MA 1115

Solutions

Please Do NOT Remove from the vicinity of Sp 270

MA1115	Homework Problems
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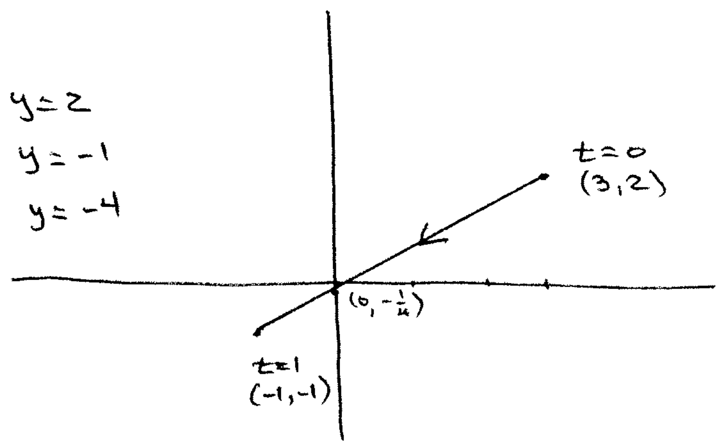
10.1 #5, 11, 13, 31

#5. a $x = 3-4t$ $y = 2-3t$

$t=0$ $x=3$ $y=2$

$t=1$ $x=-1$ $y=-1$

$t=2$ $x=-5$ $y=-4$



b. $t = \frac{3-x}{4} \Rightarrow y = 2 - 3\,\frac{3-x}{4} = 2 - \frac{9-3x}{4}$

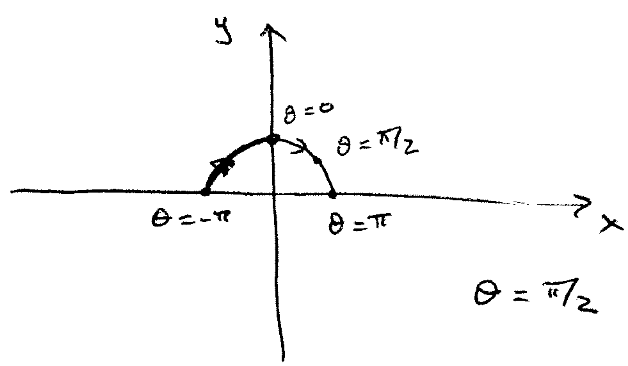
$4y = 8-9+3x = -1+3x$

$y = \frac{3}{4}x - \frac{1}{4}$ line with slope $\frac{3}{4}$

11. $x = \sin(\frac{1}{2}\theta)$ $y = \cos(\frac{1}{2}\theta)$ $-\pi \le \theta \le \pi$

a. $x^2+y^2=1$ Circle with radius $=1$ centered at origin

b.



$\theta = 0$

$x = 0$

$y = 1$

$\theta = \pi/2$ $x = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} = .7$

$y = \cos \pi/4 = \frac{\sqrt{2}}{2} = .7$

$\theta = -\pi$ $x = \sin(-\pi/2) = -1$

$y = \cos(-\pi/2) = 0$

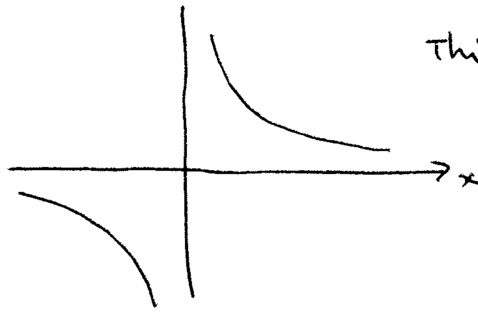
$\theta = \pi$ $x = \sin(\pi/2) = 1$

$y = \cos(\pi/2) = 0$

#13 $x = \sin t$

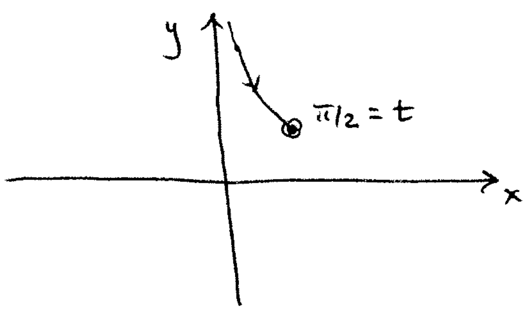
$y = \csc t \qquad 0 < t < \pi/2$

a. $y = \csc t = \frac{1}{\sin t} = \frac{1}{x}$



This is the Cartesian sketch

b. sketch the parametric curve:



$t = 0 \quad x = 0 \quad y \to \infty$

$t = \pi/2 \quad x = 1 \quad y = 1$

31. a. $x = x_1 + (x_2 - x_1)t \qquad y = y_1 + (y_2 - y_1)t \qquad 0 \le t \le 1$

$t = 0 \quad x = x_1 \quad y = y_1 \Rightarrow P_1$

$t = 1 \quad x = x_2 \quad y = y_2 \Rightarrow P_2$

This is a straight line since:

$$t = \frac{x - x_1}{x_2 - x_1} \Rightarrow y = y_1 + \frac{x - x_1}{x_2 - x_1}(y_2 - y_1)$$

