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A new sixth-order scheme for nonlinear equations

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ABSTRACT

In this paper we present a new efficient sixth-order scheme for nonlinear equations. The method is compared to several members of the family of methods developed by Neta (1979) [B. Neta, A sixth-order family of methods for nonlinear equations, Int. J. Comput. Math. 7 (1979) 157–161]. It is shown that the new method is an improvement over this well known scheme.

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1. Introduction

Solving nonlinear equations is one of the most important problems in numerical analysis. To solve nonlinear equations, iterative methods such as Newton's method are usually used. Throughout this paper we consider iterative methods to find a simple root ξ, i.e., $f(\xi) = 0$ and $f'(\xi) \neq 0$, of a nonlinear equation $f(x) = 0$, where $f : I \subset \mathbb{R} \to \mathbb{R}$ for an open interval I.

Newton's method for the calculation of ξ is probably the most widely used iterative scheme defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

It is well known (see e.g. [1]) that this method is quadratically convergent.

Some modifications of Newton's method to achieve higher order and better efficiency have been suggested and analyzed in the literature. See e.g. the books by Ostrowski [2], Traub [1] and Neta [3]. See also more recent results by Kim [4] who discussed a wide collection of sixth-order methods, Soleymani [5] and Khattri and Argyros [6]. This last paper gives a family of three step methods free from derivatives.

Most of the methods improve the order of convergence and computational efficiency of Newton's method with an additional evaluation of the function or its derivatives. To be more precise, we define informational efficiency E by

$$E = \frac{p}{d}$$

where p is the order of the method and d is the number of function- (and derivative-) evaluations per step. We also mention another measure, the efficiency index I

$$I = p^{1/d}.$$

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Here we compare the sixth-order family of methods [7] given by

$$\begin{aligned} w_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= w_n - \frac{f(w_n)}{f'(x_n)} \frac{f(x_n) + \beta f(w_n)}{f(x_n) + (\beta - 2) f(w_n)}, \\ x_{n+1} &= z_n - \frac{f(w_n)}{f'(x_n)} \frac{f(x_n) - f(w_n) + \gamma f(z_n)}{f(x_n) - 3f(w_n) + \gamma f(z_n)}, \end{aligned} \tag{2}$$

to our new sixth-order scheme. Both of these methods have the same efficiency index $6^{1/4} \approx 1.565$ which is better than that of Newton's method. This method has an error term

$$\epsilon_{n+1} = c_2 c_3 \left[c_3 - (2\beta + 1) c_2^2\right] \epsilon_n^6 + O(\epsilon_n^7), \tag{3}$$

where $\epsilon_n = x_n - \xi$ and

$$c_i = \frac{f^{(i)}(\xi)}{i! f'(\xi)}, \quad i \geq 1. \tag{4}$$

Note that γ does not appear in the error constant and therefore we can take $\gamma = 0$. The first two substeps constitute King's fourth-order scheme [8]. For the parameter β, we note that if $\beta = 0$ then the first two steps are Ostrowski's fourth-order method [2]. If we choose $\beta = -1$ then the factor in the second and third substeps is identical, and thus we can save on computation. The choice $\beta = -1/2$ minimizes the error term,

$$\epsilon_{n+1} = c_3^2 c_2 \epsilon_n^6 + O(\epsilon_n^7).$$

In the numerical experiments section we will use these three parameters for comparison.

2. Development of method and convergence analysis

We suggest replacing the first two substeps in (2) by the fourth-order method due to Kung and Traub [9]

$$\begin{aligned} w_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= w_n - \frac{f(w_n)}{f'(x_n)} \frac{1}{\left[1 - \frac{f(w_n)}{f(x_n)}\right]^2}, \end{aligned} \tag{5}$$

and then consider the method

$$\begin{aligned} w_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= w_n - \frac{f(w_n)}{f'(x_n)} \frac{1}{\left[1 - \frac{f(w_n)}{f(x_n)}\right]^2}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)} \frac{1}{\left[1 - \frac{f(w_n)}{f(x_n)} - \frac{f(z_n)}{f(x_n)}\right]^2}. \end{aligned} \tag{6}$$

For the method defined by (6), we have the following analysis of convergence.

