Journal of Computational and Applied Mathematics 10 (1984) 33–38
North-Holland

Families of methods for ordinary differential equations based on trigonometric polynomials

B. NETA and C.H. FORD
Department of Mathematics, Texas Tech University, Lubbock, TX 79409, U.S.A.

Received 8 October 1982
Revised 25 September 1983

Abstract: We consider the construction of methods based on trigonometric polynomials for the initial value problems whose solutions are known to be periodic. It is assumed that the frequency w can be estimated in advance. The resulting methods depend on a parameter $\nu = hw$, where h is the step size, and reduce to classical multistep methods if $\nu \to 0$. Gautschi [4] developed Adams and Störmer type methods. In our paper we construct Nyström's and Milne–Simpson's type methods. Numerical experiments show that these methods are not sensitive to changes in w, but require the Jacobian matrix to have purely imaginary eigenvalues.

Keywords: Periodic initial value problems, linear multistep methods.

1. Introduction

There are few numerical methods available for the solution of initial value problems which take advantage of special properties of the solution. For example, Brock and Murray [2], and Dennis [3] developed methods for exponential type solutions. Urabe and Mise [8] designed a method for solutions in whose Taylor expansion the most significant terms are of relatively high order. Gautschi [4] constructed methods of the Adams and Störmer type for problems with oscillatory solutions whose frequency is known.

In this paper we consider the initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0, \tag{1}$$

whose solution is known to oscillate with a known frequency. We construct k-step Nyström (explicit) and generalized Milne–Simpson (implicit) methods. We numerically compare these methods to Adams' methods constructed by Gautschi [4]. Our methods are restricted to problems whose Jacobian matrix have purely imaginary eigenvalues.

Let us recall some definitions and notations, see for example [5]. Let $C^s[a, b]$ $(s \geq 0)$ denote the linear space of functions $y(x)$ having s continuous derivatives in the finite closed interval $[a, b]$. We assume the space is normed by

$$\|y\| = \sum_{i=0}^{s} \max_{a \leq x \leq b} |y^{(i)}(x)|. \tag{2}$$

Definition 1. A linear functional $\mathscr{L}$ in $C^s[a, b]$ is said to be of *algebraic order* p, if

$$\mathscr{L}x^r \equiv 0, \quad r = 0, 1, \ldots, p, \qquad \mathscr{L}x^{p+1} \not\equiv 0. \tag{3}$$

Definition 2. The method

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j(\nu) f_{n+j},$$
$$\nu = wh, \ \alpha_k = +1, \tag{4}$$

is said to be of *trigonometric order* q *relative to the frequency* w if the associated linear difference operator

$$\mathscr{L}y(x) = \sum_{j=0}^{k} \left[\alpha_j y(x + jh) - h\beta_j(\nu) y'(x + jh)\right] \tag{5}$$

satisfies

$$\mathscr{L}1 \equiv 0,$$
$$\mathscr{L}\cos rwx \equiv \mathscr{L}\sin rwx \equiv 0, \quad r = 1, 2, \ldots, q, \tag{6}$$

$\mathscr{L}\cos((q+1)wx)$ and $\mathscr{L}\sin((q+1)wx)$ not both identically zero.

Definition 3. The method (4) is called *explicit* if $\beta_k(\nu) \equiv 0$, otherwise it is *implicit*. An explicit method for which

$$\alpha_k = +1, \qquad \alpha_{k-2} = -1,$$
$$\alpha_j = 0, \quad j = 0, 1, \ldots, k-3, k-1, \tag{7}$$

is called a *Nyström* method. An implicit method satisfying (7) is called a *generalized Milne–Simpson* method.

2. Construction of methods

The methods we construct are in the form

$$y_{n+k}-y_{n+k-2}=h\sum_{j=0}^{l}\beta_j(\nu)y'_{n+j} \tag{8}$$

where $l=k$ for implicit methods and $l=k-1$ for explicit ones. The operator l is then defined by

$$\mathscr{L}y(x)=y(x+kh)-y(x+(k-2)h)-h\sum_{j=0}^{l}\beta_j(\nu)y'(x+jh). \tag{9}$$

Clearly, this satisfies the first relation of (6). The other two equations in (6) imply

$$\cos r\nu k-\cos r\nu(k-2)+r\nu\sum_{j=1}^{l}\beta_j(\nu)\sin r\nu j=0, \tag{9a}$$

$$-\sin r\nu k+\sin r\nu(k-2)+r\nu\sum_{j=0}^{l}\beta_j(\nu)\cos r\nu j=0 \tag{9b}$$

for $r=1, 2,\dots,q$. These equations can be rewritten in the form

$$2\sin r\nu\sin(r(k-1)\nu)-r\nu\sum_{j=1}^{l}\beta_j(\nu)\sin r\nu j=0, \tag{10a}$$

$$2\sin r\nu\cos(r(k-1)\nu)-r\nu\sum_{j=0}^{l}\beta_j(\nu)\cos r\nu j=0. \tag{10b}$$

Note that the sum in (9a) and (10a) starts from $j=1$.

