Publications and Abstracts

1. B. Neta, G. J. Fix, Finite Element Approximation of a Nonlinear Diffusion Problem, *Computers and Mathematics with Applications*, **3**, (1977), 287–298.

   This paper deals with the finite element approximation of the nonlinear diffusion problem
   $$-\nabla \cdot \left(|\nabla u|^{p-2}\nabla u\right) = f.$$
   Glowinski and Marrocco [3] have shown that the rate of convergence decreases as p increases. In this paper we show that the rate of convergence is optimal and independent of p. This theoretical result agrees with the numerical experiments reported in the last section.

2. B. Neta, Finite Element Approximation of a Nonlinear Parabolic Problem, *Computers and Mathematics with Applications*, **4**, (1978), 247–255.

   Galerkin-type method is considered for approximating solution of the nonlinear parabolic problem
   $$u_t - \nabla \cdot \left(|\nabla u|^{p-2}\nabla u\right) = f.$$
   Error bounds were derived showing that the rate of convergence with respect to the space variables is optimal and independent of p. This result has been confirmed by the numerical experiments reported in section 4.

3. B. Neta, A Sixth-Order Family of Methods for Nonlinear Equations, *International J. Computer Mathematics*, **7**, (1979), 157–161.

   A one-parameter family of sixth-order methods for finding simple zeros of nonlinear functions is develepoed. Each member of the family requires three evaluations of the given function and only one evaluation of the derivative per step.

4. B. Neta, On 3 Inequalities, *Computers and Mathematics with Applications*, **6**, (1980),301–304.

   In this paper we obtain the best possible constants in three inequalities between two vectors. The inequalities depend on a parameter p. These inequalities were used to bound the error in the finite element approximation of a nonlinear diffusion problem.

5. B. Neta, On Determination of Best-Possible Constants in Integrals Inequalities Involving Derivatives, *Mathematics of Computations*, **35**, (1980), 1191–1194.

   This paper is concerned with the numerical approximation of the best possible constants $\gamma_{n,k}$ in the inequality
   $$\|F^{(k)}\|^2 \le \gamma_{n,k}^{-1}\left\{\|F\|^2 + \|F^{(n)}\|^2\right\},$$
   where
   $$\|F\|^2 = \int_0^\infty |F(x)|^2 dx.$$
   A list of all constants $\gamma_{n,k}$ for $n \le 10$ is given.

6. B. Neta, A New Iterative Method for the Solution of Systems of Nonlinear Equations, in Approximation Theory and Applications (Z. Ziegler, ed.), Academic Press, (1981), pp. 249-263.

   A new quasi-Newton method for the solution of systems of nonlinear algebraic equations is introduced. This method is a generalization of a sixth-order one developed by the author for approximating the solution of one nonlinear equation. The R-order of the method is four. Numerical experiments comparing the method to Newton's show that one can save over 20% of the cost of solving a system of algebraic equations. The saving is greater when the dimension is higher or the number of iterations needed is larger.

7. B. Neta, H. D. Victory, On the Development of a Convergence Theory of Synthesis Methods for Solving Diffusion Equations, *Progress in Nuclear Energy*, **8**, (1981), 283–293.

   This article is a survey of the theoretical questions arising in the development of a convergence theory and error analysis of synthesis methods for solving neutron diffusion problems. For simplicity, we discuss convergence and the error analysis for spectral synthesis methods, in which the trial functions are functions solely of energy. The diffusion coefficient, the total and scattering cross-section data for the diffusion model are assumed spatially and energy dependent, and interior interfaces (i.e. spatial discontinuities in the diffusion coefficients and cross-section data) are present. The boundary conditions imposed are homogeneous Dirichlet conditions.

8. B. Neta, Note on Error Analysis for the Spectral Synthesis Method with Interfaces, Technical Report INTT-12, Institute for Numerical Transport Theory, Texas Tech University, Lubbock, TX, 1981.

   The purpose of this note is to prove the existence and uniqueness and to derive an error estimate for spectral synthesis approximations relative to the continuous energy, continuous space, time-independent neutron diffusion equation, with given source and zero flux at the boundary. This estimate was derived by Meyer and Nelson [1] under different, and perhaps less useful conditions.

9. B. Neta, On a Family of Multipoint Methods for Nonlinear Equations, *International J. Computer Mathematics*, **9**, (1981), 353–361.

   A new one-parameter family of methods for finding simple zeros of non-linear functions is developed. Each member of the family requires four evaluations of the given function and only one evaluation of the derivative per step. The order of the method is 16.

10. B. Neta, Note on Error Analysis for the Spectral Synthesis Method with Interfaces, *Zeitschrift fur Angewandte Mathematik und Physik*, **32**, (1981), 603–608.

    The purpose of this note is to prove the existence and uniqueness and to derive an error estimate for spectral synthesis approximations relative to the continuous energy, continuous space, time-independent neutron diffusion equation, with given source and zero flux at the boundary. This estimate was derived by Meyer and Nelson [1] under different, and perhaps less useful conditions.

11. B. Neta, H. D. Victory, A New Fourth Order Finite-Difference Method for Solving Discrete-Ordinate Slab Transport Equations, Technical Report INTT-17, Institute for Numerical Transport Theory, Texas Tech University, Lubbock, TX, 1981.

    This work is concerned with a theoretical study of a new fourth-order finite-difference scheme for spatially discretizing the discrete-ordinates equations for solving numerically the slab transport (Boltzman) equation. This analysis considers the *quadratic continuous method*, whose derivation parallels that of the commonly used diamond difference and linear discontinuous schemes from balance equations for particle conservation across a spatial cell. We provide a convergence analysis of the method and prove that superconvergence phenomena are present for cell-edge and cell-average fluxes. We also present results from an S_2-test problem to show that the asymptotic convergence rates are observed on rather coarse spatial meshes.

12. B. Neta, Numerical Solution of a Nonlinear Integro-Differential Equation, Technical Report INTT-21, Institute for Numerical Transport Theory, Texas Tech University, Lubbock, TX, 1982.

    Galerkin's method with appropriate discretization in time is considered for approximating the solution of the nonlinear integro-differential equation

    $$u_t(x,t) = \int_0^t a(t-\tau)\frac{\partial}{\partial x}\sigma\left(u_x(x,\tau)\right)d\tau + f(x,t), \qquad 0 < x < 1, \quad 0 < t < T.$$

    An error estimate in a suitable norm will be derived for the difference $u - u^h$ between the exact solution u and the approximant u^h. It turns out that the rate of convergence of u^h to u as $h \to 0$ is optimal. This result was confirmed by the numerical experiments.

13. B. Neta, H. D. Victory, The Convergence Analysis for Sixth-Order Methods for Solving Discrete-Ordinate Slab Transport Equations, Technical Report INTT-20, Institute for Numerical Transport Theory, Texas Tech University, Lubbock, TX, 1982.

    Functional analytic methods, employing the *collectively compact operator approximation theory* of P. M. Anselone, are utilized to study the convergence properties of four improvements to the rather promising *quadratic method* of Gopinath, Natarajan, and Sundararaman [Nucl. Sci. Eng. **75** (1980), 181-184] for effecting slab transport calculations. This method displays a global discretization error of order four; our alterations too possess global discretization errors of order four, but display *superconvergence phenomena* in that the discretization errors in computed cell-edge fluxes are of order six. The stability of our methods results from applying Anselone's theory of sequences of collectively compact operators. A numerical example is provided which shows that our asymptotic convergence rates are observed on rather coarse meshes.

14. H. D. Victory, B. Neta, A Higher Order Method for Multiple Zeros of Nonlinear Functions, Technical Report INTT-19, Institute for Numerical Transport Theory, Texas Tech University, Lubbock, TX, 1982.

A method of order three for finding multiple zeros of nonlinear functions is developed. The method requires two evaluations of the function and one evaluation of the derivative per step.

15. B. Neta, Numerical Solution of a Nonlinear Integro-Differential Equation, *J. of Mathematical Analysis and Applications*, **89**, (1982), 598–611.

    Galerkin's method with appropriate discretization in time is considered for approximating the solution of the nonlinear integro-differential equation

    $$u_t(x,t) = \int_0^t a(t-\tau)\frac{\partial}{\partial x}\sigma\left(u_x(x,\tau)\right)d\tau + f(x,t), \qquad 0 < x < 1, \quad 0 < t < T.$$

    An error estimate in a suitable norm will be derived for the difference $u - u^h$ between the exact solution u and the approximant u^h. It turns out that the rate of convergence of u^h to u as $h \to 0$ is optimal. This result was confirmed by the numerical experiments.

16. B. Neta, H. D. Victory, The Convergence Analysis for Sixth-Order Methods for Solving Discrete-Ordinate Slab Transport Equations, *Numerical Functional Analysis*, **5**, (1982), 85-126.

    Functional analytic methods, employing the *collectively compact operator approximation theory* of P. M. Anselone, are utilized to study the convergence properties of four improvements to the rather promising *quadratic method* of Gopinath, Natarajan, and Sundararaman [Nucl. Sci. Eng. **75** (1980), 181-184] for effecting slab transport calculations. This method displays a global discretization error of order four; our alterations too possess global discretization errors of order four, but display *superconvergence phenomena* in that the discretization errors in computed cell-edge fluxes are of order six. The stability of our methods results from applying Anselone's theory of sequences of collectively compact operators. A numerical example is provided which shows that our asymptotic convergence rates are observed on rather coarse meshes.

17. B. Neta, S. C. Lee, Hybrid Methods for a Special Class of Second-Order Differential Equations, *Congressus Numerantium*, **38**, (1983), 203–225.

    Hybrid methods for the numerical solution of second order ordinary differential equations <u>not</u> containing y' are developed. The order p of such stable k-step methods is <u>not</u> limited to $p = k + 1 \quad (k + 2)$. The customary linear k-step schemes are modified by including the values of the second derivative at one "offstep" point. It is shown that the order of these hybrid methods is not subject to the above restrictions. Numerical experiments are presented. It is shown that the maximal order is achieved.

18. B. Neta, H. D. Victory, A New Fourth Order Finite-Difference Method for Solving Discrete-Ordinate Slab Transport Equations, *SIAM J. on Numerical Analysis*, **20**, (1983), 94-105.

    This work is concerned with a theoretical study of a new fourth-order finite-difference scheme for spatially discretizing the discrete-ordinates equations for solving numerically the slab transport (Boltzman) equation. This analysis considers the *quadratic* *continuous method*, whose derivation parallels that of the commonly used diamond difference and linear discontinuous schemes from balance equations for particle conservation across a spatial cell. We provide a convergence analysis of the method and prove that superconvergence phenomena are present for cell-edge and cell-average fluxes. We also present results from an S_2-test problem to show that the asymptotic convergence rates are observed on rather coarse spatial meshes.

19. H. D. Victory, B. Neta, A Higher Order Method for Multiple Zeros of Nonlinear Functions, *International J. Computer Mathematics*, **12**, (1983), 329–335.

    A method of order three for finding multiple zeros of nonlinear functions is developed. The method requires two evaluations of the function and one evaluation of the derivative per step.

20. B. Neta, A New Family of Higher Order Methods for Solving Equations, *International J. Computer Mathematics*, **14**, (1983), 191–195.

    Neta's three step sixth order family of methods for solving nonlinear equations require 3 function and 1 derivative evaluation per iteration. Using exactly the same information, another three step method can be obtained with convergence rate 10.81525 which is much better than the sixth order.

21. M. M. Chawla, B. Neta, A New Class of Linearly Implicit Single-Step P-Stable Methods for Second Order Initial-Value Problems, Technical Report INTT-28, Institute for Numerical Transport Theory, Texas Tech University, Lubbock, TX, 1984.

    A new class of linearly implicit single-step methods for the numerical solution of second order initial value problems is constructed. Methods of order two, three and four are developed and their P-stability is investigated. Numerical experiments show that our methods compare favorably with previously developed methods.

22. B. Neta, Higher Order Hybrid Störmer-Cowell Methods for Ordinary Differential Equations, *Congressus Numerantium*, **42**, (1984), 251–264.

    Hybrid Störmer-Cowell methods are constructed for the numerical solution of second order ordinary differential equations <u>not</u> containing y'. The order p of such stable k-step method is <u>not</u> limited to $k+1(k+2)$. These methods are combined with predictor k-step methods of the same order as the corretor or higher. The coefficients, the order and the error constant for the corrector and predictor(s) will be given.

23. B. Neta, Adaptive Method for the Numerical Solution of Fredholm Integral Equations of the Second Kind. Part II: Singular Kernels in Numerical Solution of Singular Integral Equations, Proceedings of an IMACS International Symposium held at Lehigh University, Bethlehem, PA, June 21-22, 1984, (A. Gerasoulis, R. Vichnevetsky, eds.), pp.68–72.

    An adaptive method based on a product integration rule for the numerical solution of Fredholm integral equation of the second kind with singular kernel is developed. We discuss two types of sigular kernels, i.e. $\log|x-y|$ and $|x-y|^{-\alpha}, \quad \alpha < 1$. The choice of mesh points is made automatically so as to equidistribute both the change in the discrete solution and its gradient. Some numerical experiments with this method are presented.

24. B. Neta, Review of Methods for Numerical Mathematics by G. I. Marchuk (Trans. by A. A. Brown), Springer-Verlag, New york, 1982, *Transport Theory and Statistical Physics*, **13**, (1984), 663–665.

25. B. Neta, C. H. Ford, Families of Methods for Ordinary Differential Equations Based on Trigonometric Polynomials, *J. Computational and Applied Mathematics*, **10**, (1984), 33–38.

    We consider the construction of methods based on trigonometric polynomials for the initial value problems whose solutions are known to be periodic. It is assumed that the frquency ω can be estimated in advance. The resulting methods depend on a parameter $\nu = h\omega$, where h is the step size, and reduce to classical multistep methods if $\nu \to 0$. Gautschi [4] developed Adams and Störmer type methods. In our paper we construct Nyström's and Milne-Simpson's type methods. Numerical experiments show that these methods are not sensitive to changes in ω, but require the Jacobian matrix to have purely imaginary eigenvalues.

26. B. Neta, H. D. Victory, A Higher Order Method for Determining Nonisolated Solutions of a System of Nonlinear Equations, *Computing*, **32**, (1984), 163–166.

    In this note, we obtain a method of order at least four to solve a singular system of nonlinear algebraic equations. This is achieved by enlarging the system to a higher dimensional one whose solution is isolated. For the larger system we use a method developed by B. Neta.

27. B. Neta, Hybrid Predictors and Correctors for Solving a Special Class of Second Order Differential Equations, *Congressus Numerantium*, **46**, (1985), 241–248.

    Hybrid predictors and correctors of order $k + k' + 1$ are constructed for the numerical solution of second order differential equations not containing the first derivative explicitly. These methods are based on both explicit ($k' = k - 1$) and implicit ($k' = k$) linear k-step methods.

28. B. Neta, B. P. Sommeijer, P. J. van der Houwen, Symmetric Linear Multistep Methods for Second Order Differential Equations with Periodic Solutions, Report NM-R8501, Center for Mathematics and Computer Science, Amsterdam, The Netherlands, 1985.

    Special symmetric linear multistep methods for second-order differential equations without first derivatives are proposed. The methods can be tuned to a possibly a priori knowledge of the user on the location of the frequencies, that are dominant in the exact solution. On the basis of such extra information the truncation error can considerably be reduced in magnitude. Numerical results are compared with results produced by the symmetric methods of Lambert and Watson and the method of Gautschi.

29. B. Neta, J. O. Igwe, Finite Differences versus Finite Elements for Solving Nonlinear Integro-Differential Equations, *J. of Mathematical Analysis and Applications*, **112**, (1985), 607–618.

Consider the nonlinear integro-differential equation

$$u_t(x,t) = \int_0^t a(t-\tau)\frac{\partial}{\partial x}\sigma\left(u_x(x,\tau)\right)d\tau + f(x,t), \qquad 0 < x < 1, \quad 0 < t < T.$$

with appropriate initial and boundary conditions. This problem serves as a model for one-dimensional heat flow in material with memory. The numerical solution via finite elements was discussed in B. Neta [*J. Math. Anal. Appl.* **89** (1982), 598-611]. In this paper we compare the results obtained there with finite difference approximation from the point of view of accuracy and computer storage. It turns out that the finite difference method yields comparable results for the same mesh spacing using less computer storage.

30. B. Neta, H. M. Tai, LU Factorization on Parallel Computers, *Computers and Mathematics with Applications*, **11**, (1985), 573–579.

A new parallel algorithm for the LU factorization of a given dense matrix A is described. The case of banded matrices is also considered. This algorithm can be combined with Sameh and Brent's [*SIAM J. Numer. Anal.* **14**, 1101–1113, (1977)] to obtain the solution of a linear system of algebraic equations. The arithmetic complexity for the dense case is $\frac{1}{3}n^2$ ($\frac{1}{2}bn$ in the banded case), using $3(n-1)$ processors and no square roots.

31. B. Neta, R. T. Williams, D. E. Hinsman, Studies in a Shallow Water Fluid Model with Topography, in Numerical Mathematics and Applications (R. Vichnevetsky, J. Vignes, eds.), Elsevier Sci. Pub., (1986), 347–354.

In this paper we develop a numerical model with the shallow water approximation for a single layer of fluid which flows over varible bottom topography. The motion is confined in a channel with cyclic boundary conditions. The Galerkin finite element method is used for the spatial variation and the time discretization is accomplished with semi-implicit finite differencing. In our experiments we use bilinear basis functions on rectangles.