Theorem 2.1. *Let $\xi \in I$ be a simple zero of a sufficiently differentiable function $f : I \to \mathbb{R}$ for an open interval I. Let $\epsilon_n = x_n - \xi$. Then the new method defined by (6) is of sixth-order. The error at the $(n+1)$ st step, ϵ_{n+1}, satisfies the relation*

$$\epsilon_{n+1} = \left(-5c_3c_2^3 + 6c_2^5 + c_2c_3^2\right) \epsilon_n^6 + O(\epsilon_n^7), \tag{7}$$

where c_i $i = 1, 2, 3$ are given by (4).

Proof. Let $\epsilon_n = x_n - \xi$, $u_n = w_n - \xi$ and $v_n = z_n - \xi$. Using the Taylor expansion of $f(x)$ around $x = \xi$ and taking $f(\xi) = 0$ into account, we get

$$f(x_n) = f'(\xi) \left[\epsilon_n + c_2\epsilon_n^2 + c_3\epsilon_n^3 + c_4\epsilon_n^4 + O(\epsilon_n^5)\right], \tag{8}$$

$$f'(x_n) = f'(\xi) \left[1 + 2c_2\epsilon_n + 3c_3\epsilon_n^2 + 4c_4\epsilon_n^3 + O(\epsilon_n^4)\right]. \tag{9}$$

Dividing (8) by (9) gives

$$u_n = \epsilon_n - \frac{f(x_n)}{f'(x_n)} = c_2\epsilon_n^2 - (-2c_3 + 2c_2^2)\epsilon_n^3 - (-3c_4 + 7c_2c_3 - 4c_2^3)\epsilon_n^4 - (10c_2c_4 + 6c_3^2 - 20c_3c_2^2 + 8c_2^4)\epsilon_n^5 + O(\epsilon_n^6), \tag{10}$$

so that, after elementary calculation,

$$\begin{aligned} f(w_n) &= f'(\xi)[u_n + c_2u_n^2 + c_3u_n^3 + c_4u_n^4 + O(u_n^5)] \\ &= f'(\xi)[c_2\epsilon_n^2 + (2c_3 - 2c_2^2)\epsilon_n^3 + (3c_4 - 7c_2c_3 + 5c_2^3)\epsilon_n^4 + (-12c_2^4 - 6c_3^2 - 10c_2c_4 + 24c_3c_2^2)\epsilon_n^5 + O(\epsilon_n^6)]. \end{aligned} \tag{11}$$

Using (8)–(11), we find

$$\begin{aligned} v_n &= u_n - \frac{f(w_n)}{f'(x_n)}\frac{1}{[1 - f(w_n)/f(x_n)]^2} \\ &= (2c_2^3 - c_2c_3)\epsilon_n^4 + (-2c_3^2 - 10c_2^4 - 2c_2c_4 + 14c_3c_2^2)\epsilon_n^5 + (21c_4c_2^2 - 7c_4c_3 + 30c_2c_3^2 + 31c_2^5 - 72c_3c_2^3)\epsilon_n^6 \\ &\quad + (-100c_4c_2^3 + 88c_4c_2c_3 - 188c_3^2c_2^2 + 246c_3c_2^4 - 6c_4^2 + 20c_3^3 - 74c_2^6)\epsilon_n^7 + O(\epsilon_n^8), \end{aligned} \tag{12}$$

so that, after elementary calculation,

$$\begin{aligned} f(z_n) &= f'(\xi)[v_n + c_2v_n^2 + c_3v_n^3 + O(v_n^4)] \\ &= f'(\xi)\left[(-c_2c_3 + 2c_2^3)\epsilon_n^4 + (-10c_2^4 - 2c_2c_4 - 2c_3^2 + 14c_3c_2^2)\epsilon_n^5 \right. \\ &\quad \left. + (31c_2^5 + 30c_2c_3^2 + 21c_4c_2^2 - 7c_4c_3 - 72c_3c_2^3)\epsilon_n^6 + O(\epsilon_n^7)\right]. \end{aligned} \tag{13}$$

