On the development of iterative methods for multiple roots

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A B S T R A C T

There are very few optimal fourth order methods for solving nonlinear algebraic equations having roots of multiplicity m. Here we compare 4 such methods, two of which require the evaluation of the \((m - 1)^{st}\) root. We will show that such computation does not affect the overall cost of the method.

1. Introduction

There is a vast literature on the solution of nonlinear equations and nonlinear systems, see for example Ostrowski [1], Traub [2], Neta [3] and the recent book by Petković et al. [4] and references therein. Most of the algorithms are for finding a simple root of a nonlinear equation \(f(x) = 0\), i.e. for a root \(x\) we have \(f(x) = 0\) and \(f'(x) \neq 0\). In this paper we are interested in the case that \(x\) is a root of multiplicity \(m > 1\). Clearly, one can use the quotient \(f(x)/f'(x)\) which has a simple root where \(f(x)\) has a multiple root. Such an idea will not require a knowledge of the multiplicity, but on the other hand will require higher derivatives. For example, Newton’s method for the function \(F(x) = f(x)/f'(x)\) will be

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n)f''(x_n)}{f'(x_n)^2}
\]

If we define the efficiency index of a method of order, \(p\) as

\[
I = p^{1/d},
\]

where \(d\) is the number of function- (and derivative-) evaluation per step then this method has an efficiency of \(2^{1/3} = 1.2599\) instead of \(\sqrt{2} = 1.4142\) for Newton’s method for simple roots.

There are very few methods for multiple roots when the multiplicity is known. The first one is due to Schröder [5] and it is also referred to as modified Newton,

\[
x_{n+1} = x_n - mu_n,
\]

where

\[
u_n = \frac{f(x_n)}{f'(x_n)}.
\]

The method is based on Newton’s method for the function \(G(x) = \sqrt{f(x)}\) which obviously has a simple root at \(x\), the multiple root with multiplicity \(m\) of \(f(x)\).

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Another method based on the same $G$ is Laguerre’s-like method
\[ x_{n+1} = x_n - \frac{\lambda u_n}{1 + \text{sgn}(\lambda - m) \sqrt{(\frac{m}{\lambda})} \left( (\lambda - 1) - \lambda u_n f'(x_n) \right)} \]

(5)

where $\lambda (\neq 0, m)$ is a real parameter. When $f(x)$ is a polynomial of degree $n$, this method with $\lambda = n$ is the ordinary Laguerre method for multiple roots, see Bodewig [6] and Neta and Chun [7]. This family of methods converges cubically.

We now list optimal fourth order methods for multiple roots. The first paper is by Li et al. [8]. Their method is a special case of one of the families found later by Li et al. [9].

Li et al. [9] have developed six fourth order methods based on the results of Neta and Johnson [10] and Neta [11]. Here we list the two optimal fourth order.

• **LCN5**

\[ y_n = x_n - \frac{2m}{m+2} u_n, \]

\[ x_{n+1} = x_n - a_3 \frac{f(x_n)}{f'(y_n)} \cdot f(x_n) - b_1 f(x_n) + b_2 f(y_n), \]

where

\[ a_3 = -\frac{1}{2} \frac{m(m+2)^3}{m^3-4m+8}, \]

\[ b_1 = -\frac{1}{m(m+2)^3-4m^2+16m+16}, \]

\[ b_2 = \frac{m^3-4m^2+8}{m^2(m+2)^3-4m+8}. \]

• **LCN6**

\[ y_n = x_n - \frac{2m}{m+2} u_n, \]

\[ x_{n+1} = x_n - a_3 \frac{f(x_n)}{f'(y_n)} \cdot f(x_n) - b_1 f(x_n) + b_2 f(y_n), \]

where

\[ a_3 = -\frac{1}{2} \frac{m(m+2)^3}{m^3-4m+8}, \]

\[ b_1 = -\frac{1}{m}, \]

\[ b_2 = \frac{m^3-4m^2+8}{m(m^3-4m^2+16m+16)}. \]

Zhou et al. [14] have also developed fourth-order optimal methods for multiple roots but they will not be included in the comparison given here. We now give the optimal methods due to Liu and Zhou [12]. These methods require the computation of the \( \frac{m-1}{m} \frac{f'(y_n)}{f(x_n)} \).

Two methods from the family developed by Liu and Zhou [12]

\[ y_n = x_n - mu_n, \]

\[ x_{n+1} = x_n - mH(w_n) f(x_n) \]

(8)

where

\[ w_n = \frac{m-1}{m} \frac{f'(y_n)}{f'(x_n)} \]

(9)

and $H(0) = 0$, $H'(0) = 1$, $H''(0) = \frac{4m}{m-1}$.

The two members given there are.

• **LZ11**
The last two methods (denoted LZ11 and LZ12) are the only ones known to the authors where the root of the function is required at each step. It may be that this will increase the cost of the method, therefore in the next section we ran these two
methods for multiple roots with multiplicity \( m > 2 \) in order to see the effect of having to evaluate \( (m-1) \sqrt[n]{f'(x_n)} \). We compare that to other optimal fourth order methods listed above (namely, LCN5 and LCN6).

In the next section we give timing comparison for optimal fourth order methods that require \( (m-1) \sqrt[n]{f'(x_n)} \) and those that do not. We close with concluding remarks.

2. Comparison of optimal fourth order methods

Kung and Traub [13] conjectured that multipoint iterative methods without memory, requiring \( n+1 \) function-evaluations per iteration, have order of convergence at most \( 2^n \). Multipoint methods that satisfy the Kung-Traub conjecture are called optimal methods. Their efficiency index is \( \frac{2^n}{n+1} \). Of all the methods listed above we have only 4 optimal methods of order 4, namely \( (6), (7), (10), \) and \( (11) \). We refer to these methods as LCN5, LCN6, LZ11 and LZ12, respectively. We have ran these 4 methods on 31 cases having roots of multiplicity \( m = 2, 3, 4, 5 \). Table 1 gives the functions, the initial guess, the root and its multiplicity.

We have ran Maple using \( \text{Digits} = 128 \), the tolerance for convergence is \( 10^{-25} \) and no more than 1000 iterations. The timing required for all examples for each method is given in Table 2.

It is clear that LZ12 requires more than three times as much CPU time than the other three methods use. Since LZ11 does not require more CPU time than LCN5 and LCN6, we conclude that the reason for the extra cost is due to a slow convergence of LZ12 and not that the \( (m – 1)^{st} \) root is costly. As suggested by the referees, we have looked at the reason for the time consumption of LZ12 and found that the method requires more iterations to converge. In a follow-on manuscript we will compare the basins of attraction which hopefully will show that unless one starts very close to the root, LZ12 will require more iterations.

Conclusion

The computation of \( (m-1) \sqrt[n]{f'(x_n)} \) does not significantly increase the CPU time required for convergence.

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