We also analyze the linearized version of the model. In this analysis we compare four spatial schemes, bilinear basis functions on rectangles, linear basis functions on isosceles triangles and second and fourth order finite differences. The time will not be discretized in the analysis.

32. B. Neta, Special Methods for Problems whose Oscillatory Solution is Damped, NPS–53–86–012,Technical Report Naval Postgraduate School, Monterey, CA, 1986.

This paper introduces methods tailored especially for problems whose solution behave like $e^{\lambda x}$, where λ is complex. The shallow water equations with topography admit such solution.

This paper complements the results of Pratt and others on exponential-fitted methods and those of Gautschi, Neta, van der Houwen and others on trigonometric-fitted methods.

33. M. M. Chawla, B. Neta, Families of Two-Step Fourth Order P-Stable Methods for Second Order Differential Equations, *J. Computational Applied Mathematics*, **15**, (1986), 213–223.

    This paper deals with a class of symmetric (hybrid) two-step fourth order P-stable methods for the numerical solution of special second order initial value problems. Such methods were proposed independently by Cash [1] and Chawla [3] and normally require three function evaluations per step. The purpose of this paper is to point out that there are some values of the (free) parameters available in the methods proposed which can reduce this work; we study two classes of such methods. The first is a class of 'economical' methods (see Definition 3.1) which reduce this work to two function evaluations per step, and the second is the class of 'efficient' methods (see Definition 3.2) which reduce this work with respect to implementation for nonlinear problems. We report numerical experiments to illustrate the order, accuracy and implementational aspects of these two classes of methods.

34. B. Neta, Families of Backward Differentiation Methods based on Trigonometric Polynomials, *International J. of Computer Mathematics*, **20**, (1986), 67–75.

    Backward differentiation methods based on trigonometric polynomials for the initial value problems whose solutions are known to be periodic are constructed. It is assumed that the frequency ω can be estimated in advance. The resulting methods depend on a parameter $\nu = h\omega$, where h is the step size, and reduce to classical backward methods if $\nu \to 0$. Neta anf Ford [6] considered Nyström and generalized Milne-Simpson type methods. Those methods require the Jacobian matrix to have purely imaginary eigenvalues. The methods we construct here will not suffer of this deficiency.

35. B. P. Sommeijer, P. J. van der Houwen B. Neta, Symmetric Linear Multistep Methods for Second Order Differential Equations with Periodic Solutions, *Applied Numerical Mathematics*, **2**, (1986), 69–77.

    Special symmetric linear multistep methods for second-order differential equations without first derivatives are proposed. The methods can be tuned to a possibly a priori knowledge of the user on the location of the frequencies, that are dominant in the exact solution. On the basis of such extra information the truncation error can considerably be reduced in magnitude. Numerical results are compared with results produced by the symmetric methods of Lambert and Watson and the method of Gautschi.

36. B. Neta, R. T. Williams, Stability and Phase Speed for Various Finite Element Formulations of the Advection Equation, *Computers and Fluids*, **14**, (1986), 393–410.

    This paper analyzes the stability and accuracy of various finite element approximations to the linearized two-dimensional advection equation. Four triagular elements with linear basis functions are included along with a rectangular element with bilinear basis functions. In addition, second- and fourth-order finite difference schemes are examined for comparison. Time is discretized with the leapfrog method. The criss-cross triangle formulation is found to be unstable. The best schemes are the isosceles triangles with linear basis functions and the rectangles with bilinear basis functions.

37. M. M. Chawla, P. S. Rao, B. Neta, Two-Step Fourth Order P-Stable Methods with Phase-Lag of Order Six for $y'' = f(t, y)$, *J. Computational and Applied Mathematics*, **16**, (1986), 233–236.

    We present a *new* family of two-step fourth-order methods which when applied to the test equation: $y'' = -\lambda^2 y, \quad \lambda > 0$, are at once P-stable and have a phase-lag of order H^6 ($H = \lambda h$, h is the step-size).

38. B. Neta, R. T. Williams, Finite Elements versus Finite Differences for Fluid Flow Problems, Proceedings International Conf. Numer. Meth. Engnrg: Theory and Applics Vol. II Transient Dynamic Anal. and Constitutive Laws for Engnrg Materials, G. N. Pande, J. Middleton (eds) Martinus Nijhoff Pub., (1987), T11, 1-8.

    In this article, we discuss finite element methods based on bilinear basis functions on rectangles and linear functions on various triangular elements. We compare the results with second and fourth order finite differences and the so-called A, B, and C schemes. The model used for this comparative study is the quasi-geostrophic approximation in the shallow water equations on a β-plane.

    It is shown that the finite element methods (on isosceles triangles or rectangles) produce better estimates to the frequency than the finite differences.

39. B. Neta, I. M. Navon, The Transfer Function Analysis of the Turkel-Zwas Scheme for the Shallow-Water equations, FSU–SCRI–87–30, Technical Report Florida State University, Tallahassee, FL, (1987).

    A transfer function analysis is used to analyze the Turkel-Zwas explicit large time step scheme applied to the shallow water equations. The transfer function concept leads to insight into the behavior of this discretization scheme in terms of comparison between continuous and discrete amplitude, phase, and group velocity coefficients. The dependence of the distortion increases with the increase in the time-step size taken for the Turkel-Zwas scheme, which depends on the ratio between a coarse and a fine mesh. A comparison with earlier results of Scoenstadt (Naval Postgraduate School Report NPS-53-79-001, 1978 (unpublished)) shows the Turkel-Zwas scheme to give reasonable results up to time steps three times larger than the CFL limit.

40. B. Neta, P. Nelson, An Adaptive Method for the Numerical Solution of a Fredholm Integral Equation of the Second Kind. Part I: Regular Kernels, *Applied Mathematics and Computation*, **21**, (1987), 171–184.

    An adaptive method based on the trapezoidal rule for the numerical solution of Fredholm integral equations of the second kind is developed. The choice of mesh points is made automatically so as to equidistribute both the change in the discrete solution and its gradient. Some numerical experiments with this method are presented.

41. B. Neta, Several New Schemes for Solving Equations, *International J. Computer Mathematics*, **23**, (1987), 265–282.

    Several new methods for solving one nonlinear equation are developed. Most of the methods are of order three and they require the knowledge of f, f' and f''. The methods will be compared to others in the literature. An extensive bibliography is given.

42. I. M. Navon and B. Neta, Application of Optimal Control Methods in Meteorology- 4-D Data Assimilation Problems, ICIAM–87: Proceedings of the First International Conference on Industrial and Applied Mathematics, James McKenna and Roger Temam (eds), SIAM, Philadelphia, (1988), 282-284.

43. B. Neta, The Effect of Spatial Discretization on the Steady-State Solution of the Shallow Water Equations, Computational Methods in Flow Analysis, Proc. International Conference on Computational Methods in Flow Analysis, Okayama, Japan, September 4-8 (H. Niki and M. Kawahara, eds.), **2**, (1988), 1041–1048.

    Fourier transforms are used to obtain the steady-state solution of the discretized two-dimensional shallow water equations. Several finite element and finite difference schemes were considered. This analysis extends the results given in Schoenstadt for the one-dimensional case.

44. B. Neta, Y. Ilan-Lipowski, A New Conservative Scheme for Solving the Two Body Problem, NPS–53–88–009, Technical Report Naval Postgraduate School, Monterey, CA, (1988).

    The well known Störmer-Cowell class of linear k-step methods for the solution of second order initial value problems suffer from orbital instability. The solution of a problem describing a uniform motion in a circular orbit, spirals inward. Several modifications were suggested, some require the a-priori knowledge of the frequency. Here we develop a method based on the conservation of energy and momentum. This method overcomes the aforementioned difficulty.

45. B. Neta, C. L. DeVito, The Transfer Function Analysis of Various Schemes for the Two Dimensional Shallow-Water equations, *Computers and Mathematics with Applications*, **16**, (1988), 111–137.

    In this paper various finite difference and finite element approximations to the linearized two-dimensional shallow-water equations are analyzed. This analysis complements previous results for the one-dimensional case.

46. B. Neta, Analysis of the Turkel-Zwas Scheme for the Shallow Water Equations, in Numerical and Applied Mathematics W. F. Ames (ed) Proc. 12th IMACS World Congress on Scientific Computation, Paris, France, July 18-22, **1 sec. 3**, (1989), 257–264.

In this note the Turkel-Zwas explicit large time step scheme for the two dimensional shallow-water equations is analyzed. This analysis complements the results of Scoenstadt and Neta and Navon for the one dimensional case and of Neta and DeVito in the two dimensional case. It is shown that the frequencies of the approximate solution approach those of the exact solution as Δx and Δy tend to zero. But the approximations suggested for the Coriolis terms will not allow the frequency to attain the value zero as in the continuous case. It is suggested that a symmetric approximation of these terms be used to overcome this difficulty.

47. R. T. Williams, B. Neta, A Comparative Study of Finite Elements and Finite Differences for Weather Prediction, Proc. Fifth International Symposium on Numerical Methods in Engineering, Lausanne, Switzerland, September 11-15, 1989 R. Gruber, J. Periaux and R. P. Shaw, (eds), **1**, (1989), 659–669.

    The vertical discretization in a linearized baroclinic prediction model is analyzed by comparing various finite element and finite difference solutions following Jordan [1] and Shapiro [2]. The baroclinic instability experiments of Shapiro [2] were augmented to include the unstaggered vertical scheme from Jordan [1]. Two basic wind profiles were used and the experiments were run with various resolution models and horizontal wavelengths. For a given wind profile and vertical resolution, different models performed better. The finite element models for the vertical grids did not perform up to their possibilities due to the boundary elements. However, for the unstaggered vertical grid, the finite element model did better than the finite difference schemes.

48. B. Neta, Analysis of Finite Element and Finite Differences for Shallow Water Equations, Seminar on Finite Element Fluid Flow Analysis, Chuo University, Tokyo, Japan, 24 August 1989.

    In this review article we discuss analyses of finite-element and finite-difference approximations of the shallow water equations. An extensive bibliography is given.

49. B. Neta, C. P. Katti, Solution of Linear Initial Value Problems on a Hypercube, NPS–53–89–001, Technical Report Naval Postgraduate School, Monterey, CA, (1989).

    There are many articles discussing the solution of boundary value problems on various parallel machines. The solution of initial value problems does not lend itself to parallelism, since in this case one uses methods that are sequential in nature.

    Here we develop a parallel scheme for initial value problems based on the box scheme and a modified recursive doubling technique.

    Fully implicit Runge Kutta methods were discussed by Jackson and Nörsett (1986) and Lie (1987). Lie assumes that each processor of the parallel computer having vector capabilities.

50. W. Gragg, B. Neta, Fortran Subroutines for the Evaluation of the Confluent Hypergeometric Functions, NPS–53–89–014,Technical Report Naval Postgraduate School, Monterey, CA, (1989).

In this report we list the Fortran subroutines for evaluating the confluent hypergeometric functions $M(a, b; x)$ and $U(a, b; x)$. These subroutines use the stable recurrence relations given e.g. in Wimp.

51. B. Neta, Special Methods for Problems whose Oscillatory Solution is Damped, *Applied Mathematics and Computation*, **31**, (1989), 161–169.

    This paper introduces methods tailored especially for problems whose solution behave like $e^{\lambda x}$, where λ is complex. The shallow water equations with topography admit such solution.

    This paper complements the results of Pratt and others on exponential-fitted methods and those of Gautschi, Neta, van der Houwen and others on trigonometric-fitted methods.

52. B. Neta, R. T. Williams, Rossby Wave Phase Speed and Group Velocities for Finite Element and Finite Difference Approximations to the Vorticity-Divergence Form of the Shallow Water Equations, *Monthly Weather Review*, **117**, (1989), 1439–1457.

    In this paper Rossby wave frequencies and group velocities are analyzed for various finite element and finite difference approximations to the vorticity-divergence form of the shallow water equations. Also included are finite difference solutions for the primitive equations for the staggered grids B and C from Wajsowicz and for the unstaggered grid A. The results are presented for three ratios between the grid size and the Rossby radius of deformation. The vorticity-divergence equation schemes give superior solutions to those based on the primitive equations. The best results come from the finite element schemes that use linear basis functions on isosceles triangles and bilinear basis functions on rectangles. All of the primitive equation finite difference schemes have problems for at least one Rossby deformation-grid size ratio.

53. B. Neta, I. M.Navon, Analysis of the Turkel-Zwas Scheme for the Shallow-Water Equations, *J. Computational Physics*, **81**, (1989), 277–299.

    A transfer function analysis is used to analyze the Turkel-Zwas explicit large time step scheme applied to the shallow water equations. The transfer function concept leads to insight into the behavior of this discretization scheme in terms of comparison between continuous and discrete amplitude, phase, and group velocity coefficients. The dependence of the distortion increases with the increase in the time-step size taken for the Turkel-Zwas scheme, which depends on the ratio between a coarse and a fine mesh. A comparison with earlier results of Scoenstadt (Naval Postgraduate School Report NPS-53-79-001, 1978 (unpublished)) shows the Turkel-Zwas scheme to give reasonable results up to time steps three times larger than the CFL limit.

54. B. Neta, R. T. Williams, Analysis of Finite Element Methods for the Solution of the Vorticity-Divergence Form of the Shallow Water Equations. in Numerical Methods in Fluid Dynamics, Proc. International Symposium on Computational Fluid Dynamics, Nagoya, Japan, August 28-31, 1989 (M. Yasuhara, H. Daiguji, K. Oshima, eds), Japan Society of Comp. Fluid Dynamics, (1990), 402-407.

Various finite element approximations to the vorticity-divergence form of the shallow water equations are analyzed. The vorticity-divergence equation schemes give superior solutions to those which are based on the primitive equations. The best results come from the finite element schemes on isosceles triangles and rectangles.

55. B. Neta, N. Okamoto, On Domain Decomposition Methods for Solving Partial Differential Equations, NPS–MA–90–004, Technical Report Naval Postgraduate School, Monterey, CA, (1990).

    A domain decomposition method for solving partial differential equations is described. The conditions on interfaces will be of Dirichlet type and obtained by the boundary element method using very few (less than 10) unknowns.

56. B. Neta, I. M. Navon, J. Yu, Analysis of the Turkel Zwas Scheme for the Two Dimensional Shallow Water Equations in Spherical Coordinates, FSU–SCRI–90–91, Technical Report Florida State University, Tallahassee, FL, (1990).

    A linear analysis of the shallow-water equations in spherical coordinates for the Turkel-Zwas (T-Z) [1] explicit large time-step scheme was carried out. This paper complements the results of Schoenstadt [2], Neta nad Navon [3], and other in 1-D, and of Neta and DeVito [4] in 2-D, but applied to the spherical coordinate case of the T-Z scheme as documented and analyzed by Navon and de Villiers [5]. This coordinate system is more realistic in meteorology and more complicated to analyze since the coefficients are no longer constants.

57. P. Nelson, C. P. Katti, B. Neta, Convergence of Inner–Outer Source Iterations with Finite Termination of the Inner Iterations, *J. of Integral Equations and Applications*, **2**, (1990), 147–174.

    A two-stage (nested) iteration strategy, in which the outer iteration is analogous to a block Gauss-Seidel method and the inner iteration to a Jacobi method for each of these blocks, often are used in the numerical solution of discretized approximations to the neutron transport equation. This paper is concerned with the effect, within a continuous space model, of errors from finite termination of the inner iterations upon the overall convergence. The main result is that convergence occurs, to the solution of the original problem, in the limit that both the outer iteration index and the minimum over all outer indices and all groups ("blocks") jointly become large. Positivity properties (in the sense of cone preserving) are used extensively.

58. B. Neta and I. M. Navon, Analysis of the Turkel Zwas Scheme for the Shallow Water Equations on the Sphere, *Proc. 13th IMACS World Congress on Scientific Computation, Dublin, Ireland,*, R. Vichnevetsky, J. J. H. Miller (eds.) July 22-26, (1991), pp. 305-306.

    A linear analysis of the shallow-water equations in spherical coordinates for the Turkel-Zwas (T-Z) [1] explicit large time-step scheme was carried out. This paper complements the results of Schoenstadt [2], Neta nad Navon [3], and other in 1-D, and of Neta and DeVito [4] in 2-D, but applied to the spherical coordinate case of the T-Z scheme as documented and analyzed by Navon and de Villiers [5]. This coordinate system is more realistic in meteorology and more complicated to analyze since the coefficients are no longer constants.

59. L. Lustman, B. Neta, Software for the Parallel Solution of Systems of Ordinary Differential Equations, NPS–MA–91–009 Technical Report, (1991).

    This report contains software for the solution of systems of ordinary differential equations on an INTEL iPSC/2 hypercube. A diskette is available upon request from the second author.

60. L. Lustman, B. Neta, C. P. Katti, Solution of Linear Ordinary Differential Systems on an INTEL Hypercube, *SIAM J. on Scientific and Statistical Computing*, **12**, (1991) 1480–1485.

    Here we develop and test a parallel scheme for the solution of linear systems of ordinary initial value problems based on the box scheme and a modified recursive doubling technique. The box scheme may be replaced by any stable integrator. The algorithm can be modified to solve boundary value problems. Software for both problems is available upon request.

61. B. Neta, Solution of Ordinary Differential Initial Value Problems on an INTEL Hypercube with Applications to Orbit Determination, *Proc. of the 1992 Space Surveillance Workshop, M.I.T. Lincoln Laboratory, Lexington, MA,* April 7-9, 1992, A. J. Coster and K. P. Schwan (eds), **1**, (1992), 205–208.