By doing simple calculations with (11)–(13) we obtain

$$\begin{aligned} \epsilon_{n+1} &= v_n - \frac{f(z_n)}{f'(x_n)}\frac{1}{[1 - f(w_n)/f(x_n) - f(z_n)/f(x_n)]^2} \\ &= (-5c_3c_2^3 + 6c_2^5 + c_2c_3^2)\epsilon_n^6 + O(\epsilon_n^7), \end{aligned} \tag{14}$$

which means that the method defined by (6) is at least sixth-order. This completes the proof. □

Remark. Kung and Traub [9] have conjectured that one can get an optimal eighth-order scheme using the same information as our scheme. That is for three function- and one derivative-evaluation one can get an eighth-order method. Therefore our scheme is not optimal, but the computational cost of our additional step is not as high as the optimal eighth-order obtained by interpolation.

3. Numerical examples

In this section we present some numerical experiments using our new method and compare these results to the three members of Neta's family of schemes. All computations were done using MAPLE using 128 digit floating point arithmetics (Digits := 128). We accept an approximate solution rather than the exact root, depending on the precision (ϵ) of the computer. We use the following stopping criteria for computer programs: (i) $|x_{n+1} - x_n| < \epsilon$, (ii) $|f(x_{n+1})| < \epsilon$, and so, when the stopping criterion is satisfied, x_{n+1} is taken as the exact root ξ computed. For numerical illustrations in this section we used the fixed stopping criterion $\epsilon = 10^{-25}$. We used the following 23 test functions, some are taken from [10] and some from [11].

Test function	x_0	x_*
$f_1(x) = x^3 + 4x^2 - 10$	1.5	1.3652300134140968457608068290
$f_2(x) = \sin^2(x) - x^2 + 1$	1.371	1.4044916482153412260350868178
$f_3(x) = (x-1)^3 - 1$	2.5	2.0
$f_4(x) = x^3 - 10$	4.0	2.1544346900318837217592935665
$f_5(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5$	−1.5	−1.2076478271309189270094167584
$f_6(x) = e^{x^2+7x-30} - 1$	4.0	3.0
$f_7(x) = \sin(x) - \frac{x}{2}$	2.0	1.8954942670339809471440357381
$f_8(x) = x^5 + x - 10000$	4.0	6.3087771299726890947675717718

$f_9(x) = \sqrt{x} - \dfrac{1}{x} - 3$	1.0	9.633595562832695192406312709 2
$f_{10}(x) = e^x + x - 20$	0.0	2.842438953784447067816585940 2
$f_{11}(x) = \ln(x) + \sqrt{x} - 5$	1.0	8.309432694231571795346955682 7
$f_{12}(x) = x^3 - x^2 - 1$	0.5	1.465571231876768026656731225 2
$f_{13}(x) = x^2 - e^x - 3x + 2$	0.5	0.257530285439860760455367304 9
$f_{14}(x) = \arctan(x)$	0.15	0
$f_{15}(x) = e^x \sin(x) + \ln(1 + x^2)$	1.0	0
$f_{16}(x) = \ln(x^2 + x + 2) - x + 1$	4.0	4.152590736757158274996989005
$f_{17}(x) = e^{-x^2+x+2} - 1$	−0.85	−1
$f_{18}(x) = x^5 + x^4 + 4x^2 - 15$	1.2	1.347428098968304981506715381
$f_{19}(x) = x^3 + 1$	−1.5	−1
$f_{20}(x) = 11x^{11} - 1$	1.0	0.804133097503664323741463498 4
$f_{21}(x) = \sqrt{2 + x^2} \sin\left(\dfrac{\pi}{x^2}\right) + \dfrac{1}{1 + x^4} - \dfrac{17\sqrt{3} + 1}{17}$	1.6	2
$f_{22}(x) = \cos\left(\dfrac{\pi}{2}x\right) + \dfrac{\ln(x^2 + 2x + 2)}{1 + x^2}$	1.6	1.435888438664446664647913828
$f_{23}(x) = x^4 + \sin\left(\dfrac{\pi}{x^2}\right) - 5$	1.2	1.414213562373095048801688724.