2.1. Explicit methods

(i) $k=2$. The parameters $\beta_0(\nu)$, $\beta_1(\nu)$ can be calculated from (9) with $r=1$

$$\beta_0=0,\quad \beta_1=\frac{2\sin\nu}{\nu} \tag{11}$$

and the method

$$y_{n+2}-y_n=\frac{2\sin\nu}{\nu}hf_{n+1}. \tag{12}$$

This is a method of trigonometric order $q=1$. The local truncation error is given by

$$\tfrac{1}{3}\left(w^2y'(x_{n+1})+y'''(x_{n+1})\right)h^3+O(h^4). \tag{13}$$

Thus the method is of algebraic order $p=2$.

(ii) $k=3$. In this case we have three parameters $\beta_0(\nu)$, $\beta_1(\nu)$ and $\beta_2(\nu)$. Using (10a) and (10b) with $r=1$, one obtains a system of two equations

$$2\sin\nu\sin 2\nu-\nu\sum_{j=1}^{2}\beta_j(\nu)\sin\nu j=0,$$

$$2\sin\nu\cos 2\nu-\nu\sum_{j=0}^{2}\beta_j(\nu)\cos\nu j=0.$$

The solution of this system is

$$\beta_1(\nu)=-2\beta_0(\nu)\cos\nu,\qquad \beta_2(\nu)=\beta_0(\nu)+\frac{2\sin\nu}{\nu}. \tag{14}$$

Thus, one has the following family of methods:

$$y_{n+3}-y_{n+1}=h\beta_0\left[y'_n-2(\cos\nu)y'_{n+1}+y'_{n+2}\right]+\frac{2h\sin\nu}{\nu}y'_{n+2}. \tag{15}$$

One can choose β_0, so as to increase the algebraic order. It can easily be shown that the local truncation error is

$$-\left(\beta_0-\tfrac{1}{3}\right)h^3\left(y_n'''+w^2y_n'\right)+\left(\tfrac{2}{3}-\beta_0\right)h^4\left(y_n^{(4)}+w^2y_n''\right)+O(h^5). \tag{16}$$

If one chooses $\beta_0=\frac{1}{3}$, then the method (15) will be of algebraic order 3. In that case (15) becomes

$$y_{n+3}-y_{n+1}=h\left[\tfrac{1}{3}y'_n-\tfrac{2}{3}(\cos\nu)y'_{n+1}+\left(\tfrac{1}{3}+\frac{2\sin\nu}{\nu}\right)y'_{n+2}\right]. \tag{17}$$

(iii) $k=4$. The 4 parameters are computed from (10) with $l=3$ and $r=1, 2$. This was done by MACSYMA (Project MAC's SYmbolic MAnipulation system written in LISP and used for performing symbolic as well as numerical mathematical manipulations [1]).

$$\beta_0=-\sin\nu/D, \tag{18a}$$

$$\beta_1=-2\sin\nu(1-2\cos\nu)(1+\cos\nu)/D, \tag{18b}$$

$$\beta_2=-\sin\nu(4\cos\nu\cos 2\nu+1)/D, \tag{18c}$$

$$\beta_3=2\sin 2\nu(1+\cos\nu)/D, \tag{18d}$$

where

$$D = \nu(1 + 2\cos\nu). \tag{18e}$$

2.2. Implicit methods

(i) $k = 2$. The three parameters are computed from (10) with $l = 2$, $r = 1$. One obtains a family of methods with

$$\beta_1 = \frac{2\sin\nu}{\nu} - 2\beta_0\cos\nu,$$

$$\beta_2 = \beta_0. \tag{19}$$

The family of methods of trigonometric order 1 is

$$y_{n+2} - y_n = h\beta_0\left(y_n' - 2(\cos\nu)y_{n+1}' + y_{n+2}'\right) + \frac{2h\sin\nu}{\nu}y_{n+1}'. \tag{20}$$