    The Naval Space Surveillance Center (NAVSPASUR) uses an analytic satellite motion model based on the Brouwer - Lyddane theory to track objects orbiting the Earth. In this paper we develop several parallel algorithms based on this model. These have been implemented on the INTEL iPSC/2 hypercube multi-computer. The speed-up and efficiency of these algorithms will be obtained. We show that the best of these algorithms achieves 87% efficiency if one uses a 16-node hypercube.

62. L. Lustman, B. Neta, Software for the parallel Conservative Scheme for the Shallow Water equations, NPS–MA–92–004 Technical Report ,Naval Postgraduate School, Monterey, CA, 1992.

    This report contains the host and node programs for the solution of the shallow water equations with topography on an INTEL iPSC/2 hypercube. Finite difference scheme conserving enstrophy and energy is employed in each subdomain.

63. B. Neta, B. Mansager, Audio Detection Algorithms, NPS–MA–92-008 Technical Report, Naval Postgraduate School, Monterey, CA, 1992.

    Audio information concerning targets generally includes direction, frequencies and energy levels. One use of audio cueing is to use direction information to help determine where more sensitive visual direction and acquisition sensors should be directed. Generally, use of audio cueing will shorten times requires for visual detection, although there could be circumstances where the audio information is misleading and degrades visual

performance. Audio signatures can also be useful for helping classify the emanating platform, as well as to provide estimates of its velocity.

64. L. Lustman, B. Neta, W. Gragg, Solution of Ordinary Differential Initial Value Problems on an INTEL Hypercube, *Computers and Mathematics with Applications*, **23**, (1992), 65–72.

    Schemes for the solution of *linear* initial or boundary value problems on a hypercube were developed by Katti and Neta [1] and tested and improved by Lustman, Neta and Katti [2]. Among other procedures for parallel computers, fully implicit Runge-Kutta methods were discussed by Jackson and Norsett [3] and Lie [4].

    Here we develop a method based on extrapolation to the limit, which is useful even for nonlinear problem. Numerical experiments show excellent accuracy when low order schemes are combined with polynomial extrapolation.

65. B. Neta, Analysis of Finite Element and Finite Differences for Shallow Water Equations: A Review, *Mathematics and Computers in Simulation*, **34**, (1992), 141–161.

    In this review article we discuss analyses of finite-element and finite-difference approximations of the shallow water equations. An extensive bibliography is given.

66. B. Neta, D. A. Danielson, Parallel Computing May Improve Space Surveillance, *ONR Naval Research Review*, **44**, (1992), 54.

67. B. Neta, R. Thanakij, Finite Element Approximation of the Shallow Water Equations on the MasPar, in Advances in Finite Element Analysis in Fluid Dynamics, (M.N. Dhaubhadel, M. S. Engelman and W. G. Habashi, eds), FED - Vol 171, Amer. Soc. Mech. Eng., N.Y., (1993) pp. 47-52.

    Here we report on development of a high order finite element code for the solution of the shallow water equations on the massively parallel computer MP-1104. We have compared the parallel code to the one available on the Amdahl serial computer. It is suggested that one uses a low order finite element to reap the benefit of the massive number of processors available.

68. B. Neta, Parallel Solution of Initial Value Problems, Proc. Fourth International Colloquium on Differential Equations, (D. Bainov, V. Covachev, A. Dishliev, eds) Vol. 2, pp. 19-42, Plovdiv, Bulgaria, August 18-23, (1993).

    This paper will be divided to three parts. In the first part we discuss the solution of linear ordinary differential systems of equations. Initial value as well as boundary value problems will be considered. The second part will cover nonlinear ordinary differential systems. In this case the algorithm is based on extrapolation and as such is efficient only on small parallel computers. We close with the method of lines and its use to solve partial differential equations on parallel computers.

69. W. E. Phipps, B. Neta, D. A. Danielson, Parallelization of the Naval Space Surveillance Satellite Motion Model, *Proc. 1993 Space Surveillance Workshop, M. I. T. Lincoln*

Laboratory, Lexington, MA, March 30- April 1, 1993, R. W. Miller, R. Sridharan, (eds), **2**, (1993), 71-79.

70. B. Neta, L. Lustman, Parallel Conservative Scheme for Solving the Shallow Water Equations, *Monthly Weather Review*, **121**, (1993), 305–309.

    To improve the simulation of nonlinear aspects of the flow over steep topography, a potential enstrophy and energy conserving finite difference scheme for the shallow water equations was derived by Arakawa and Lamb.

    Here a parallel algorithm is developed for the solution of these equations which is based on Arakawa and Lamb's scheme. It is shown that the efficiency of the scheme on an eight-node INTEL iPSC/2 hypercube is 81%. Forty mesh points in the x direction and 19 in the y direction were used in each subdomain.

71. A. Staniforth, R. T. Williams, B. Neta, Influences of Linear Depth Variations on Barotropic Kelvin and Poincaré Waves, *J. of the Atmospheric Sciences*, **50**, (1993), 929-940.

    Exact solutions to the linearized shallow-water equations in a channel with linear depth variation and a mean flow are obtained in terms of confluent hypergeometric functions. These solutions are the generalization to finite s (depth variation parameter) of the approximate solutions for infinitesimal s. The equations also respect an energy conservation principle (and the normal modes are neutrally stable) in contradistinction to those of previous studies. They are evaluated numerically for a range in s from $s = 0.1$ to $s = 1.95$, and the range of validity of previously derived approximate solutions is established. For small s the Kelvin and Poincaré solutions agree with those of Hyde, which were obtained by expanding in s. For finite s the solution differ significantly from the Hyde expansions, and the magnitude of the phase speed decreases as s increases. The Rossby wave phase speeds are close to those obtained when the depth is linearized although the difference increases with s. The eigenfunctions become more distorted as s increases so that the largest amplitude and the smallest scale occur near the shallowest boundary. The negative Kelvin wave has a very unusual behavior as s increases.

72. W. E. Phipps, B. Neta, D. A. Danielson, Parallelization of the Naval Space Surveillance Satellite Motion Model, *J. Astronautical Sciences*, **41**, (1993), 207-216.

    The Naval Space Surveillance Center (NAVSPASUR) uses an analytic satellite motion model based on the Brouwer - Lyddane theory to track objects orbiting the Earth. In this paper we develop several parallel algorithms based on this model. These have been implemented on the INTEL iPSC/2 hypercube multi-computer. The speed-up and efficiency of these algorithms will be obtained. We show that the best of these algorithms achieves 87% efficiency if one uses a 16-node hypercube.

73. B. Neta, R. Thanakij, Finite Element Approximation of the Shallow Water Equations on the MasPar, NPS–MA–93–014 Technical Report , Naval Postgraduate School, Monterey, CA, 1993.

Here we report on development of a high order finite element code for the solution of the shallow water equations on the massively parallel computer MP-1104. We have compared the parallel code to the one available on the Amdahl serial computer. It is suggested that one uses a low order finite element to reap the benefit of the massive number of processors available.

74. B. Neta, B. Mansager, Benefit of Sound Cueing in Comabt Simulation, NPS–MA–94-003 Technical Report ,Naval Postgraduate School, Monterey, CA, 1993.

    Here we report on the incorporation of a sound algorithm into Janus (A) combat simulation model and the inclusion of the Tactical Unmanned Ground Vehicle (TUGV) for sound acquisition. The benefit of TUGV will be demonstrated and some ideas for model improvements will be given.

75. B. Neta, D. A. Danielson, S. Ostrom, S. K. Brewer, Performance of Analytic Orbit Propagator on a Hypercube and a Workstation Cluster, Proceedings AIAA-AAS Astrodynamics Program, in Scottsdale, AZ 1-3 August, 1994, Paper Number AIAA 94-3706.

    In this paper we discuss the benefit of parallel computing in propagating orbits of objects. Several analytic methods are now in use operationally. We will discuss three such schemes. We demonstrate the benefit of parallelism by using an INTEL iPSC/2 hypercube and by using a cluster of Unix-based workstations running Parallel Virtual Machine (PVM). The software PVM allows a heterogeneous set of networked workstations to appear as a multicomputer.

    We will show that one can achieve near 100% efficiency on the hypercube.

76. D. A. Danielson, B. Neta, and L. W. Early, Semianalytic Satellite Theory (SST): Mathematical Algorithms, NPS–MA–94-001, Technical Report , Naval Postgraduate School, Monterey, CA, 1994.

    Modern space surveillance requires fast and accurate orbit predictions for myriads of objects in a broad range of Earth orbits. Conventional Special Perturbations orbit propagators, based on numerical integration of the osculating equations of motion, are accurate but extremely slow. Conventional General Perturbations orbit propagators, based on fully analytical theories like those of Brouwer, are faster but contain large errors due to inherent approximations in the theories. New orbit generators based on Semianalytic Satellite Theory have been able to approach the accuracy of Special Perturbations propagators and the speed of General Perturbations propagators. Semianalytic Satellite Theory has been originated by P. J. Cefola and his colleagues, but the theory is scattered throughout the published and unpublished literature. In this document the theory is simplified, assembled, unified and extended.

77. L. C. Stone, S. B. Shukla, B. Neta, Parallel Satellite Orbit Prediction Using a Workstation Cluster, *International J. Computer and Mathematics with Applications*, **28**, (1994), 1-8.

In this paper, the benefits of parallel computing using a workstation cluster are explored for satellite orbit prediction. Data and function decomposition techniques are used. Speedup and throughput are the performance metric studied.

The software employed for parallelization was the Parallel Virtual Machine (PVM) developed by the Oak Ridge National Laboratory. PVM enables a network of heterogeneous workstations to appear as a parallel multicomputer to the user programs.

A speedup of almost 6 was achieved when using 8 SUN workstations.

78. Daniel J. Fonte, Jr., Beny Neta, D. A. Danielson, Major W. R. Dyar, USMC, Comparison of Orbit Propagators in the Research and Development Goddard Trajectory Determination System (R & D GTDS). Part I: Simulated Data, Proceedings AAS-AIAA Astrodynamics Conference, Halifax, Nova Scotia, Canada, 14-17 August 1995, Paper Number AAS 95-431.

    This paper evaluates the performance of various orbit propagation theories for artificial earth satellites in different orbital regimes. Specifically, R&D GTDS's Cowell (numerical technique), DSST (semianalytical technique), SGP, SGP4, and Brouwer-Lyddane (analytic techniques) orbit propagators are compared for decaying circular (280 km altitude), low altitude circular (599 km altitude), high altitude circular (1336 km altitude), Molniya, and geosynchronous orbits. All test cases implement a one orbital period differential correction fit to simulated data derived from a Cowell truth trajectory. These fits are followed by a one orbital period predict with the DC solve-for vector. Trajectory comparisons are made with the Cowell "truth" trajectory over both the fit and predict spans. Computation time and RMS errors are used as comparison metrics. The Unix-based version of R&D GTDS (NPS SUN Sparc 10) is the test platform used in this analysis.

79. D. A. Danielson, C. P. Sagovac, B. Neta, and L. W. Early, Semianalytic Satellite Theory (SST): Mathematical Algorithms, NPS–MA–95-002, Technical Report , Naval Postgraduate School, Monterey, CA, 1995.

    Modern space surveillance requires fast and accurate orbit predictions for myriads of objects in a broad range of Earth orbits. Conventional Special Perturbations orbit propagators, based on numerical integration of the osculating equations of motion, are accurate but extremely slow. Conventional General Perturbations orbit propagators, based on fully analytical theories like those of Brouwer, are faster but contain large errors due to inherent approximations in the theories. New orbit generators based on Semianalytic Satellite Theory have been able to approach the accuracy of Special Perturbations propagators and the speed of General Perturbations propagators. Semianalytic Satellite Theory has been originated by P. J. Cefola and his colleagues, but the theory is scattered throughout the published and unpublished literature. In this document the theory is simplified, assembled, unified and extended.

80. F. X. Giraldo, and B. Neta, Finite Element Approximation of Large Air Pollution Problems I: Advection, NPS–MA–95–005 Technical Report, Naval Postgraduate School, Monterey, CA, 1995.

An Eulerian and semi-Lagrangian finite element methods for the solution of the two dimensional advection equation were developed. Bilinear rectangular elements were used. Linear stability analysis of the method is given. Software is available on WWW using URL address http://math.nps.navy.mil/∼bneta/.

81. C. P. Sagovac, D. A. Danielson, J. R. Clynch and B. Neta, Fast Interpolation for Global Positioning System (GPS) Satellite Orbits, NPS–MA–95–006, Technical Report, Naval Postgraduate School, Monterey, CA, 1995.

    In this report, we discuss and compare several methods for polynomial interpolation of Global Positioning System ephemeris data.

82. F. X. Giraldo, B. Neta and C. P. Katti, Parallel Solutions of Tridiagonal and Pentadiagonal Systems, NPS–MA–95–007, Technical Report, Naval Postgraduate School, Monterey, CA, 1995.

    We present an algorithm for the parallel solution of tridiagonal and pentadiagonal linear systems having nonzero elements at the top right and bottom left corners. Tridiagonal systems of this kind arise from the solution of two point boundary value problems with periodic boundary conditions. Pentadiagonal systems of this kind arise from e.g the approximation of the shallow water equations by the two-stage Galerkin method combined with a high accuracy compact approximation to the first derivative (Navon, 1983).

83. F. X. Giraldo and B. Neta, Software for the Staggered and Unstaggered Turkel-Zwas Schemes for the Shallow Water Equations on the Sphere, NPS–MA–95–008, Technical Report, Naval Postgraduate School, Monterey, CA, 1995.

    A linear analysis of the shallow water equations in spherical coordinates for the Turkel-Zwas explicit large time-step scheme was presented by Neta, Giraldo and Navon. This report presents the software developed to test the staggered as well as the unstaggered Turkel-Zwas scheme for the solution of the shallow water equations on the sphere.

84. B. Neta, and D. A. Danielson, R & D GTDS SST:
    Code Flowcharts and Input, NPS–MA–95-009, Technical Report, Naval Postgraduate School, Monterey, CA, 1995.

    This document describes the major capabilities of the Research and Development Goddard Trajectory Determination System (R&D GTDS) Code. Only the Ephemeris Generation and Differential Corrections are described in detail with several examples. This is a follow-on to our document collecting the mathematical algorithms used in R&D GTDS.

85. B. Neta, C. P. Sagovac, D. A. Danielson, and J. R. Clynch,Fast Interpolation for Global Positioning System (GPS) Satellite Orbits, AAS/AIAA Astrodynamics Specialist Conference, July 29-31, 1996, San Diego, Paper Number AIAA 96-3658.

    In this report, we discuss and compare several methods for polynomial interpolation of Global Positioning System ephemeris data. We show that the use of difference tables is more efficient than the method currently in use to sonstruct and evaluate the Lagrange polynomials

86. B. Neta, and D. Vallado, On Satellite Umbra/Penumbra Entry and Exit Positions, NPS–MA–96-005, Technical Report, Naval Postgraduate School, Monterey, CA, 1996.

    The problem of computing Earth satellite (in elliptical orbit) entry and exit positions through the Earth's umbra and penumbra is a problem dating from the earliest days of the space age, but it is still of the utmost importance to many space projects for thermal and power considerations (Mullins, 1991). It is also important for optical tracking of a satellite. To a lesser extent, the satellite external torque history and the sensor systems are influenced by the time the satellite spends in the Earth's shadow.

    The umbra is the conical total shadow projected from the Earth on the side opposite the sun. In this region, the intensity of the solar radiation is zero. The penumbra is the partial shadow between the umbra and the full-light region. In the penumbra, the light of the sun is only partially cut off by the Earth, and the intensity is between 0 and 1. All textbooks discussing the problem, even the recent work by Mullins (1991), suggest the use of a quartic equation analytic solution. Since the quartic is a result of squaring the equation of interest, one must check the 4 solutions and discard the spurious ones. In this note, we suggest to solve the original equation numerically. We will give a condition for the existence of a solution, discuss the initial guess for the iterative scheme and compare the complexity of the two schemes (ours versus the analytic solution of the quartic).

87. D. Cersovsky, E. Kleinschmidt, B. Mansager, B. Neta, Audio Detection Algorithms in Combat Simulations, *International J. Mathematical and Computer Modelling*, **23**, (1996), 65-72.

    Audio information concerning targets generally includes direction, frequencies and energy levels. One use of audio cueing is to use direction information to help determine where more sensitive visual detection and acquisition sensors should be directed. Generally, use of audio cueing will shorten times requires for visual detection, although there could be circumstances where the audio information is misleading and degrades visual performance. Audio signatures can also be useful for helping classify the emanating platform, as well as to provide estimates of its velocity.

    The Janus combat simulation is the premier high resolution model used by the Army and other agencies to conduct research. This model has a visual detection model which essentially incorporates algorithms as described by Hartman [3]. The model in its current form does not have any sound cueing capabilities. We have modified Janus combat simulation model to include the Tactical Unmanned Ground Vehicle (TUGV) for sound acquisition.

88. B. Neta, and J. B. Knorr, Running NEC4 on the Cray at N.P.S., *Applied Computational Electromagnetic Society Newsletter*, **11 No. 3**, (1996), 12-15.

NEC4 is the latest version of Numerical Electromagnetic Code developed at Lawrence Livermore National Laboratory to analyze electromagnetic responses of antennas and scaterrers. The code is based on the method of moments to solve integral equations.

Several routines had to be modified and parallelization of the factorization is implemented.