Table 1
Comparison of sixth-order iterative schemes.

f		$N0$	$N1$	Nh	New
f_1	IT	3	3	3	3
	$f(x_*)$	−6e−127	−6e−127	−6e−127	−6e−127
f_2	IT	3	3	3	3
	$f(x_*)$	−1e−127	−1e−127	−1e−127	−1e−127
f_3	IT	3	4	3	4
	$f(x_*)$	0	0	0	0
f_4	IT	4	4	4	4
	$f(x_*)$	0	0	0	0
f_5	IT	4	4	4	4
	$f(x_*)$	−1e−126	−1.1e−126	−1.2e−126	−1.1e−126
f_6	IT	11	div	6	9
	$f(x_*)$	0		0	0
f_7	IT	3	3	3	3
	$f(x_*)$	−2e−128	−2e−128	−2e−128	−2e−128
f_8	IT	div	div	7	5
	$f(x_*)$			0	0
f_9	IT	div	div	div	4
	$f(x_*)$				0
f_{10}	IT	div	div	div	7
	$f(x_*)$				0
f_{11}	IT	5	div	div	4
	$f(x_*)$	−1e−127			−1e−127
f_{12}	IT	13	18	15	11
	$f(x_*)$	−1e−127	−1e−127	−1e−127	−1e−127
f_{13}	IT	3	3	3	3
	$f(x_*)$	−1e−127	−1e−127	1e−127	−1e−127
f_{14}	IT	3	3	3	3
	$f(x_*)$	0	0	0	0
f_{15}	IT	4	4	4	4
	$f(x_*)$	0	0	0	0
f_{16}	IT	3	3	3	3
	$f(x_*)$	0	0	0	0

Table 1 (*continued*)

f		$N0$	$N1$	Nh	New
f_{17}	IT	3	3	3	3
	$f(x_*)$	0	0	0	0
f_{18}	IT	3	3	3	div
	$f(x_*)$	−1e−126	−1e−126	−1e−126	
f_{19}	IT	3	4	3	4
	$f(x_*)$	0	0	0	0
f_{20}	IT	6	div	4	4
	$f(x_*)$	−5e−128		−5e−128	1e−127
f_{21}	IT	4	4	4	4
	$f(x_*)$	0	0	0	0
f_{22}	IT	3	3	3	3
	$f(x_*)$	−7e−128	−7e−128	−7e−128	−7e−128
f_{23}	IT	3	4	3	3
	$f(x_*)$	5e−127	5e−127	5e−127	5e−127

In Table 1 we presented the results for $N0$ (the case for $\beta = 0$), $N1$ (the case for $\beta = -1$), Nh (the case for $\beta = -1/2$) and our new scheme. The number of iterations IT is given along with the value of the function at the last iteration $f(x_*)$. Notice that out of 23 cases our method diverged only in one case but for the three members of Neta's family we found divergence in 3–5 cases. In 11 cases the methods gave the same answer with the same number of iterations. The $\beta = -1/2$ and the new method were superior in 5 cases whereas the other methods were superior in 3 or 4 cases. Therefore we can conclude that the new method is competitive with Neta's family of sixth-order schemes.

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