{\partial}{\partial y} u(y, E') \right|^2 dE' \leq |\mathcal{I}| ||u||^2_{1, \mathcal{R}^n}. 

Since \( L_2((0, \infty); C_0^\infty \mathcal{R}^n) \) is dense in \( L_2((0, \infty); H^1(\mathcal{R}^n)) \), we have the inequality holding for any \( u \in L_2((0, \infty); H^1(\mathcal{R}^n)) \) so, for \( |x| \to 0 \) and \( u \in \mathcal{I} \) deduce by virtue of the Calderón Extension theorem.
\[
\int_{\Omega} \int_{0}^{\infty} |\tilde{u}(x + y, E') - u(y, E')|^2 dE' dy \to 0
\]

uniformly as |x| \to 0.

Finally, (c) follows from the inequalities

\[
\int_{\Omega} \int_{0}^{\infty} \int_{0}^{\infty} k(x, E, E') \Sigma(x, E') u(x, E') dE' dx \leq \int_{\Omega} \int_{0}^{\infty} \int_{0}^{\infty} k^2(x, E, E') \Sigma^2(x, E') dE' dx \leq \left( \sup_{\Omega} \int_{0}^{\infty} k^2(x, E, E') \Sigma^2(x, E') dE' \right) \|u\|^2.
\]

Thus we can deduce that the mapping K is compact.

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