89. B. Neta, and D. Vallado, On Satellite Umbra/Penumbra Entry and Exit Positions, Proc. AAS/AIAA Space Flight Mechanics Meeting, Huntsville, AL, February 10-12, 1997, Paper Number AAS 97-155.

    The problem of computing Earth satellite (in elliptical orbit) entry and exit positions through the Earth's umbra and penumbra is a problem dating from the earliest days of the space age, but it is still of the utmost importance to many space projects for thermal and power considerations (Mullins, 1991). It is also important for optical tracking of a satellite. To a lesser extent, the satellite external torque history and the sensor systems are influenced by the time the satellite spends in the Earth's shadow.

    The umbra is the conical total shadow projected from the Earth on the side opposite the sun. In this region, the intensity of the solar radiation is zero. The penumbra is the partial shadow between the umbra and the full-light region. In the penumbra, the light of the sun is only partially cut off by the Earth, and the intensity is between 0 and 1. All textbooks discussing the problem, even the recent work by Mullins (1991), suggest the use of a quartic equation analytic solution. Since the quartic is a result of squaring the equation of interest, one must check the 4 solutions and discard the spurious ones. In this note, we suggest to solve the original equation numerically. We will give a condition for the existence of a solution, discuss the initial guess for the iterative scheme and compare the complexity of the two schemes (ours versus the analytic solution of the quartic).

90. F. X. Giraldo and B. Neta, A Comparison of a Family of Eulerian and Semi-Lagrangian Finite Element Methods for the Advection-Diffusion Equation, in *Computer Modelling of Seas and Coastal Regions III*, J. R. Acinas and C. A. Brebbia (eds), Computational Mechanics Publications, Southampton, U. K., (1997), 217-229.

    This paper compares a family of Eulerian and semi-Lagrangian methods. Numerical experiments are performed on the two-dimensional advection-diffusion equations having a known analytic solution. The numerical results demonstrate that the semi-Lagrangian method is superior to the Eulerian method while using time steps two to four times greater. This property makes them more attractive than Eulerian methods particularly for integrating atmospheric and ocean equations because long time histories are sought for such problems.

91. B. Neta, Parallel Version of Special Perturbations Orbit Propagator, *Proc. AAS/AIAA Astrodynamics Conference*, Sun Valley, ID, August 4-7, 1997, Paper Number AAS 97-688.

Parallelization can be achieved by either control or domain decomposition. The latter was tried for analytic (by Neta et al) and semianalytic (by Wallace) propagators. Neal and Coffey will discuss the domain decomposition for special perturbations. The control decomposition idea is inefficient for analytic propagators (Neta et al), because the computation time is too short. In this paper we discuss a control decomposition approach to parallelize a numerical orbit propagator which is more computationally intensive.

92. B. Neta, F. X. Giraldo and I. M. Navon, Analysis of the Turkel-Zwas Scheme for the Two-Dimensional Shallow Water Equations in Spherical Coordinates, *Journal of Computational Physics*, **133**, (1997), 102-112.

    A linear analysis of the shallow water equations in spherical coordinates for the Turkel-Zwas (T-Z) explicit large time-step scheme is presented. This paper complements the results of Schoenstadt, Neta and Navon and others in 1-D, and of Neta and DeVito in 2-D, but applied to the spherical coordinate case of the T-Z scheme. This coordinate system is more realistic in meteorology and more complicated to analyze, since the coefficients are no longer constant. The analysis suggests that the T-Z scheme must be staggered in a certain way in order to get eigenvalues and eigenfunctions approaching those of the continuous case. The importance of such an analysis is the fact that it is also valid for non-constant coefficients and thereby applicable to any numerical scheme. Numerical experiments comparing the original (unstaggered) and staggered versions of the T-Z scheme are presented. These experiments corroborate the analysis by showing the improvements in accuracy gained by staggering the Turkel-Zwas scheme.

93. B. Neta, Parallelization of Satellite Motion Models, it SIAM News, **30**, November 1997.

    This paper summarizes previous results on parallelization of analytic propagators.

94. B. Neta, and D. Vallado, On Satellite Umbra/Penumbra Entry and Exit Positions, *J. Astronautical Sciences*, **46**, (1998), 91-103.

    The problem of computing Earth satellite entry and exit positions through the Earth's umbra and penumbra, for satellites in elliptical orbits, is solved without the use of a quartic equation. A condition for existence of a solution in the case of a cylindrical shadow is given. This problem is of interest in case one would like to include perturbation force resulting from solar radiation pressure. Most satellites (including geosynchronous) experience periodic eclipses behind the Earth. Of course when the satellite is eclipsed, it's not exposed to solar radiation pressure. When we need extreme accuracy, we must develop models that turn the solar radiation calculations "on" and "off," as appropriate, to account for these periods of inactivity.

95. B. Neta, Trajectory Propagation Using Information on Periodicity, *Proc. AIAA/AAS Astrodynamics Specialist Conference*, Boston, MA, August 10-12, 1998, Paper Number AIAA 98-4577.

    Families of methods to integrate first and second order ordinary differential equations whose solution known to be periodic will be discussed. The methods can be tuned to a possibly a-priori knowledge of the user on the location of the frequencies, that are dominant in the exact solution. On the basis of such extra information the truncation error can considerably be reduced in magnitude. The paper compares these methods to well known integrators and discusses a simple mechanism to estimate the frequency during the integration process.

96. J. B. Knorr and B. Neta, Signal Processing for Aperture Antenna Beam Compression, NPS–EC–98-016, Technical Report, Naval Postgraduate School, Monterey, CA, 1998.

    This report presents the results of a preliminary investigation of the application of correlation signal processing to aperture antennas. Correlation is equivalent to matched filtering. In this application the result is spatial filtering. Processing a signal incident on an antenna in this way results in compression of the antenna beam solid angle and permits the angle of arrival of the incident signal to be determined. Thus, correlation signal processing can be employed to determine angle of arrival within the wider field of view of the antenna. Using this approach, one can simultaneously achieve both a wide instantaneous field of view and the ability to more precisely determine angle of arrival.

97. B. Neta, Parallel Version of Special Perturbations Orbit Propagator, NPS–MA–98-003, Technical Report Naval Postgraduate School, Monterey, CA, 1998.

98. F. X. Giraldo and B. Neta, Stability Analysis for Eulerian and Semi-Lagrangian Finite Element Formulation of the Advection-Diffusion Equation, *Computers and Mathematics with Applications*, **38**, (1999), 97-112.

    This paper analyzes the stability of the finite element approximation to the linearized two-dimensional advection-diffusion equation. Bilinear basis functions on rectangular elements are considered. This is one of the two best schemes as was shown by Neta and Williams. Time is discretized with the theta algorithms that yield the explicit ($\theta = 0$), semi-implicit ($\theta = \frac{1}{2}$), and implicit ($\theta = 1$) methods. This paper extends the results of Neta and Williams for the advection equation. Giraldo and Neta have numerically compared the Eulerian and semi-Lagrangian finite element approximation for the advection-diffusion equation. This paper analyzes the finite element schemes used there.

    The stability analysis shows that the semi-Lagrangian method is unconditionally stable for all values of θ while the Eulerian method is only unconditionally stable for $\frac{1}{2} \leq \theta \leq 1$. This analysis also shows that the best methods are the semi-implicit ones ($\theta = \frac{1}{2}$). In essence this paper analytically compares a semi-implicit Eulerian method with a semi-implicit semi-Lagrangian method. It is concluded that (for small or no diffusion) the semi-implicit semi-Lagrangian method exhibits better amplitude, dispersion and group velocity errors than the semi-implicit Eulerian method thereby achieving better results. In the case the diffusion coefficient is large, the semi-Lagrangian loses its competitiveness.

99. D. Mortari, and B. Neta, Optimal best fitting of numerical data: Part II, *Proc. AAS/AIAA Space Flight Mechanics Meeting*, Breckenridge, CO, February 7-10, 1999, Paper Number AAS 99-183.

    This paper presents the extension to both the trigonometric basis functions and the Legendre polynomials, of an alternative method which was presented previously and applied to m-degree polynomials as well as to exponential functions, to optimize the best fitting of numerical data. The proposed procedures are provided in a closed-form and therefore represent a good alternative solution tool to the QR decomposition, to tackle ill-conditioned best fitting problems.

100. J. R. Clynch, R. Franke and B. Neta, Improvements In Dynamic GPS Positions Using Track Averaging, NPS–MA–99-004, Technical Report, Naval Postgraduate School, Monterey, CA, 1999.

    The issue of improving a Global Positioning System (GPS), Precise Positioning System (PPS) solution under dynamic conditions through averaging is investigated. Static and Dynamic data from the Precision Lightweight GPS Receiver (PLGR) were used to analyze the error characteristics and deign an averaging technique for dynamic conditions.

    It was found that the errors in PPS solutions are dominated by the satellite broadcast ephemeris parameters. The solution errors are highly correlated for a given set of satellites/ephemeris. The variation can be as low as 0.4 m in dynamic conditions, but a slowly changing "bias" of several meters is also present.

    For fitting the location of a road observed repeatedly with a PPS receiver a technique based on "space curves" was developed. Here the solutions are transformed from functions of time to functions of space (location). These then are used. Curves could be fit with a Bezier polynomial easily to the 0.4 m level. These analytic curves were then used to form an ensemble average. The bias vectors between the solutions were found with least squares estimation. These vectors were averaged using several techniques. This idea was applied to a short rad segment. Using 9 independent measurements taken over 6 months, the road was surveyed at the submeter level.

101. J. H. Gordis, and B. Neta, Fast transient analysis for locally nonlinear structures by recursive block convolution, *Proc. 70th Shock and Vibration Symposium*, Albuquerque, NM, November 15-19, 1999.

    The transient analysis of large structural systems with localized nonlinearities is a computationally demanding process, inhibiting dynamic redesign and optimization. A previously developed integral equation formulation for transient structural synthesis has demonstrated the ability to solve large locally nonlinear transient problems in a fraction of the time required by traditional direct integration methods, with equivalent or better accuracy. A recursive block-by-block convolution algorithm is developed for the solution of the governing integral equations which farther reduces the solution times required. Examples using realistically-sized finite element models are presented, demonstrating the performance of the formulation.

102. J. H. Gordis, and B. Neta, Efficient Nonlinear Transient Dynamic Analysis for Structural Optimization Using an Exact Integral Equation Formulations, NPS–ME–99-009, Technical Report Naval Postgraduate School, Monterey, CA, 1999.

103. D. Mortari, and B. Neta, k-vector range searching techniques, *Proc. AAS/AIAA Space Flight Mechanics Meeting*, Clearwater, FL, January 23-26, 2000, Paper Number AAS 00-128.

Various k-vector range searching techniques are presented here. These methods accomplish the range search by taking advantage of an n-long vector of integers, called the k-vector, to minimize the search time. The price is increased memory requirement for the k-vector allocation. However, it is possible to balance the extra memory required and the speed attained by choosing a step parameter h which samples the k-vector. A two-level k-vector technique is also presented to minimize the speed of the admissible data identification associated with a given range. The proposed methods are compared with the well known "binary search" technique, and demonstrate a high speed gain rate (from 9 to more than 18 times). Finally, just to show one of the wide-range possible applications, a method to compute the *arcsin* function, based on the k-vector technique and a look-up table, is presented.

104. J. R. Clynch, R. Franke and B. Neta, Improvements In Dynamic GPS Positions Using Track Averaging, Proc. ION Tech. Meeting, January 26-28, 2000, Anaheim, CA.

The issue of improving a Global Positioning System (GPS), Precise Positioning System (PPS) solution under dynamic conditions through averaging is investigated. Static and Dynamic data from the Precision Lightweight GPS Receiver (PLGR) were used to analyze the error characteristics and deign an averaging technique for dynamic conditions.

It was found that the errors in PPS solutions are dominated by the satellite broadcast ephemeris parameters. The solution errors are highly correlated for a given set of satellites/ephemeris. The variation can be as low as 0.4 m in dynamic conditions, but a slowly changing "bias" of several meters is also present.

For fitting the location of a road observed repeatedly with a PPS receiver a technique based on "space curves" was developed. Here the solutions are transformed from functions of time to functions of space (location). These then are used. Curves could be fit with a Bezier polynomial easily to the 0.4 m level. These analytic curves were then used to form an ensemble average. The bias vectors between the solutions were found with least squares estimation. These vectors were averaged using several techniques. This idea was applied to a short rad segment. Using 9 independent measurements taken over 6 months, the road was surveyed at the submeter level.

105. B. Neta, T. Fukushima, Obrechkoff versus super-implicit methods for the integration of Keplerian orbits, Astrodynamics Specialist Conference, Denver, CO, 14-17 August 2000. Paper number AIAA-2000-4029.

This paper discusses the numerical solution of first order initial value problems and a special class of second order ones (those not containing first derivative). Two classes of methods are discussed, super-implicit and Obrechkoff. We will show equivalence of super-implicit and Obrechkoff schemes. The advantage of Obrechkoff methods is that they are high order one-step methods and thus will not require additional starting values. On the other hand they will require higher derivatives of the right hand side. In case the right hand side is complex, we may prefer super-implicit methods. The disadvantage of super-implicit methods is that they, in general, have a larger error constant. To get the same error constant we require one or more extra future values. We can use these extra values to increase the order of the method instead of decreasing the error constant.

106. J. Gordis and B. Neta, An adaptive method for the numerical solution of Volterra integral equations, *Recent Advances in Applied and Theoretical Mathematics*, N. Mastorakis, editor, World Scientific and Engineering Society International Conference, Athens, Greece, December 1-3, 2000, pp. 1-8.

    In this paper we introduce an adaptive method for the solution of Volterra integral equations of the second kind. We demonstrate the benefit of adaptivity and apply the idea to the nonlinear equation arises in transient structural synthesis.

107. C. A. Kluever, B. Neta, C. D. Hall, and J. M. Hanson, Spaceflight Mechanics 2000, Advances in the Astronautical Sciences, Vol. 105, Univelt, Inc., San Diego, 2000.

108. J. H. Gordis, and B. Neta, Fast transient analysis for locally nonlinear structures by recursive block convolution, *ASME J. Vibrations and Acoustics*, **123**, (2001) 545-547.

    The transient analysis of large structural systems with localized nonlinearities is a computationally demanding process, inhibiting dynamic redesign and optimization. A previously developed integral equation formulation for transient structural synthesis has demonstrated the ability to solve large locally nonlinear transient problems in a fraction of the time required by traditional direct integration methods, with equivalent or better accuracy. A recursive block-by-block convolution algorithm is developed for the solution of the governing integral equations which farther reduces the solution times required. Examples using realistically-sized finite element models are presented, demonstrating the performance of the formulation.

109. J. B. Knorr, and B. Neta, Plotting Circularly Polarized Field Patterns Using Processed NEC 4 Output Files, *Applied Computational Electromagnetic Society Newsletter*, **16, No. 2**, (2001), 26-33.

    NEC4 computes E_θ and E_ϕ, the linearly polarized components of the far field radiated by an antenna. Software packages such as GNEC, which are built around the NEC4 engine [Ref. 1], allow the user to plot far field patterns for E_θ and E_ϕ from the data in the NEC4 output file. When the far field of an antenna is elliptically polarized, the NEC4 output file indicates the sense of polarization, right or left hand. While this information is useful, antennas that radiate a field that is basically circularly polarized are better characterized by their right and left hand circularly polarized field patterns. The cavity backed spiral, for example, is an antenna that is used in applications where circular polarization is desired and one would prefer to see the far field patterns of the right and left hand components, E_{right} and E_{left}. This note describes a simple approach to plotting the patterns for E_{right} and E_{left} by processing the data in the standard NEC4 output file.

110. B. Neta, Y. Ilan-Lipowski, A new scheme for trajectory propagation, *Proc. AIAA/AAS Astrodynamics Specialist Conference*, Quebec City, Quebec, Canada, July 31 - August 3, 2001, Paper Number AAS 01-446.

    In this paper we develop a method for the solution of the equations of motion of an object acted upon by several gravitational masses. In general the motion can be described by a special class (for which y' is missing) of second order initial value poblems (IVPs)
    $$y''(x) = f(x, y(x)), \qquad y(0) = y_0,\, y'(0) = y'_0.$$
    The numerical integration methods for this can be divided into two distinct classes: (a) problems for which the solution period is known (even approximately) in advance; (b) problems for which the period is not known. Here we only consider some methods of the second class. Numerical methods of Runge-Kutta type as well as linear multistep methods can be found in the literature.

    Our idea here is to develop a new method that conserves the energy per unit mass in the case of perturbation-free flight and use the energy in other cases to approximate the angular variation. The generalization to cases were the energy is not conserved is given. We close with numerical experiments for both cases and compare the solution to well established methods.

111. J. G. Taylor and B. Neta, Support of JCATS Limited V&V, NPS–MA–01-001, Technical Report Naval Postgraduate School, Monterey, CA, 2001.

    The goal of this study effort was to assess the ability of the Joint Conflict and Tactical Simulation (JCATS) to simulate the capabilities of non-lethal weapons (NLW) and to provide a product that can be incorporated into the full VV&A of JCATS. This work investigated the first 32 algorithms on the JNLWD V&V Priority List. It evaluated JCATS algorithms in two ways:

    (a) verification of computer code against algorithm documentation,

    (b) appropriateness of algorithms within context of U.S. Army current model standards.