One can choose β_0 so as to increase the order. The local truncation error is

$$\begin{aligned}&\left(\tfrac{1}{3} - \beta_0\right)\left[y''' + w^2y'\right]h^3 \\ &+\left(\tfrac{1}{3} - \beta_0\right)\left[y^{(4)} + w^2y''\right]h^4 \\ &+\left[\left(\tfrac{4}{5} - \tfrac{2}{3}\beta_0\right)y^{(5)} + \left(\tfrac{1}{12}\beta_0 - \tfrac{1}{60}\right)w^4y' \right.\\ &\left.+\left(\tfrac{1}{6} - \tfrac{1}{2}\beta_0\right)w^2y'''\right]h^5 + O(h^6).\end{aligned} \tag{21}$$

If one chooses $\beta_0 = \frac{1}{3}$ the method obtained is

$$y_{n+2} - y_n = = \tfrac{1}{3}h\left[y_n' - 2\left(\cos\nu - \frac{3\sin\nu}{\nu}\right)y_{n+1}' + y_{n+2}'\right]. \tag{22}$$

This method is of algebraic order 4. (Note that the explicit method with the same step number $k = 2$ is only of algebraic order 2.)

(ii) $k = 3$. The four parameters are computed by MACSYMA.

$$\beta_0 = 0, \tag{23a}$$

$$\beta_1 = \sin\nu/D, \tag{23b}$$

$$\beta_2 = 2\sin\nu(1 + \cos\nu)/D, \tag{23c}$$

$$\beta_3 = \beta_1, \tag{23d}$$

where

$$D = \nu(1 + 2\cos\nu). \tag{23e}$$

The method is

$$y_{n+3} - y_{n+1} = h\left(\beta_1 y_{n+1}' + \beta_2 y_{n+2}' + \beta_1 y_{n+3}'\right). \tag{24}$$

It is of trigonometric order 2. Note that by shifting the index one has an implicit 2-step method of trigonometric order 2.

(iii) $k = 4$. In this case one obtains a family of methods. We were not able to reduce the expressions for β_i, $i \neq 1$ (because of disk space). The expression for β_1 in terms of the parameter β_4 is:

$$\beta_1 = \left(\sin\nu/\nu\left(\tfrac{1}{2} + \cos\nu\right) - 2\beta_4\right) \times (1 + \cos\nu)(2\cos\nu - 1). \tag{25}$$

With this expression for β_1 one can solve a system of two equations for the two unknowns β_2, β_3 in terms of β_4. This system is (10a) for $r = 1, 2$. Once this is done the expression for β_0 can be obtained from (10b) with $r = 1$.

3. Numerical experiments

In our numerical experiments we approximated the solution of the following system of initial value problems:

$$\boldsymbol{Y}'(t) = \boldsymbol{F}(t, \boldsymbol{Y}), \tag{26}$$

$$\boldsymbol{Y}(0) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \tag{27}$$

where

$$\boldsymbol{Y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \tag{28}$$

and

$$\boldsymbol{F} = \begin{bmatrix} y_2 \\ -y_1/r^3 \\ y_4 \\ -y_3/r^3 \end{bmatrix} \tag{29}$$

and

$$r^2 = y_1^2 + y_3^2. \tag{30}$$

The exact solution is

$$\boldsymbol{Y}_e = \begin{bmatrix} \sin t \\ \cos t \\ \cos t \\ -\sin t \end{bmatrix}. \tag{31}$$

Clearly, $w = 1$. We have computed the solution at $t = 12\pi$ using various values of w and various methods as given here and in [4]. In Table 1 we have compared the L_2 norm of the error at $t = 12\pi$

Table 1
$q=1$

w	Adams		Nyström	Generalized Milne–Simpson
	Explicit	Implicit		
0.90	0.287 − 1	0.231 − 2	0.460 − 2	0.230 − 5
0.95	0.148 − 1	0.119 − 2	0.236 − 2	0.124 − 5
1.00	0.975 − 11	0.189 − 12	0.256 − 11	0.262 − 10
1.05	0.155 − 1	0.125 − 2	0.248 − 2	0.144 − 5
1.10	0.318 − 1	0.256 − 2	0.508 − 2	0.310 − 5

using $h=\pi/60$ and various values of w. Note that the explicit Adams method ($q=1$) didn't give as good results as Nyström's ($q=1$). Note also that the generalized Milne–Simpson gave 3 digits of accuracy more than Adams implicit. As one should expect all methods gave the correct answer for $w=1$.