    All 32 algorithms were verified, with very few discrepancies with the documentation being found. Of these 32 algorithms, only 25 were documented already by LLNL in the JCATS Algorithm Manual so documentation for the remaining 7 was developed with the help of LLNL from documentation internal to the JCATS computer code. Evaluation of these algorithms (actually a subset of five or so key algorithms) within

the context of a compendium of algorithms developed for the Close Combat Tactical Trainer (CCTT) developed by AMSAA revealed that several key algorithms (particularly target acquisition) should be upgraded, if possible. This research also revealed a document that could be used to provide the theoretical basis of most of the AMSAA algorithms, particularly those for attrition. Such a document was never available to LLNL. Although some key algorithms should be upgraded (mainly because of modeling and simulation developments of the last five years or so), all JCATS algorithms (including its target-acquisition algorithm) were at one time more than adequate for analysis purposes. Moreover, overall the algorithms reviewed are deemed to be adequate (particularly in comparison with Janus Army) for playing close combat with non-lethal weapons in urban terrain for purposes of analysis. Further work (particularly along the lines of the issues raised by this work) is necessary, however, to document these modeling issues. Some research is required to better articulate the technical issues raised here, particularly if future V&V efforts are to build on the work at hand.

112. J. G. Taylor and B. Neta, Explicit Analytical Expression for a Lanchester Attrition-Rate Coefficient for Bonder and Farrell's m-Period Target-Engagement Policy, Working Group Paper # 5, Defense Threat Reduction Agency (DTRA) Project, MOVES Academic Group, Naval Postgraduate School, Monterey, CA, July 2001.

    The purpose of this working paper is to give an explicit analytical expression for a Lanchester-type attrition-rate coefficient for direct-fire combat in a heterogeneous-target environment with serial acquisition of targets for Bonder and Farrell's m-period target-acquisition policy.

113. I. M. Navon, B. Neta, M. Y. Hussaini, A Perfectly Matched Layer Formulation for the Nonlinear Shallow Water Equations Models: The Split Equation Approach, FSU–CSIT–01–41, Technical Report Florida State University, Tallahassee, FL, 2001.

    A limited-area model of nonlinear shallow water equations (SWE) with the Coriolis term in a rectangular domain is considered. The rectangular domain is extended to include the so-called perfectly matched layer (PML). Following the proponent of the original method, the PML equations are obtained by splitting the shallow water equations in the coordinate directions and introducing the damping terms. The efficacy of the PML boundary treatment is demonstrated in the case where a Gaussian pulse is initially introduced at the center of the rectangular physical domain. A systematic study is carried out for different mean convection speeds, and various values of the PML width and the damping coefficients. For the purpose of comparison, a reference solution is obtained on a fine grid on the extended domain with the characteristic boundary conditions. The L2 difference in the height field between the solution with the PML boundary treatment and the reference solution along a line at a downstream position in the interior domain is computed. The PML boundary treatment is found to yield better accuracy compared with both the characteristic boundary conditions and the Engquist-Majda absorbing boundary conditions on an identical grid.

114. B. Neta, S. Reich, and H. D. Victory, Galerkin Spectral Synthesis Methods for Diffusion

Equations with General Boundary Conditions, *Annals of Nuclear Energy*, **29**, (2002), 913-927.

An existence and uniqueness theory is developed for the energy dependent, steady state neutron diffusion equation with inhomogeneous oblique boundary conditions imposed. Also, a convergence theory is developed for the Galerkin Spectral Synthesis Approximations which arise when trial functions depending only on energy are utilized. The diffusion coefficient, the total and scattering cross-sectional data are all assumed to be both spatially and energy dependent. Interior interfaces defined by spatial discontinuities in the cross-section data are assumed present. Our estimates are in a Sobolev-type norm, and our results show that the spectral synthesis approximations are optimal in the sense of being of the same order as the error generated by the best approximation to the actual solution from the subspace to which the spectral synthesis approximations belong.

115. D. Givoli and B. Neta, High-Order Higdon Non-Reflecting Boundary Conditions for the Shallow Water Equations, NPS–MA–02-001, Technical Report Naval Postgraduate School, Monterey, CA, 2002.

     In this report we document the implementation of high order Higdon nonreflecting boundary conditions. We suggest a way to choose the parameters and demonstrate numerically the efficiency of our choice. The model we used is the shallow water equations and as a special case the Klein-Gordon equation. These equations are solved by the finite difference method. We comment on the use of finite elements and demonstrate a new, more efficient method. The case of curved boundary is discussed. We close with a list of topics for research.

116. D. Givoli and B. Neta, High-Order Non-Reflecting Boundary Conditions for Dispersive Wave Problems, Proceeding of the International Conference on Computational and Mathematical Methods in Science and Engineering, Alicante, Spain, 20-25 September 2002.

     Problems of linear time-dependent dispersive waves in an unbounded domain are considered. The infinite domain is truncated via an artificial boundary $\mathcal{B}$, and a high-order Non-Reflecting Boundary Condition (NRBC) is imposed on $\mathcal{B}$. Then the problem is solved by a Finite Difference scheme in the finite domain bounded by $\mathcal{B}$. The sequence of NRBCs proposed by Higdon is used. However, in contrast to the original low-order implementation of the Higdon conditions, a new scheme is devised which allows the easy use of a Higdon-type NRBC of *any* desired order. In addition, a procedure for the automatic choice of the parameters appearing in the NRBC is proposed. The performance of the scheme is demonstrated via a numerical example.

117. K. T. Alfriend, B. Neta, K. Luu, and C. A. H. Walker, Spaceflight Mechanics 2002, Advances in the Astronautical Sciences, Vol. 112, Univelt, Inc., San Diego, 2002.

118. B. Neta, T. Fukushima, Obrechkoff versus super-implicit methods for the solution of first and second order initial value problems, in special issue on Numerical Methods in

Physics, Chemistry and Engineering, *Computers and Mathematics with Applications*, **45**, (2003), 383-390.

This paper discusses the numerical solution of first order initial value problems and a special class of second order ones (those not containing first derivative). Two classes of methods are discussed, super-implicit and Obrechkoff. We will show equivalence of super-implicit and Obrechkoff schemes. The advantage of Obrechkoff methods is that they are high order one-step methods and thus will not require additional starting values. On the other hand they will require higher derivatives of the right hand side. In case the right hand side is complex, we may prefer super-implicit methods. The disadvantage of super-implicit methods is that they, in general, have a larger error constant. To get the same error constant we require one or more extra future values. We can use these extra values to increase the order of the method instead of decreasing the error constant.

119. B. Neta, Y. Lipowski, A New Scheme for Trajectory Propagation, *J. Astronautical Sciences*, **50**, (2003), 255–268.

In this paper we develop a new numerical method to integrate the equations of motion of a celestial body. The idea is to replace the differential equation for the fast moving component by an equation for the energy per unit mass. We use a simple first order explicit method for the approximation of the new system. It is shown that the radial error is much smaller than that of some numerical schemes. It will be of interest to have a more extensive comparison with state-of-the-art methods currently in use for long term trajectory propagation. The evaluation of energy is also more accurate than in other known schemes. This method also conserves the energy per unit mass in the case of perturbation-free flight. The idea can be extended to higher order methods and implicit schemes.

120. D. Givoli and B. Neta, High-Order NRBCs for Dispersive Waves, *Wave Motion*, **37**, (2003), 257-271.

Problems of linear time-dependent dispersive waves in an unbounded domain are considered. The infinite domain is truncated via an artificial boundary $\mathcal{B}$, and a high-order Non-Reflecting Boundary Condition (NRBC) is imposed on $\mathcal{B}$. Then the problem is solved by a Finite Difference scheme in the finite domain bounded by $\mathcal{B}$. The sequence of NRBCs proposed by Higdon is used. However, in contrast to the original low-order implementation of the Higdon conditions, a new scheme is devised which allows the easy use of a Higdon-type NRBC of *any* desired order. In addition, a procedure for the automatic choice of the parameters appearing in the NRBC is proposed. The performance of the scheme is demonstrated via numerical examples.

121. D. Givoli and B. Neta, High-Order Non-Reflecting Boundary Scheme for Time-Dependent waves, *J. Computational Physics*, **186**, (2003), 24–46.

A new non-reflecting boundary scheme is proposed for time-dependent wave problems in unbounded domains. The linear time-dependent wave equation, with or without a dispersive term, is considered outside of an obstacle or in a semi-infinite wave guide. The infinite domain is truncated via an artificial boundary $\mathcal{B}$, and a high-order Non-Reflecting Boundary Condition (NRBC) is imposed on $\mathcal{B}$. Then the problem is solved numerically in the finite domain bounded by $\mathcal{B}$. The new boundary scheme is based on a reformulation of the sequence of NRBCs proposed by Higdon. In contrast to the original formulation of the Higdon conditions, the scheme constructed here does not involve any high derivatives beyond second order. This is made possible by introducing special auxiliary variables on $\mathcal{B}$. As a result, the new NRBCs can easily be used up to *any* desired order. They can be incorporated in a finite element or a finite difference scheme; in the present paper the latter is used. The parameters appearing in the NRBC are chosen automatically via a special procedure. Numerical examples concerning a semi-infinite wave guide are used to demonstrate the performance of the new method.

122. D. Givoli, B. Neta, and Igor Patlashenko, Finite Element Analysis of Time-Dependent Semi-Infinite Wave-Guides with High-Order Boundary Treatment, *International Journal Numerical Methods in Engineering*, **58**, (2003), 1955–1983.

    A new Finite Element (FE) scheme is proposed for the solution of exterior time-dependent wave problems, in dispersive or non-dispersive media. The infinite domain is truncated via an artificial boundary $\mathcal{B}$, and a high-order Non-Reflecting Boundary Condition (NRBC) is developed and applied on $\mathcal{B}$. The new NRBC does not involve any high derivatives beyond second order, but its order of accuracy is as high as one desires. It involves some parameters which are chosen automatically as a pre-process. A C^0 semi-discrete FE formulation incorporating this NRBC is constructed for the problem in the finite domain bounded by $\mathcal{B}$. Augmented and split versions of this FE formulation are proposed. The semi-discrete system of equations is solved by the Newmark time-integration scheme. Numerical examples concerning dispersive waves in a semi-infinite wave guide are used to demonstrate the performance of the new method.

123. V. van Joolen, D. Givoli and B. Neta, High-Order Non-Reflecting Boundary Conditions for Dispersive Wave in Cartesian and Cylindrical Coordinate Systems, *International J. Computational Fluid Dynamics*, **17(4)**, (2003), 263–274.

    Among the many areas of research that Prof. Kawahara has been active in is the subject of *open boundaries* in which linear time-dependent dispersive waves are considered in an unbounded domain. The infinite domain is truncated via an artificial boundary $\mathcal{B}$ on which an Open Boundary Condition (OBC) is imposed. In this paper Higdon OBC's and Hagstrom-Hariharan OBC's are considered. Higdon-type conditions, originally implemented as low-order OBC's, are made accessible for any desired order via a new scheme. The higher-order Higdon OBC is then reformulated using auxiliary variables and made compatible for use with Finite Element (FE) methods. Methodology for selecting Higdon parameters are also proposed. The performance of these schemes are demonstrated in two numerical examples. This is followed by a discussion of the Hagstrom-Hariharan OBC, which is applicable to non-dispersive media on cylindrical and spherical geometries. The paper extends this OBC to the "slightly dispersive" case.

124. D. Givoli and B. Neta, High-Order Non-Reflecting Boundary Conditions for the Dispersive Shallow Water Equations, *J. Compuational Applied Mathematics*, **158**, (2003), 49–60.

Problems of linear time-dependent dispersive waves in an unbounded domain are considered. The infinite domain is truncated via an artificial boundary $\mathcal{B}$, and a high-order Non-Reflecting Boundary Condition (NRBC) is imposed on $\mathcal{B}$. Then the problem is solved by a Finite Difference scheme in the finite domain bounded by $\mathcal{B}$. The sequence of NRBCs proposed by Higdon is used. However, in contrast to the original low-order implementation of the Higdon conditions, a new scheme is devised which allows the easy use of a Higdon-type NRBC of *any* desired order. In addition, a procedure for the automatic choice of the parameters appearing in the NRBC is proposed. The performance of the scheme is demonstrated via a numerical example.

125. V. van Joolen, B. Neta and D. Givoli, High-Order Non-Reflecting Boundary Conditions for Dispersive Wave Problems in Stratified Media, Proceedings Sixth International Conference on Computer Modelling and Experimental Measurements of Seas and Coastal Regions, Cadiz, Spain, 23-25 June 2003, (*C.A. Brebbia, D. Almorza and F. López–Aguayo*, eds), pp. 73-82.

Problems of linear time-dependent dispersive waves in an unbounded domain are considered. The infinite domain is truncated via an artificial boundary $\mathcal{B}$. A high-order Non-Reflecting Boundary Condition (NRBC) is imposed on $\mathcal{B}$, and the problem is solved by a Finite Difference (FD) scheme in the finite domain. The sequence of NRBCs proposed by Higdon is used. However, in contrast to the original low-order implementation, a new scheme is devised which allows the easy use of a Higdon-type NRBC of *any* desired order. In addition, the problem is considered for a *stratified* media. The performance of the scheme is demonstrated via numerical example.

126. V. van Joolen, B. Neta, and D. Givoli, A Stratified Dispersive Wave Model with High-Order Non-Reflecting Boundary Conditions, *Computers and Mathematics with Applications*, **48**, (2004), 1167–1180.

A layered-model is introduced to approximate the effects of stratification on linearized shallow water equations. This time-dependent dispersive wave model is appropriate for describing geophysical (e.g. atmospheric or oceanic) dynamics. However, computational models that embrace these very large domains that are global in magnitude can quickly overwhelm computer capabilities. The domain is therefore truncated via artificial boundaries, and robust non-reflecting boundary conditions (NRBC) devised by Higdon are imposed. A scheme previously proposed by Neta and Givoli that easily discretizes high-order Higdon NRBC's is used. The problem is solved by Finite Difference (FD) methods. Numerical examples follow the discussion.

127. I.M. Navon, B. Neta, M.Y. Hussaini, A perfectly matched layer approach to the linearized shallow water equations models, *Monthly Weather Review*, **132 No.6**, (2004), 1369 – 1378.

A limited-area model of linearized shallow water equations (SWE) on an f-plane for a rectangular domain is considered. The rectangular domain is extended to include the so-called perfectly matched layer (PML) as an absorbing boundary condition. Following the proponent of the original method, the equations are obtained in this layer by splitting the shallow water equations in the coordinate directions and introducing the absorption coefficients. The performance of the PML as an absorbing boundary treatment is demonstrated using a commonly employed bell-shaped Gaussian initially introduced at the center of the rectangular physical domain.

Three typical cases are studied:

- A stationary Gaussian where adjustment waves radiate out of the area.
- A geostrophically balanced disturbance being advected through the boundary parallel to the PML. This advective case has an analytical solution allowing us to compare forecasts.
- The same bell being advected at an angle of 45 degrees so that it leaves the domain through a corner.

For the purpose of comparison, a reference solution is obtained on a fine grid on the extended domain with the characteristic boundary conditions. We also compute the r.m.s. difference between the 48-hour forecast and the analytical solution as well as the 48-hour evolution of the mean absolute divergence which is related to geostrophic balance. We found that the PML equations for the linearized shallow water equations on an f-plane support unstable solutions when the mean flow is not unidirectional. Use of a damping term consisting of a 9-point smoother added to the discretized PML equations stabilizes the PML equations. The reflection/transmission is analyzed along with the case of instability for glancing propagation of the bell disturbance. A numerical illustration is provided showing that the stabilized PML for glancing bell propagation performs well with the addition of the damping term.

128. V. van Joolen, B. Neta, and D. Givoli, High-Order Boundary Conditions for Linearized Shallow Water Equations with Stratification, Dispersion and Advection, *International J. Numerical Methods in Fluids*, **46(4)**, (2004), 361–381.

The two-dimensional linearized shallow water equations are considered in unbounded domains with density stratification. Wave dispersion and advection effects are also taken into account. The infinite domain is truncated via a rectangular artificial boundary $\mathcal{B}$, and a high-order Open Boundary Condition (OBC) is imposed on $\mathcal{B}$. Then the problem is solved numerically in the finite domain bounded by $\mathcal{B}$. A recently developed boundary scheme is employed, which is based on a reformulation of the sequence of OBCs originally proposed by Higdon. The OBCs can easily be used up to *any* desired order. They are incorporated here in a finite difference scheme. Numerical examples are used to demonstrate the performance and advantages of the computational method, with an emphasis on the effect of stratification.

129. J.G. Taylor, B. Neta, and P.A. Shugart, An Analytical Model That Provides Insights into Various C^2 Issues, Proceedings of the 2004 Command and Control Research and Technology Symposium, to appear.

This paper develops an analytical model that can very simply provide important insights into the consequences (in terms of combat outcomes) generated by different C2 architectures for information processing. A Lanchester-type model of force-on-force combat that reflects C2 architecture at the platform level is developed through detailed analysis of the target-engagement cycle for a single typical firer in modern tank combat. The most significant new aspect of this model is the con-sideration of so-called parallel acquisition of targets, i.e. new targets can be acquired while a previ-ously acquired target is being engaged. Computational results are given that show that being able to effect parallel acquisition of targets can not only significantly increase a tank force's infliction of casualties on an enemy tank force, but also significantly reduce the number of casualties that are suf-fered. The model given here is developed by use of Taylor's new methodology for Lanchester attri-tion-rate coefficients under conditions of stochastic line of sight. This methodology allows one to play significantly more micro-combat detail than has ever been possible in Lanchester-type models. Hence, it has opened up new vistas for the mathematical modeling of force-on-force combat.