In Table 2 we compared Adams implicit methods of trigonometric orders 2 and 3 with generalized Milne–Simpson methods of the same order. Note that for Adams methods we used the coefficients as given in [4], i.e. the Taylor series expansion. Our second order method shows slightly better results than Adams implicit second order. The insensitivity to overestimation or underestimation of w is demonstrated in both methods. This is an excellent feature since w is not known exactly beforehand. The results are summarized in Table 2.

In our next experiment, we consider the following 'almost periodic' problem studied by Stiefel and Bettis [7]:

$$z''+z=0.001\ \mathrm{e}^{\mathrm{i}t},\quad \mathrm{i}=\sqrt{-1},\ 0\leqslant t\leqslant 40\pi, \tag{32}$$

$$z(0)=1, \tag{33}$$

$$z'(0)=0.9995\mathrm{i}, \tag{34}$$

whose theoretical solution is

$$z(t)=\cos t+0.0005t\sin t + \mathrm{i}(\sin t-0.0005t\cos t). \tag{35}$$

The solution represents motion on a perturbation of a circular orbit in the complex plane; the point $z(t)$ spirals slowly outwards. We write the equations in the equivalent form

$$y_1'-y_2=0,\qquad y_2'+y_1=0.001\cos t,$$

$$y_3'-y_4=0,\qquad y_4'+y_3=0.001\sin t, \tag{36}$$

$$y_1(0)=1,\qquad y_2(0)=0,$$

$$y_3(0)=0,\qquad y_4(0)=0.9995. \tag{37}$$

The exact solution of this system is

$$\begin{aligned} y_1(t)&=\cos t+0.0005t\sin t,\\ y_2(t)&=-0.9995\sin t+0.0005t\cos t,\\ y_3(t)&=\sin t-0.0005t\cos t,\\ y_4(t)&=0.9995\cos t+0.0005t\sin t. \end{aligned} \tag{38}$$

This first order system was solved numerically for $0\leqslant t\leqslant 40\pi$ (which corresponds to 20 orbits) using methods of trigonometric orders 2 and 3. The results for $h=\pi/60$ are presented in Table 3. The insensitivity to changes in w is manifested again in this example. Note that since we haven't obtained the parameters $\beta_i(\nu)$ for a third order method we solve the system (10) numerically for given w and h using Gaussian elimination with partial pivoting). For each w and h we have to solve a system of six equations for the six parameters $\beta_i(\nu)$. All numerical experiments were performed in double precision on IBM 3033 computer.

Table 2

w	Adams implicit		Generalized Milne–Simpson	
	$q=2$	$q=3$	$q=2$	$q=3$
0.90	0.166 − 4	0.311 − 5	0.285 − 5	0.298 − 5
0.95	0.100 − 4	0.369 − 5	0.169 − 5	0.201 − 5
1.00	0.220 − 6	0.435 − 5	0.239 − 10	0.119 − 11
1.05	0.132 − 4	0.510 − 5	0.232 − 5	0.344 − 5
1.10	0.308 − 4	0.594 − 5	0.535 − 5	0.878 − 5

Table 3

w	Adams implicit		Generalized Milne–Simpson	
	$q=2$	$q=3$	$q=2$	$q=3$
0.90	0.842 − 3	0.108 − 5	0.134 − 3	0.446 − 7
0.95	0.841 − 3	0.129 − 5	0.129 − 3	0.295 − 7
1.00	0.839 − 3	0.153 − 5	0.115 − 3	0.103 − 7
1.05	0.837 − 3	0.181 − 5	0.133 − 3	0.612 − 7
1.10	0.835 − 3	0.211 − 5	0.120 − 3	0.148 − 6

In Table 4 we present a comparison of results obtained by the generalized Milne–Simpson method ($q=3$) with a sixth order symmetric method developed by Lambert and Watson [6] for the problem (32)–(34).

In our next experiment we solved the following differential equation:

$$y''' + \lambda y'' + y' + \lambda y = 0, \quad 0 \leqslant t \leqslant 12\pi, \tag{39}$$

whose exact solution is

$$y(x) = c_1 \cos x + c_2 \sin x + c_3 \,\mathrm{e}^{-\lambda x}. \tag{40}$$

In order to get a small perturbation to the periodic solution we choose the initial values

$$y(0) = y'(0) = 1 + 10^{-10}, \qquad y''(0) = -1 + 10^{-10}. \tag{41}$$

Therefore the constants in (40) are

$$\begin{aligned} c_1 &= 1 + 10^{-10} - 2 \cdot 10^{-10}/(1+\lambda^2), \\ c_2 &= 1 + 10^{-10} + 2 \cdot 10^{-10}\lambda/(1+\lambda^2), \\ c_3 &= 2 \cdot 10^{-10}/(1+\lambda^2). \end{aligned} \tag{42}$$

In Table 5 we compare results obtained by Adams methods of trigonometric orders 1 and 2 with the Nyström method of order 1 and generalized Milne–Simpson of order 2 for various values of λ and $h = \pi/60$, $w = 1$.