130. D. Givoli, B. Neta, and V.J. van Joolen, Application of Higdon non-reflecting boundary conditions to shallow water models, in Proceedings ICOSAHOM 2004, Brown University, Providence, RI, 21-25 June (T. Hagstrom and T. Warburton)

A shallow water model with linear time-dependent dispersive waves in an unbounded domain is considered. The domain is truncated with artificial boundaries B where a sequence of high-order non-reflecting boundary conditions (NRBCs) proposed by Higdon are applied. Methods devised by Givoli and Neta that afford easy implementation of Higdon NRBCs are refined in order to reduce computational expenses. The new refinement makes the computational effort associated with the boundary tratment quadratic rather than exponential (as in the original scheme) with the order. This allows for implementations of NRBCs of higher orders than previously. A numerical example for a semi-infinite channel truncated on one side is presented. Finite difference schemes are used throughout.

131. V. van Joolen, B. Neta, and D. Givoli, High-Order Higdon-Like Boundary Conditions for Exterior Transient Wave Problems, *International Journal Numerical Methods in Engineering*, **63**, (2005), 1041–1068.

Recently developed non-reflecting boundary conditions are applied for exterior time-dependent wave problems in unbounded domains. The linear time-dependent wave equation, with or without a dispersive term, is considered in an infinite domain. The infinite domain is truncated via an artificial boundary $\mathcal{B}$, and a high-order Non-Reflecting Boundary Condition (NRBC) is imposed on $\mathcal{B}$. Then the problem is solved numerically in the finite domain bounded by $\mathcal{B}$. The new boundary scheme is based on a reformulation of the sequence of NRBCs proposed by Higdon. We consider here two reformulations: one that involves high-order derivatives with a special discretization scheme, and another that does not involve any high derivatives beyond second order. The latter formulation is made possible by introducing special auxiliary variables on $\mathcal{B}$. In both formulations the new NRBCs can easily be used up to *any* desired order. They can be incorporated in a finite element or a finite difference scheme; in the present paper the latter is used. In contrast to previous papers using similar formulations, here the method is applied to a fully exterior two-dimensional problem, with a rectangular boundary. Potential difficulties with corner instability are shown not to arise. Numerical examples in infinite domains are used to demonstrate the performance and advantages of the new method.

132. B. Neta, P-stable Symmetric Super-Implicit Methods for Periodic Initial Value Problems, *Computers and Mathematics with Applications*, **50**, (2005), 701–705.

The idea of super-implicit methods (requiring not just past and present but also future values) was suggested by Fukushima recently. Here we construct P-stable super-implicit methods for the solution of second order initial value problems. The benefit of such methods is realized when using vector or parallel computers.

133. B. Neta, Variational Data Assimilation and Optimal Control, Introduction to a special issue of *Computers and Mathematics with Applications*, **52 (8-9)**, (2006), xiii – xv.

134. B. Neta, Variational Data Assimilation and Optimal Control, Editorial for a special issue of *Computers and Mathematics with Applications*, **52 (8-9)**, (2006) , xvii – xxi.

135. B. Neta, P-stable High Order Super-Implicit and Obrechkoff Methods for Periodic Initial Value Problems, *Computers and Mathematics with Applications*, **54**, (2007), 117 – 126.

This paper discusses the numerical solution of periodic initial value problems. Two classes of methods are discussed, super-implicit and Obrechkoff. The advantage of Obrechkoff methods is that they are high order one-step methods and thus will not require additional starting values. On the other hand they will require higher derivatives of the right hand side. In case the right hand side is very complex, we may prefer super-implicit methods. We develop super-implicit P-stable method of order 12 and Obrechkoff method of order 18.

136. J. R. Dea, F. X. Giraldo, B. Neta, High-Order Higdon Non-Reflecting Boundary Conditions for the Linearized Euler Equations, NPS–MA–07-001, Technical Report Naval Postgraduate School, Monterey, CA, 2007.

In this report we document the implementation of high-order Higdon nonreflecting boundary conditions. We suggest a way to choose the parameters and demonstrate numerically the efficiency of our choice. The model we used is the linearized 2-D Euler equations with zero advection. These equations are solved by the finite difference method. We close with a list of topics for research.

137. B. Neta, V. J. vanJoolen, J. R. Dea, and D. Givoli, Application of High-Order Higdon Non-Reflecting Boundary Conditions to Linear Shallow Water Models, *Communications in Numerical Methods in Engineering*, **24**, (2008), 1459-1466. doi:10.1002/cnm.1044.

A shallow water model with linear time-dependent dispersive waves in an unbounded domain is considered. The domain is truncated with artificial boundaries $\mathcal{B}$ where a sequence of high-order non-reflecting boundary conditions (NRBCs) proposed by Higdon are applied. Methods devised by Givoli and Neta that afford easy implementation of Higdon NRBCs are refined in order to reduce computational expenses. The new refinement makes the computational effort associated with the boundary treatment quadratic rather than exponential (as in the original scheme) with the order. This allows for implementation of NRBCs of higher orders than previously. A numerical example for a semi-infinite channel truncated on one side is presented. Finite difference schemes are used throughout.

138. B. Neta and A. N. Johnson, High Order Nonlinear Solver for Multiple Roots, *Computers and Mathematics with Applications*, **55**, (2008), 2012-2017.

A method of order four for finding multiple zeros of nonlinear functions is developed. The method is based on Jarratt's fifth order method (for simple roots) and it requires one evaluation of the function and three evaluations of the derivative. The informational efficiency of the method is the same as previously developed schemes of lower order. For the special case of double root, we found a family of fourth order methods requiring one less derivative. Thus this family is more efficient than all others. All these methods require the knowledge of the multiplicity.

139. B. Neta, On Popovski's method for nonlinear equations, *Applied Mathematics and Computation*, **201**, (2008), 710-715, doi:10.1016/j.amc.2008.01.012.

Two different modifications of Popovski's method are developed, both are free of second derivatives. In the first modified scheme we traded the second derivative by an additional function evaluation. In the second method we replaced the second derivative by a finite difference and thus reducing the order slightly and reducing the number of evaluations per step by one. Therefore the second modification is more efficient.

140. B. Neta and A. N. Johnson, High order nonlinear solver, *J. Computational Methods in Science and Engineering*, **8 No. 4-6**, (2008), 245–250.

An eighth order method for finding simple zeros of nonlinear functions is developed. The method requires two function- and three derivative-evaluation per step. If we define informational efficiency of a method as the order per function evaluation, we find that our method has informational efficiency of 1.6.

141. B. Neta, New Third Order Nonlinear Solvers for Multiple Roots, *Applied Mathematics and Computation*, **202**, (2008), 162-170, doi:10.1016/j.amc.2008.01.031.

Two third order methods for finding multiple zeros of nonlinear functions are developed. One method is based on Chebyshev's third order scheme (for simple roots) and the other is a family based on a variant of Chebyshev's which doesn't require the second derivative. Two other more efficient methods of lower order are also given. These last two methods are variants of Chebyshev's and Osada's schemes. The informational efficiency of the methods is discussed. All these methods require the knowledge of the multiplicity.

142. C. Chun and B. Neta, Some modification of Newton's method by the method of undetermined coefficients, *Computers and Mathematics with Applications*, **56**, (2008), 2528–2538, doi:10.1016/j.camwa.2008.05.005.

     In this paper, we construct some modifications of Newton's method for solving nonlinear equations, which is based on the method of undetermined coefficients. It is shown by way of illustration that the method of undetermined coefficients is a promising tool for developing new methods, and reveals its wide applicability by obtaining some existing methods as special cases. Two new sixth order methods are developed. Numerical examples are given to support that the methods thus obtained can compete with other iterative methods.

143. John H. Seago, B. Neta, Thomas J. Eller, and Frederic J. Pelletier, Spaceflight Mechanics 2008, Advances in the Astronautical Sciences, Vol. 130, Univelt, Inc., San Diego, 2008.

144. C. Chun and B. Neta, A third-order modification of Newton's method for multiple roots, *Applied Mathematics and Computation*, **211**, (2009), 474-479.

     In this paper, we present a new third-order modification of Newton's method for multiple roots, which is based on existing third-order multiple root-finding methods. Numerical examples show that the new method are competitive to other methods for multiple roots.

145. T. Jangveladze, Z. Kiguradze, B. Neta, Large time behavior of solutions and finite difference scheme to a nonlinear integro-differential equation, *Computers and Mathematics with Applications*, **57**, (2009), 799–811.

     The large-time behavior of solutions and finite difference approximations of the nonlinear integro-differential equation associated with the penetration of a magnetic field into a substance are studied. Asymptotic properties of solutions for the initial-boundary value problem with homogeneous Dirichlet boundary conditions is considered. The rates of convergence are given too. The convergence of the semidiscrete and the finite difference schemes are also proved.

146. J. R. Dea, F. X. Giraldo, B. Neta, High-Order Non-Reflecting Boundary Conditions for the Linearized 2-D Euler Equations: No Mean Flow Case, *Wave Motion*, **46**, (2009), 210–220, doi:10.1016/j.wavemoti.2008.11.002.

     Higdon-type non-reflecting boundary conditions (NRBCs) are developed for the 2-D linearized Euler equations with Coriolis forces. This implementation is applied to a simplified form of the equations, with the NRBCs applied to two (opposing) sides of the domain. We demonstrate the validity of the NRBCs to high order and consider their long-time stability. We close with a list of areas for further research.

147. T. Jangveladze, Z. Kiguradze, B. Neta, Large Time Behavior of Solutions to a Nonlinear Integro-Differential System, *J. Math. Anal. Appl.*, **351**, (2009), 382-391.

 Asymptotic behavior of solutions as $t \to \infty$ to the nonlinear integro-differential system associated with the penetration of a magnetic field into a substance is studied. Initial-boundary value problems with two kind of boundary data are considered. The first with homogeneous conditions on whole boundary and the second with nonhomogeneous boundary data on one side of lateral boundary. The rates of convergence are given too.

148. C. Chun, H. J. Bae, B. Neta, New families of nonlinear third-order solvers for finding multiple roots, *Computers and Mathematics with Applications*, **57**, (2009), 1574–1582, doi: 10.1016/j.camwa.2008.10.070.

 In this paper, we present two new families of iterative methods for multiple roots of nonlinear equations. One of the families require one-function and two-derivative evaluation per step and the other family requires two-function and one-derivative evaluation. It is shown that both are third-order convergent for multiple roots. Numerical examples suggest that each family member can be competitive to other third-order methods and Newton's method for multiple roots. In fact the second family Is even better than the first.

149. T. Jangveladze, Z. Kiguradze, B. Neta, Finite Difference Approximation of a Nonlinear Integro-Differential System, *Applied Mathematics and Computation*, **215**, (2009), 615-628, doi: 10.1016/j.amc.2009.05.061.

 Finite difference approximation of the nonlinear integro-differential system associated with the penetration of a magnetic field into a substance is studied. The convergence of the finite difference scheme is proved. The rate of convergence of the discrete scheme is given. The decay of the numerical solution is compared with the analytical results proven earlier.

150. C. Chun and B. Neta, Certain improvements of Newton's method with fourth-order convergence, *Appl. Math. Comp.*, **215**, (2009), 821-828.

 In this paper we present two new schemes, one is third-order and the other is fourth-order. These are improvements of second order methods for solving nonlinear equations and are based on the method of undetermined coefficients. We show that the fourth-order method is more efficient than the fifth-order method due to Kou et al [J. Kou, Y. Li, X. Wang, Some modifications of Newton's method with fifth-order covergence, J. Comput. Appl. Math., **209**, (2007), 146-152.] Numerical examples are given to support that the methods thus obtained can compete with other iterative methods.

151. B. Neta, J. M. Lindquist, F. X. Giraldo, Analytic Solutions of the Klein-Gordon Equation in a Semi-infinite Channel, NPS-MA-009-002, Technical Report, Naval Postgraduate School, Department of Applied Mathematics, Monterey, CA 93943.

In this report we show how to construct analytic solutions of the Klein-Gordon equation in a semi-infinite channel. The Klein-Gordon equation can be derived from the shallow water equations. The analytic solutions are given for various choices of initial and boundary conditions.

152. B. Neta, Extension of Murakami's High Order Nonlinear Solver to Multiple Roots, *International Journal of Computer Mathematics*, **8**, (2010), 1023–1031, DOI: 10.1080/0020716080227226

Several fourth order methods for finding multiple zeros of nonlinear functions are developed. The methods are based on Murakami's fifth order method (for simple roots) and they require one evaluation of the function and three evaluations of the derivative. The informational efficiency of the methods is the same as previously developed methods of lower order. All these methods require the knowledge of the multiplicity.

153. J. M. Lindquist, B. Neta, F. X. Giraldo, A spectral element solution of the Klein-Gordon equation with high-order treatment of time and non-reflecting boundary, *Wave Motion*, **47**, (2010) 289–298, doi:10.1016/j.wavemoti.2009.11.007.

A spectral element (SE) method is proposed for the solution of the Klein-Gordon equation. The infinite domain is truncated via an artificial boundary $\mathcal{B}$, and a high-order Non-Reflecting Boundary Condition (NRBC) is developed and applied on $\mathcal{B}$. The new NRBC does not involve any high derivatives, but its order of accuracy is as high as one desires. Numerical examples, in various configurations, concerning the propagation of a pressure pulse are used to demonstrate the performance of the new method. Effects of time integration techniques and long term results are discussed. Specifically, we show that in order to achieve the full benefits of high-order accuracy requires balancing all errors involved; this includes the order of accuracy of the spatial discretization method, time-integrators, and boundary conditions.

154. T. Jangveladze, Z. Kiguradze, B. Neta, Large time asymptotic and numerical solution of a nonlinear diffusion model with memory, *Computers and Mathematics with Applications*, **59**, (2010), 254–273, doi: 10.1016/j.camwa.2009.07.052.

Large time behavior of solutions and finite difference approximation of a nonlinear system of integro-differential equations associated with the penetration of a magnetic field into a substance is studied. Two initial-boundary value problems are investigated. The first with homogeneous conditions on whole boundary and the second with non-homogeneous boundary data on one side of lateral boundary. The rates of convergence are given too. Mathematical results presented show that there is a difference between stabilization rates of solutions with homogeneous and nonhomogeneous boundary conditions. The convergence of the corresponding finite difference scheme is also proved. The decay of the numerical solution is compared with the analytical results.

155. S. G. Li, L. Z. Cheng, B. Neta, Some fourth-order nonlinear solvers with closed formulae for multiple roots, *Computers and Mathematics with Applications*, **59**, (2010), 126–135, doi: 10.1016/j.camwa.2009.08.066.

In this paper, we present six new fourth-order methods with closed formulae for finding multiple roots of nonlinear equations. The first four of them require one-function and three-derivative evaluation per iteration. The last two require one-function and two-derivative evaluation per iteration. Several numerical examples are given to show the performance of the presented methods compared with some known methods.

156. J. M. Lindquist, F. X. Giraldo, B. Neta, Klein-Gordon equation with advection on unbounded domains using spectral elements and high-order non-reflecting boundary conditions, *Appl. Math. Appl.*, **217**, (2010), 2710–2723.

A reduced shallow water model under constant, non-zero advection in the infinite channel is considered. High-order (Givoli-Neta) non-reflecting boundary conditions are introduced in various configurations to create a finite computational space and solved using a spectral element formulation with high-order time integration. Numerical examples are used to demonstrate the synergy of using high-order spatial, time, and boundary discretization. We show that by balancing all numerical errors involved, high-order accuracy can be achieved for unbounded domain problems.

157. B. Neta, M. S. Petković, Construction of optimal order nonlinear solvers using inverse interpolation, *Applied Mathematics and Computation*, **217** (2010), 2448-2455.

There is a vast literature on finding simple roots of nonlinear equations by iterative methods. These methods can be classified by order, by the information used or by efficiency. There are very few optimal methods, that is methods of order 2^m requiring $m + 1$ function evaluations per iteration. Here we give a general way to construct such methods by using inverse interpolation and any optimal two-point method. The presented optimal multipoint methods are tested on numerical examples and compared to existing methods of the same order of convergence.

158. C. Chun, B. Neta, P. Stanica, Recurrence relations for a third-order family of methods in Banach spaces, *Computers and Mathematics with Applications*, **61**, (2011), 1665-1675.

Recently, Parida and Gupta [J. Comp. Appl. Math. **206** (2007), 873-877] used Rall's recurrence relations approach (from 1961) to approximate roots of nonlinear equations, by developing several methods, the latest of which is free of second derivative and it is of third order. In this paper, we use an idea of Kou and Li [Appl. Math. Comp. **187** (2007), 1027-1032] and modify the approach of Parida and Gupta, obtaining yet another third-order method to approximate a solution of a nonlinear equation in a Banach space. We give several applications to our method.

159. Temur Jangveladze, Zurab Kiguradze, Beny Neta, Galerkin Finite Element Method for One Nonlinear Integro-Differential Mode, *Applied Mathematics and Computation*, **217**, (2011), 6883-6892, doi:10.1016/j.amc.2011.01.053.

Galerkin finite element method for the approximation of a nonlinear integro-differential equation associated with the penetration of a magnetic field into a substance is studied. First type initial-boundary value problem is investigated. The convergence of the finite element scheme is proved. The rate of convergence is given too. The decay of the numerical solution is compared with the analytical results.

160. M. S. Petković, J. Džunić, B. Neta, Interpolatory multipoint methods with memory for solving nonlinear equations, *Applied Mathematics and Computation*, 218, (2011), 2533–2541 .

A general way to construct multipoint methods for solving nonlinear equations by using inverse interpolation is presented. The proposed methods belong to the class of multipoint methods with memory. In particular, a new two-point method with memory with the order $(5 + \sqrt{17})/2 \approx 4.562$ is derived. Computational efficiency of the presented methods is analyzed and their comparison with existing methods with and without memory is performed on numerical examples. It is shown that a special choice of initial approximations provides a considerably great accuracy of root approximations obtained by the proposed interpolatory iterative methods.

161. M. Scott, B. Neta, C. Chun, Basin attractors for various methods, *Applied Mathematics and Computation*, 218, (2011), 2584–2599 , DOI 10.1016/j.amc.2011.07.076.

There are many methods for the solution of a nonlinear algebraic equation. The methods are classified by the order, informational efficiency and efficiency index. Here we consider other criteria, namely the basin of attraction of the method and its dependence on the order. We discuss several methods of various orders and present the basin of attraction for several examples. It can be seen that not all higher order methods were created equal. Newton's, Halley's, Murakami's and Neta-Johnson's methods are consistently better than the others. In two of the examples Neta's sixteenth order scheme was also as good.

162. C. Chun, B. Neta, A new sixth-order scheme for nonlinear equations, *Applied Mathematics Letters*, 25, (2012) , 185–189 , doi:10.1016/j.aml.2011.08.012.

In this paper we present a new efficient sixth-order scheme for nonlinear equations. The method is compared to several members of the family of methods developed by Neta (Intern. J. Computer Math. **7**, (1979), 157-161). It is shown that the new method is an improvement over this well known scheme.

163. J. M. Lindquist, B. Neta, F. X. Giraldo, High-order Non-reflecting Boundary Conditions for Dispersive Waves in Polar Coordinates Using Spectral Elements,*Applied Mathematics and Computation*, **218**, (2012), 6666–6676 .

High-order non-reflecting boundary conditions are introduced to create a finite computational space and for the solution of dispersive waves using a spectral element formulation with high-order time integration. Numerical examples are used to demonstrate the synergy of using high-order spatial, time, and boundary discretization. We show that by balancing all numerical errors involved, high-order accuracy can be achieved for unbounded domain problems in polar coordinate systems.

164. C. Chun, M. Y. Lee, B. Neta, J. Džunić, On optimal fourth-order iterative methods free from second derivative and their dynamics, *Applied Mathematics and Computation*, **218**, (2012), 6427–6438.

In this paper new fourth order optimal root-finding methods for solving nonlinear equations are presented. The classical Jarratt's family of fourth-order methods are obtained as special cases. We then present results which describe the conjugacy classes and dynamics of the presented optimal method for complex polynomials of degree two and three. The basins of attraction of existing optimal methods and our method are presented and compared to illustrate their performance.

165. B. Neta, M. Scott, C. Chun, Basin attractors for various methods for multiple roots, *Applied Mathematics and Computation*, **218**, (2012) , 5043–5066, doi:10.1016/j.amc.2011.10.071.

There are several methods for approximating the multiple zeros of a nonlinear function when the multiplicity is known. The methods are classified by the order, informational efficiency and efficiency index. Here we consider other criteria, namely the basin of attraction of the method and its dependence on the order. We discuss all known methods of orders two to four and present the basin of attraction for several examples.

166. B. Neta, C. Chun, M. Scott, A Note on the modified super-Halley method, *Applied Mathematics and Computation*, **218**, (2012), 9575-9577.

There are many methods for solving nonlinear algebraic equations. Some of these methods are just rediscovered old ones. In this note we show that the modified super Halley scheme is the same as one of Jarratt's methods.

167. B. Neta, M. Scott, C. Chun, Basins of attraction for several methods to find simple roots of nonlinear equations, *Applied Mathematics and Computation*, **218**, (2012), 10548–10556.

There are many methods for solving a nonlinear algebraic equation. The methods are classified by the order, informational efficiency and efficiency index. Here we consider other criteria, namely the basin of attraction of the method and its dependence on the order. We discuss several third and fourth order methods to find simple zeros. The relationship between the basins of attraction and the corresponding conjugacy maps will be discussed in numerical experiments. The effect of the extraneous roots on the basins is also discussed.

168. T. Jangveladze, Z. Kiguradze, B. Neta, S. Reich, Finite element approximation of a nonlinear diffusion model with memory, *Numerical Algorithms*, **64**, (2013), 127–155.

The convergence of a finite element scheme approximating a nonlinear system of integro-differential equations is proven. This system arises in mathematical modeling of the process of a magnetic field penetrating into a substance. The decay of the numerical solution is compared with both the analytical and finite difference results.

169. B. Neta, M. Scott, On a family of Halley-like methods to find simple roots of nonlinear equations, *Applied Mathematics and Computation*, **219**, (2013), 7940-7944.

There are many methods for solving a nonlinear algebraic equation. Here we introduce a family of Halley-like methods and show that EulerChebyshev and BSC are just members of the family. We discuss the conjugacy maps and the effect of the extraneous roots on the basins of attraction.

170. B. Neta, C. Chun, On a family of Laguerre methods to find multiple roots of nonlinear equations, *Applied Mathematics and Computation*, **219**, (2013), 10987–11004.

    There are several methods for solving a nonlinear algebraic equation having roots of a given multiplicity m. Here we compare a family of Laguerre methods of order three as well as two others of the same order and show that Euler-Cauchy's method is best. We discuss the conjugacy maps and the effect of the extraneous roots on the basins of attraction.

171. B. Neta, C. Chun, M. Scott, On the development of iterative methods for Multiple Roots, *Applied Mathematics and Computation*, **223**, (2013), 358–361.

    There are very few optimal fourth order methods for solving nonlinear algebraic equations having roots of multiplicity m. Here we compare 4 such methods, two of which require the evaluation of the $(m-1)^{st}$ root. We will show that such computation does not affect the overall cost of the method.

172. M. S. Petković, B. Neta, L. D. Petković, J. Džunić, Multipoint methods for solving nonlinear equations: a survey, *Applied Mathematics and Computation*, **226**, (2014), 635–640.

    Multipoint iterative methods belong to the class of the most efficient methods for solving nonlinear equations. Recently interest in the research and development of this type of methods has arisen from their capability to overcome theoretical limits of one-point methods concerning the convergence order and computational efficiency. This survey paper is a mixture of theoretical results and algorithmic aspect and it is intended as a review of the most efficient root-finding algorithms and developing techniques in a general sense. Many existing methods of great efficiency appear as special cases of presented general iterative schemes. Special attention is devoted to multipoint methods with memory that use already computed information to considerably increase convergence rate without additional computational costs. Some classical results of the 1970s that have had a great influence to the topic, often neglected or unknown to many readers, are also included not only as historical notes but also as genuine sources of many recent ideas. To a certain degree, the presented study follows in parallel main themes shown in the recently published book *Multipoint Methods for Solving Nonlinear Equations* (Elsevier/Academic Press, 2013), written by the authors of this paper.

173. C. Chun, B. Neta, J. Kozdon, M. Scott, Choosing weight functions in iterative methods for simple roots, *Applied Mathematics and Computation*, **227**, (2014), 788–800.

    Weight functions with a parameter are introduced into an iteration process to increase the order of the convergence and enhance the behavior of the iteration process. The parameter can be chosen to restrict extraneous fixed points to the imaginary axis and provide the best basin of attraction. The process is demonstrated on several examples.

174. B. Neta, C. Chun, M. Scott, Basins of attraction for optimal eighth order methods to find simple roots of nonlinear equations, *Applied Mathematics and Computation*, **227**, (2014), 567–592.

Several optimal eighth order methods to obtain simple roots are analyzed. The methods are based on two step, fourth order optimal methods and a third step of modified Newton. The modification is performed by taking an interpolating polynomial to replace either $f(z_n)$ or $f'(z_n)$. In six of the eight methods we have used a Hermite interpolating polynomial. The other two schemes use inverse interpolation. We discovered that the eighth order methods based on Jarratt's optimal fourth order methods perform well and those based on King's or Kung- Traub's methods do not. In all cases tested, the replacement of $f(z)$ by Hermite interpolation is better than the replacement of the derivative, $f'(z)$.

175. C. Chun, B. Neta, Soojin Kim, On Jarratt's family of optimal fourth-order iterative methods and their dynamics, *Fractals*,**22**, (2014), 1450013, DOI:10.1142/S0218348X14500133.

Jarratt [Some fourth-order multi-point iterative methods for solving equations, Math. Comput., 20, (1966), 434-437] has developed a family of fourth-order optimal methods. He suggested two members of the family. The dynamics of one of those was discussed in [C. Chun et al., On optimal fourth-order iterative methods free from second derivative and their dynamics, Appl. Math. Comput., 218, (2012), 6427-6438]. Here we show that the family can be written using a weight function and analyze all members of the family to find the best performer.

176. B. Neta, C. Chun, Basins of attraction for several optimal fourth order methods for multiple roots, *Mathematics and Computers in Simulations*, **103**, (2014), 39–59.

There are very few optimal fourth order methods for solving nonlinear algebraic equations having roots of multiplicity m. Here we compare 5 such methods, two of which require the evaluation of the $(m-1)^{st}$ root. The methods are usually compared by evaluating the computational efficiency and the efficiency index. In this paper all the methods have the same efficiency, since they are of the same order and use the same information. Frequently, comparisons of the various schemes are based on the number of iterations required for convergence, number of function evaluations, and/or amount of CPU time. If a particular algorithm does not converge or if it converges to a different solution, then that particular algorithm is thought to be inferior to the others. The primary flaw in this type of comparison is that the starting point represents only one of an infinite number of other choices. Here we use the basin of attraction idea to recommend the best fourth order method. The basin of attraction is a method to visually comprehend how an algorithm behaves as a function of the various starting points.

177. B. Neta, C. Chun, Corrigendum to "On a family of Laguerre methods to find multiple roots of nonlinear equations", *Applied Mathematics and Computation*, **248**, (2014), 693-696.

This is a note to correct the typographical errors in our paper. Dr Xiaojian Zhou has emailed us "...we have some confusions on the results of $S(z)$ (or $S(u)$). For example, for Osada's method, you have $S(z) = z^3[(m-1)z + 2m]$ as shown in Theorem 2.5, but we have $S(z) = \frac{z^3[(m-1)z+2m]}{2mz+m-1}$ where there is a denominator. We have checked all the conjugacy maps $S(z)$ and we present the corrected theorems.

178. B. Neta, C. Chun, An analysis of a new family of eighth-order optimal methods, *Applied Mathematics and Computation*, **245**, (2014), 86–107.

A new family of eighth order optimal methods is developed and analyzed. Numerical experiments show that our family of methods perform well and in many cases some members are superior to other eighth order optimal methods. It is shown how to choose the parameters to widen the basin of attraction.

179. B. Neta, C. Chun, Basins of attraction for Zhou-Chen-Song fourth order family of methods for multiple roots, *Mathematics and Computers in Simulations*, **109**, (2015), 74–91.

There are very few optimal fourth order methods for solving nonlinear algebraic equations having roots of multiplicity m. In a previous paper we have compared 5 such methods, two of which require the evaluation of the $(m-1)^{st}$ root. We have used the basin of attraction idea to recommend the best optimal fourth order method. Here we suggest to improve on the best of those five, namely Zhou-Chen-Song method by showing how to choose the best weight function.

180. C. Chun, B. Neta, An analysis of a family of Maheshwari-based optimal eighth order methods,*Applied Mathematics and Computation*, **253**, (2015), 294–307.

In this paper we analyze an optimal eighth-order family of methods based on Maheshwari's fourth order method. This family of methods uses a weight function. We analyze the family using the information on the extraneous fixed points. Two measures of closeness of an extraneous points set to the imaginary axis are considered and applied to the members of the family to find its best performer. The results are compared to a modified version of Wang-Liu method.

181. Y. H. Geum, Y. I. Kim, B. Neta, On developing a higher-order family of double-Newton methods with a bivariate weighting function, *Applied Mathematics and Computation*, **254**, (2015), 277–290.

In this paper we analyze an optimal eighth-order family of methods based on King's fourth order method to solve a nonlinear equation. This family of methods was developed by Thukral and Petković and uses a weight function. We analyze the family using the information on the extraneous fixed points. Two measures of closeness of an extraneous points set to the imaginary axis are considered and applied to the members of the family to find its best performer. The results are compared to a modified version of Wang-Liu method.

182. C. Chun, B. Neta, An analysis of a King-based family of optimal eighth-order methods, *Amer. J. Algorithms and Computing*, **2**, (2015), 1–17.

    In this paper we analyze an optimal eighth-order family of methods based on King's fourth order method. This family of methods was developed by Thukral and Petković and uses a weight function. We analyze the family using the information on the extraneous fixed points. Two measures of closeness of an extraneous points set to the imaginary axis are considered and applied to the members of the family to find its best performer. The results are compared to a modified version of Wang-Liu method.

183. C. Chun, B. Neta, Basins of attraction for several third-order methods to find multiple roots of nonlinear equations methods, *Applied Mathematics and Computation*, **268**, (2015), 129–137.

    There are several third order methods for solving a nonlinear algebraic equation having roots of a given multiplicity m. Here we compare a recent family of methods of order three to Euler-Cauchy's method which found to be the best in a previous work. There are even fewer fourth order methods for multiple roots but we will not include those here.

184. C. Chun, B. Neta, Comparing the basins of attraction for Kanwar-Bhatia-Kansal family to the best fourth order method, *Applied Mathematics and Computation*, **266**, (2015), 277–292.

    There are relatively few optimal fourth order methods for solving nonlinear algebraic equations having roots of known multiplicity m. In a previous paper we have compared 5 such methods, two of which require the evaluation of the $(m-1)^{st}$ root. We have used the basin of attraction idea to recommend the best optimal fourth order method. Here we compare the family of methods developed by Kanwar et al. (Kanwar, V., Bhatia, S., Kansal, M., New optimal class of higher-order methods for multiple roots, permitting $f'(x_n) = 0$, *Appl. Math. Comput.*, **222**, (2013), 564-574) to the best known method. We will also point out some mistake in deriving that family.

185. Y. H. Geum, Y. I. Kim, B. Neta, A class of two-point sixth-order multiple-zero finders of modified double-Newton type and their dynamics, *Appl. Math. Comput.*, **270**, (2015), 387–400.

    Under the assumption of the known multiplicity of zeros of nonlinear equations, a class of two-point sextic-order multiple-zero finders and their dynamics are investigated in this paper by means of extensive analysis of modified double-Newton type of methods. With the introduction of a bivariate weight function dependent on function-to-function and derivative-to-derivative ratios, higher-order convergence is obtained. Additional investigation is carried out for extraneous fixed points of the iterative maps associated with the proposed methods along with a comparison with typically selected cases. Through a variety of test equations, numerical experiments strongly support the theory developed in this paper. In addition, relevant dynamics of the proposed methods is successfully explored for various polynomials with a number of illustrative basins of attraction.

186. C. Chun, B. Neta, On the new family of optimal eighth order methods developed by Lotfi et al., *Numerical Algorithms*, **72**, (2016), 363–376.

   Recently Lotfi et al. (A new class of three-point methods with optimal convergence order eight and its dynamics, Numer. Algor., **68**, (2015), 261–288) have developed a new family of optimal order eight for the solution of nonlinear equations. They have experimented with 3 members of the family and compared them to other eighth order methods. One of the best known eight order method was not included. They also did not mention the best choice of parameters in the methods used and why. The basins of attraction were given for several examples without a quantitative comparison. It will be shown how to choose the best parameters in all these methods, and to quantitatively compare the methods.

187. C. Chun, B. Neta, Comparison of several families of optimal eighth order methods, *Appl. Math. Comput.*, **274**, (2016), 762–773..

   Several families of optimal eighth order methods to find simple roots are compared to the best known eighth order method due to Wang & Liu (Wang, X., Liu, L., Modified Ostrowski's method with eighth-order convergence and high efficiency index, *Appl. Math. Lett.*, **23**, (2010), 549–554.) We have tried to improve their performance by choosing the free parameters of each family using two different criteria.

188. C. Chun, B. Neta, The basins of attraction of Murakami's fifth order family of methods, *Appl. Numer. Math.*, **110**, (2016), 14–25.

   In this paper we analyze Murakami's family of fifth order methods for the solution of nonlinear equations. We show how to find the best performer by using a measure of closeness of the extraneous fixed points to the imaginary axis. We demonstrate the performance of these members as compared to the two members originally suggested by Murakami. We found several members for which the extraneous fixed points are on the imaginary axis, only one of these have 6 such points (compared to 8 for the other members). We show that this member is best.

189. B. Neta, C. Chun, M. Scott, Corrigendum to "Basins of attraction for optimal eighth order methods to find simple roots of nonlinear equations", *Applied Mathematics and Computation*, **218**, (2016), 396–403.

   In this note we correct the error in the order of convergence of 3 methods we mentioned in our paper in *Appl. Math. Comput.*, **227** (2014), 567–592. We will show how the order of convergence is related to the conjugation analysis.

190. C. Chun, B. Neta, An analysis of Khattri's 4th order family of methods, *Applied Mathematics and Computation*, **279**, (2016), 198–207.

In this paper we analyze an optimal fourth-order family of methods suggested by [?]. We analyze the family using the information on the extraneous fixed points. Two measures of closeness to the imaginary axis of the set of extraneous points are considered and applied to the members of the family to find its best performer. The results are compared to three best members of King's family of methods.