Note that the explicit Adams formula integrates stable for sufficiently small λ, whereas the (explicit) Nyström method tends to develop increasing instabilities as λ increases. The reason is that the explicit Adams method possesses a non-empty stability region in the left-half plane, whereas the Nyström method is only stable on a part of the

Table 4

h	Lambert and Watson order = 6	Generalized Milne–Simpson $q=3$, $w=1$
$\pi/5$	0.007300	0.001406
$\pi/6$	0.002303	0.000623
$\pi/12$	0.000033	0.000008

Table 5
$h = \pi/60$, $w = 1$

λ	Adams		Nyström $q=1$	Generalized Milne–Simpson $q=1$
	Explicit $q=1$	Implicit $q=1$		
0	0.17 − 9	0.17 − 9	0.17 − 9	0.17 − 9
0.1	0.18 − 9		0.19 − 9	0.18 − 9
0.2	0.19 − 9		0.20 − 9	0.20 − 9
0.5	0.20 − 9		0.36 − 3	0.14 − 8
1.0	0.20 − 9	0.20 − 9	unstable	unstable
5.0	0.18 − 9			
10.0	0.17 − 9			
17.5	0.17 − 9			
20.0	unstable	0.18 − 9 unconditionally stable		

Table 6

t	Absolute error	
	Störmer explicit $q=2$	Generalized Milne–Simpson $q=2$
2	0.0000030	0.0000552
3	0.0000029	0.0002530
4	0.0000011	0.0003808
5	0.0000013	0.0004067
6	0.0000038	0.0003311
7	0.0000052	0.0001860
8	0.0000053	0.0000248
9	0.0000038	0.0000947

imaginary axis. The behavior demonstrated in the table can now be explained by the fact that $-\lambda$ is an eigenvalue of the Jacobian matrix. This example is due to P.J. van der Houwen and B.P. Sommeijer.

In our last experiment we compare the generalized Milne–Simpson of trigonometric order 2 with Störmer's method of the same order for solving the equation

$$y''(t)+\left(100+\frac{1}{4t^2}\right)y(t)=0, \quad 1 \leqslant t \leqslant 9. \tag{43}$$

We choose the initial values so that the exact solution in terms of Bessel functions is

$$y(t)=\sqrt{t}\,J_0(10t). \tag{44}$$

We let $h=0.02$ and $w=10$ as in [4]. In Table 6 we present the error at every 50th point using the above mentioned two methods.

It is clear that Störmer's method is better. It is usually recommended to use direct methods for $y''=f(t, y)$ rather than apply methods to an equivalent first order system [5, p. 253].

Acknowledgment

The authors would like to thank Prof. George J. Fix for the invitation to Carnegie Mellon University and the use of the computer. We would like to thank the referees, Professors P.J. van der Houwen and B.P. Sommeijer, for their valuable comments which improved the paper.

References

[1] R. Bogen et al., *MACSYMA Reference Manual* (MIT Press, Cambridge, MA, 1975).

[2] P. Brock and F.J. Murray, The use of exponential sums in step by step integration, *Math. Tables Aids Comput.* **6** (1952) 63–78.

[3] S.C.R. Dennis, The numerical integration of ordinary differential equations possessing exponential type solutions, *Proc. Cambridge Phil. Soc.* **65** (1960) 240–246.

[4] W. Gautschi, Numerical integration of ordinary differential equations based on trigonometric polynomials, *Numer. Math.* **3** (1961) 381–397.

[5] J.D. Lambert, *Computational Methods in Ordinary Differential Equations* (Wiley, London, 1977).

[6] J.D. Lambert, and I.A. Watson, Symmetric multistep methods for periodic initial value problems, *J. Inst. Math. Appl.* **18** (1976) 189–202.

[7] E. Stiefel and D.G. Bettis, Stabilization of Cowell's method, *Numer. Math.* **13** (1969) 154–175.

[8] M. Urabe and S. Mise, A method of numerical integration of analytic differential equations, *J. Sci. Hiroshima Univ. Ser. A* **19** (1955) 307–320.