191. Y. H. Geum, Y. I. Kim, B. Neta, A sixth-order family of three-point modified Newton-like multiple-root finders and the dynamics behind their extraneous fixed points, *Applied Mathematics and Computation*, **283**, (2016), 120–140.

A class of three-point sixth-order multiple-root finders and the dynamics behind their extraneous fixed points are investigated by extending modified Newton-like methods with the introduction of the multivariate weight functions in the intermediate steps. The multivariate weight functions dependent on function-to-function ratios play a key role in constructing higher-order iterative methods. Extensive investigation of extraneous fixed points of the proposed iterative methods is carried out for the study of the dynamics associated with corresponding basins of attraction. Numerical experiments applied to a number of test equations strongly support the underlying theory pursued in this paper. Relevant dynamics of the proposed methods is well presented with a variety of illustrative basins of attraction applied to various test polynomials.

192. I. Petković, B. Neta, On an application of symbolic computation and computer graphics to root finders: the case of multiple roots of unknown multiplicity, *J. Comput. Appl. Math.*, **308**, (2016), 215–230.

The contemporary powerful mathematical software enables a new approach to handling and manipulating complex mathematical expressions and other mathematical objects. Particularly, the use of symbolic computation leads to new contribution to constructing and analyzing numerical algorithms for solving very difficult problems in applied mathematics and other scientific disciplines. In this paper we are concerned with the problem of determining multiple zeros when the multiplicity is not known in advance, a task that is seldom considered in literature. By the use of computer algebra system *Mathematica*, we employ symbolic computation through several programs to construct and investigate algorithms which both determine a sought zero and its multiplicity. Applying a recurrent formula for generating iterative methods of higher order for solving nonlinear equations, we construct iterative methods that serve (i) for approximating a multiple zero of a given function f when the order of multiplicity is unknown and, simultaneously, (ii) for finding exact order of multiplicity. In particular, we state useful cubically convergent iterative sequences that find the exact multiplicity in a few iteration steps. Such approach, combined with a rapidly convergent method for multiple zeros, provides the construction of efficient composite algorithms for finding multiple zeros of very high accuracy. The properties of the proposed algorithms are illustrated by several numerical examples and basins of attraction.

193. A. Cordero, J. R. Torregrosa, B. Neta, T. Lotfi, Real and complex dynamics of iterative methods, *Discrete Dyanamics in Nature and Society*, (2016), 4765286.

194. C. Chun, B. Neta, On the eighth order family of methods due to Khan et al., unpublished report.

Khan et al., Appl. Math. Lett. 25 (2012), 2262-2266 claimed they have developed a new general eighth-order family of iterative method for simple roots. We will show that the method is just a special case of the family of methods developed by Kou et al. Commun. Nonlinear Sci. Numer. Simulat. 15(2010), 536-544.

195. C. Chun, B. Neta, Comparative study of eighth order methods for finding simple roots of nonlinear equations, *Numerical Algorithms*, **74**, (2017), 1169–1201.

Recently there were many papers discussing the basins of attraction of various methods and ideas how to choose the parameters appearing in families of methods and weight functions used. Here we collected many of the eighth order schemes scattered in the literature and presented a quantitative comparison. We have used the average number of function evaluations per point, the CPU time and the number of black points to compare the methods. Based on 6 examples, we found that the best method based on the 3 criteria is SA8 due to Sharma and Arora.

196. M.-Y. Lee, Y. I. Kim, B. Neta, An optimal family of eighth-order simple-root finders with weight functions dependent on function-to-function ratios and their dynamics underlying extraneous fixed points, *J. Comput. Appl. Math.*, **317**, (2017), 31–54.

We introduce in this paper an optimal family of eighth-order methods developed by Petković et al. and investigate their dynamics under the relevant extraneous fixed points among which purely imaginary ones are specially treated for the analysis of the rich dynamics. Their theoretical and computational properties are fully investigated along with a main theorem describing the order of convergence and the asymptotic error constant as well as proper choices of special cases. A wide variety of relevant numerical examples are illustrated to confirm the underlying theoretical development. In addition, this paper investigates the dynamics of selected existing optimal eighth-order iterative maps with the help of illustrative basins of attraction for various polynomials.

197. Y. H. Geum, Y. I. Kim, B. Neta, A family of optimal quartic-order multiple-zero finders with a weight function of the principal k^{th} root of a derivative-to-derivative ratio and their basins of attraction, *Mathematics and Computers in Simulations*, **136**, (2017), 1–21.

Multiple-zero finders with optimal quartic convergence for nonlinear equations are proposed in this paper with a weight function of the principal k^{th} root of a derivative-to-derivative ratio. The optimality of the proposed multiple-zero finders is checked for their consistency based on Kung-Traub's conjecture established in 1974. Through various test equations, relevant numerical experiments strongly support the claimed theory in this paper. Also investigated are extraneous fixed points of the iterative maps associated with the proposed methods. Their dynamics are explored along with illustrated basins of attraction for various polynomials.

198. C. Chun, B. Neta, Comparative study of eighth order methods for finding simple roots of nonlinear equations, *Numerical Algorithms*, **74**, (2017), 1169–1201.

Recently, there were many papers discussing the basins of attraction of various methods and ideas how to choose the parameters appearing in families of methods and weight functions used. Here, we collected many of the eighth-order schemes scattered in the literature and presented a quantitative comparison. We have used the average number of function evaluations per point, the CPU time, and the number of black points to compare the methods. Based on seven examples, we found that the best method based on the three criteria is SA8 due to Sharma and Arora.

199. C. Chun, B. Neta, Comparative study of methods of various orders for finding simple roots of nonlinear equations, *Journal of Applied Analysis and Computation*, **9**, (2019), 400–427.

Recently there were many papers discussing the basins of attraction of various methods and ideas how to choose the parameters appearing in families of methods and weight functions used. Here we collected many of the results scattered and put a quantitative comparison of methods of orders from 2 to 7. We have used the average number of function evaluations per point, the CPU time and the number of black points to compare the methods. We also include the best eighth order method. Based on 7 examples, we show that there is no method that is best based on the 3 criteria. We found that the best eighth order method, SA8, and CLND are at the top.

200. C. Chun, B. Neta, How good are methods with memory for the solution of nonlinear equations, *SeMA J. (Boletin de la Sociedad Española de Matemática Aplicada)*, **74**, (2017), 613–625, DOI: 10.1007/s40324-016-0105-x.

Multipoint methods for the solution of a single nonlinear equation allow higher order of convergence without requiring higher derivatives. Such methods have an order barrier as conjectured by Kung and Traub. To overcome this barrier, one constructs multipoint methods with memory, i.e. use previously computed iterates. We compare multipoint methods with memory to the best methods without memory and show that the use of memory is computationally more expensive and the methods are not competitive.

201. M. S. Rhee, Y. I. Kim, B. Neta, An optimal eighth-order class of three-step weighted Newton's methods and their dynamics behind the purely imaginary extraneous fixed points, *Int. J. Computer Math.*, **95**, (2008), 2174–2211.

In this paper, we not only develop an optimal class of three-step eighth-order methods with higher-order weight functions employed in the second and third sub-steps, but also investigate their dynamics underlying the purely imaginary extraneous fixed points. Their theoretical and computational properties are fully described along with a main theorem stating the order of convergence and the asymptotic error constant as well as extensive studies of special cases with rational weight functions. A number of numerical examples are illustrated to confirm the underlying theoretical development. Besides, to show the convergence behavior of global character, fully explored is the dynamics of the proposed family of eighth-order methods as well as an existing competitive method with the help of illustrative basins of attraction.

202. Y. H. Geum,Y. I. Kim, B. Neta, Constructing a family of optimal eighth-order modified Newton-type multiple-zero finders along with the dynamics behind their purely imaginary extraneous fixed points, *J. Comput. Appl. Math.*, **333**, (2018), 131–156.

An optimal family of eighth-order multiple-zero finders and the dynamics behind their basins of attraction are proposed by considering modified Newton-type methods with multivariate weight functions. Extensive investigation of purely imaginary extraneous fixed points of the proposed iterative methods is carried out for the study of the dynamics associated with corresponding basins of attraction. Numerical experiments strongly support the underlying theory pursued in this paper. An exploration of the relevant dynamics of the proposed methods is presented along with illustrative basins of attraction for various polynomials.

203. L. D. Petković, M. S. Petković, B. Neta, On optimal parameter of Laguerre's zero-finding method, *International J. Computer Math.*, **96**, (2019), 692–707.

A one parameter Laguerres family of iterative methods for solving nonlinear equations is considered. This family includes the Halley, Ostrowski and Euler methods, most frequently used one-point third order methods for finding zeros. These methods are of great practical interest since they have the highest computational efficiency in the class of one-point methods. Investigation of convergence quality of these methods and their ranking reduces to searching optimal parameter of Laguerres family, which is the main goal of this paper. Although methods from Laguerres family have been extensively studied in the literature for more decades, their proper ranking was primarily discussed according to numerical experiments. Regarding that such ranking is not trustworthy even for algebraic polynomials, more reliable comparison study is presented by combining the comparison by numerical examples and the comparison using dynamic study of methods by basins of attraction that enable their graphic visualization. This combined approach has shown that Ostrowskis method possesses the best convergence behavior for most polynomial equations.

204. C. Chun, B. Neta, Developing high order methods for the solution of systems of nonlinear equations, *Appl. Math. Comput.*, **342**, (2019), 178–190.

Two families of order six for the solution of systems of nonlinear equations are developed and compared to existing schemes of order up to six. We have found that one of the methods in the literature has been rediscovered. The comparison is based on the total cost of an iteration and the performance on 14 examples of systems of dimensions 2–9.

205. C. Chun, B. Neta, Comparative study of methods of various orders for finding repeated roots of nonlinear equations, *J. Comp. Appl. Math.*, **340**, (2018), 11-42.

In this paper we are considering 19 (families of) methods for finding repeated roots of a nonlinear equation. The methods are of order up to 6. We use the idea of basin of attraction to compare the methods. We found that 4 methods performed best based on 3 quantitative criteria.

206. L. D. Petković, M. S. Petković, B. Neta, On generalized Halley-method for solving nonlinear equations, *Applicable Analysis and Discret Mathematics*, **13**, (2019), 399–422.

A generalized Halley-like one-parameter family of cubically convergent iterative method for solving nonlinear equations is constructed and studied. A simple square-root free iterative formula (in contrast to Laguerres family) is convenient for the user who can generate various third order method by changing the involved parameter. Two variants, both with cubic convergence, are developed, one for finding simple zeros and other for multiple zeros of known multiplicities. As special cases this family includes Halley-like and Chebyshev-like methods. To analyze convergence behavior of the methods from the derived family relative to the parameter, we have used two methodologies: (i) testing by numerical examples and (ii) dynamic study using basins of attraction. This is, actually, a mixture of theoretical results, algorithmic aspects, numerical experiments, and computer graphics. According to the outcomes of dynamic study, we come to the surprising result that one-point third order methods from the proposed family can have better convergent properties in practice (at least for algebraic polynomials) relative to optimal two-point methods despite of (theoretical) lower computational efficiency. Finally, starting from the proposed third order method and using an accelerating procedure, we construct a new fourth order method of Halleys type.

207. B. Neta, Dynamics of some one-point third-order methods for finding simple roots of nonlinear equations, *Comm. Numer. Anal.*, **2**, (2018), 111–130

In this paper we have considered 32 one-point methods of cubic order for finding simple roots of a nonlinear function. These methods are constructed by decomposition of previously known schemes. We have used the idea of basins of attractions to compare the performance of these methods with Halley's method on 4 polynomial functions and one non-polynomial function. Based on 3 quantitative criteria, namely average number of iterations per point, CPU time required and the number of points for which the method did not converge in 40 iterations, we have found 4 methods that performed close to best. We also show that decomposing good methods does not necessarily lead to a better one or even to a scheme as good as the original. We found only one example that gave reasonable results and it is the only one with purely imaginary repelling extraneous fixed points.

208. Y. H. Geum, Y. I. Kim, B. Neta, Developing an optimal class of generic sixteenth-order simple-root finders and investigating their dynamics, *Math.*, **7 (1)**, (2019), Article ID 8.

In this paper, we develop a class of optimal sixteenth-order simple-root finders with generic weight functions. Their computational and dynamical aspects are fully investigated along with a main theorem stating the order of convergence and the asymptotic error constant. Special cases with polynomial and rational weight functions have been extensively studied for applications to real-world problems. A number of computational experiments clearly support the underlying theory on the local convergence of the proposed methods. In addition, to investigate the convergence behavior of global character, fully explored is the dynamics of the proposed family of methods as well as existing competitive methods via illustrative basins of attraction.

209. M. Y. Lee, Y. I. Kim, B. Neta, A generic family of optimal sixteenth-order multiple-root finders and their dynamics underlying purely imaginary extraneous fixed points, **7**, (2019), 562–587.

A generic family of optimal sixteenth-order multiple-root finders are theoretically developed from general settings of weight functions under the known multiplicity. Special cases of rational weight functions are considered and relevant coefficient relations are derived in such a way that all the extraneous fixed points are purely imaginary. A number of schemes are constructed based on the selection of desired free parameters among the coefficient relations. Numerical and dynamical aspects on the convergence of such schemes are explored with tabulated computational results and illustrated attractor basins. Overall conclusion is drawn along with future work on a different family of optimal root-finders.

210. C. Chun, B. Neta, Trigonometrically-fitted methods: a review, Mathematics, **7 (12)**, (2019), 1197–1216.

Numerical methods for the solution of ordinary differential equations are based on polynomial interpolation. In 1952, Brock and Murray (Math. Tables Aids Comput., 6, (1952), 63–78) have suggested exponentials for the case that the solution is known to be of exponential type. In 1961, Gautschi (Numer. Math., 3, (1961), 381-397) came up with the idea of using information on the frequency of a solution to modify linear multistep methods by allowing the coefficients to depend on the frequency. Thus the methods integrate exactly appropriate trigonometric polynomials. This was done for both first order systems and second order initial value problems. Gautschi concluded that "the error reduction is not very substantial unless" the frequency estimate is close enough. As a result, no other work was done in this direction until 1984 when Neta and Ford (J. Comput. Appl. Math., 10, (1984), 33-38) showed that "Nyström's and Milne-Simpson's type methods for systems of first order initial value problems are not sensitive to changes in frequency." This opened the flood gates and since then there were many papers on the subject.

211. B. Neta, A new twelfth order trigonometrically-fitted Obrechkoff-like method for second order initial value problems, J. Numer. Anal. Industrial Appl. Math. (JNAIAM), **15** No. 3-4, (2021), 5370.

A new trigonometrically-fitted method of order 12 is developed and compared to an existing P-stable method of the same order. Our method fit exactly the sine and cosines functions $\sin(r\omega x)$, $\cos(r\omega x)$, $r = 1, 2$ and monomials up to degree 9. Our method is tested on several linear and nonlinear examples to demonstrate its accuracy and sensitivity to perturbation in the known frequency. We also show where it is preferable to use the trigonometrically-fitted method.

212. C. Chun, B. Neta, An efficient derivative-free method for the solution of systems of equations, submitted for publication.

We have modified our previously developed fourth order method to approximate solution of non-differentiable systems of equations. There are few derivative-free methods in the literature. The most recent article compared fourth order and seventh order methods to show the efficiency of their fourth order. We have shown that our method compares favorably with methods in the literature.

213. A. Shokri, B. Neta, M. M. Khalsaraei, A new eight-step P-stable method for the numerical solution of the radial Schrödinger equation, TWMS J. Pure Appl. Math., accepted for publication.

In this paper, we present a new eight-step singularly P-stable method with vanished phase-lag and its derivatives up to fifth order for the numerical integration of the one-dimensional radial time independent Schrödinger equation. Numerical stability and phase properties of the new methods are analyzed and the periodicity region of the method has been plotted. Finally we compare the new method to the corresponding classical ones and other known methods from the literature to demonstrate the high efficiency of our new method.

214. B. Neta, Basin attractors for derivative-free methods to find simple roots of nonlinear equations, J. Numer. Anal. Approx. Th., **49**, (2020), 177–189.

Numerous methods exist for finding zeros of nonlinear equations. Several of these schemes are derivative-free. One of the oldest is the secant method where the derivative is replaced by a divided difference. Clearly such method will need an additional starting value. Here we study the dynamics of several derivative-free methods and compare them using the idea of basin of attraction. As a results, we develop a new high-order derivative-free method and study its dynamics.

215. B. Neta, A new derivative-free method to solve nonlinear equations, Mathematics, 9 (2021), 583587, https://doi.org/10.3390/math9060583.

A new high-order derivative-free method for the solution of a nonlinear equation is developed. The novelty is the use one of Traubs method as first step. The order is proven and demonstrated. It is also shown that the method has much fewer divergent points and runs faster than an optimal eighth-order derivative-free method.

216. A. Shokri, M. M. Khalsaraei, B. Neta, A. Atashyar, An efficient symmetric eight-step singularly P-stable method for the numerical solution of second-order IVPs, Mathematics, 9 (2021), 806–827. https://doi.org/10.3390/math9080806.

In this paper, a symmetric eight-step predictor method (explicit) of 10th order is presented for the numerical integration of IVPs of second-order ordinary differential equations. This scheme has variable coefficients and can be used as a predictor stage for other implicit schemes. First, we showed the singular P-stability property of the new method, both algebraically and by plotting the stability region. Then, having applied it to well-known problems like Mathieu equation, we showed the advantage of the proposed method in terms of efficiency and consistency over other methods with the same order.