y is a linear function of x

slope is $\frac{y_2 - y_1}{x_2 - x_1}$

b. $P_1(-2, 7) \to P_2(3, -1)$

$x = -2 + 5t$

$y = 7 - 8t \qquad 0 \le t \le 1$

10.2 # 3, 6, 11, 16, 27, 31, 37, 42

#3 $x = 1 + 4t - t^2$ $\quad y = 2 - t^3$ $\quad t = 1$

$$\frac{dx}{dt} = 4 - 2t \qquad \frac{dy}{dt} = -3t^2$$

$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{-3}{4-2} = \frac{-3}{2}$$

$$x(1) = 1 + 4 - 1 = 4 \qquad y(1) = 2 - 1 = 1$$

$$\underline{y - 1 = -\frac{3}{2}(x - 4)}$$

6. $x = \sin^3\theta$ $\quad y = \cos^3\theta$ $\quad \theta = \pi/6$

$$\frac{dx}{d\theta} = 3\sin^2\theta\cos\theta \qquad \frac{dy}{d\theta} = -3\cos^2\theta\sin\theta$$

$$\left.\frac{dx}{d\theta}\right|_{\theta=\pi/6} = 3\cdot\frac{\sqrt{3}}{8} \qquad \left.\frac{dy}{d\theta}\right|_{\theta=\pi/6} = -3\cdot\frac{3}{8}$$

$$x(\pi/6) = \frac{1}{8} \qquad y(\pi/6) = \frac{3\sqrt{3}}{8}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \frac{-9/\cancel{8}}{\frac{3\sqrt{3}}{\cancel{8}}} = -\frac{3}{\sqrt{3}} = -\sqrt{3}$$

$$\underline{y - \frac{3\sqrt{3}}{8} = -\sqrt{3}\left(x - \frac{1}{8}\right)}$$

11. $x = t^2+1 \qquad y = t^2+t$

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 2t+1$$

$$\Rightarrow \frac{dy}{dx} = \frac{2t+1}{2t} = 1 + \frac{1}{2t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(1+\frac{1}{2t}\right) = \frac{d}{dt}\left(1+\frac{1}{2t}\right)\frac{dt}{dx}$$

$$= \left[0 - \frac{1}{2}\ t^{-2}\right]\frac{1}{2t} = -\frac{1}{4t^3}$$

For the curve to be concave upward we need

$$\frac{d^2y}{dx^2} > 0 \quad \Rightarrow \quad -\frac{1}{4t^3} > 0$$

$$\Rightarrow \quad \frac{1}{4t^3} < 0 \quad \Rightarrow \quad t^3 < 0$$

$$\Rightarrow \boxed{t < 0}$$

#16 p.651 10.2

$\dot{x} = -2\sin 2t$
$\dot{y} = -\sin t$

$$\frac{dy}{dx} = \frac{\cancel{\sin t}}{4\cancel{\sin t}\cos t} = \frac{1}{4}\sec t$$

$$\frac{d}{dx}\left(\quad\right) = \frac{d}{dt}\left(\frac{1}{4\cos t}\right) \Big/ (-2\sin 2t)$$

$$= \frac{\sin t}{4\cos^2 t\,(-2\sin 2t)}$$

$$= -\frac{1}{16\cos^3 t} \qquad 0 < t < \pi$$

$\cos t > 0$	$0 < t < \pi/2$	$\Rightarrow$	down
< 0	$\pi/2 < t < \pi$	$\Rightarrow$	up
$= 0$	$t = \pi/2$		inflection

#27

$x = r\theta - d\sin\theta$
$y = r - d\cos\theta$
$\dot{x} = r - d\cos\theta$
$\dot{y} = d\sin\theta$

$$\frac{dy}{dx} = \frac{d\sin\theta}{r - d\cos\theta}$$

$d = r \Rightarrow \frac{\sin\theta}{1-\cos\theta} = \frac{dy}{dx}$
cycloid

$d<r \qquad d>r$

$\downarrow$

vertical if $r = d\cos\theta$

$\cos\theta = \frac{r}{d} \leq 1 \Rightarrow r \leq d$

No vertical if $d<r$

31. $x = a\cos\theta$

$y = b\sin\theta$

$0 \le \theta \le 2\pi$

to find area

$$A = 4\int_{\pi/2}^{0} b\sin\theta\,(-a\sin\theta)\,d\theta$$

$$= +4ab\int_{0}^{\pi/2} \underbrace{\sin^2\theta}\,d\theta$$

$$= \frac{4ab}{2}\int_{0}^{\pi/2} (1-\cos 2\theta)\,d\theta \qquad \cos 2\theta = 1 - 2\sin^2\theta$$

$$\frac{\cos 2\theta - 1}{2} = -\sin^2\theta$$

$$= \frac{4ab}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\Bigg|_{0}^{\pi/2} = +\frac{\cancel{2}\pi ab}{\cancel{2}} = \pi ab$$

37. $x = t + e^{-t}$ $\quad 0 \le t \le 2$
$y = t - e^{-t}$

$\frac{dx}{dt} = 1 - e^{-t}$

$\frac{dy}{dt} = 1 + e^{-t}$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(1 - e^{-t}\right)^2 + \left(1 + e^{-t}\right)^2$

$= 1 - \cancel{2e^{-t}} + e^{-2t} + 1 + \cancel{2e^{-t}} + e^{-2t}$

$= 2 + 2e^{-2t}$

arc length $= \int_0^2 \sqrt{2(1+e^{-2t})}\, dt \cong 3.1416$ using Maple

42. $x = e^t + e^{-t}$ $\quad y = 5 - 2t$ $\quad 0 \le t \le 3$

$\frac{dx}{dt} = e^t - e^{-t}$ $\quad \frac{dy}{dt} = -2$

$\left(\frac{dx}{dt}\right)^2 = \left(e^t - e^{-t}\right)^2 = e^{2t} - 2 + e^{-2t}$

arc length $= \int_0^3 \sqrt{e^{2t} - 2 + e^{-2t} + (-2)^2}\, dt$

$= \int_0^3 \sqrt{e^{2t} + 2 + e^{-2t}}\, dt$

$= \int_0^3 \sqrt{(e^t + e^{-t})^2}\, dt = \int_0^3 (e^t + e^{-t})\, dt$

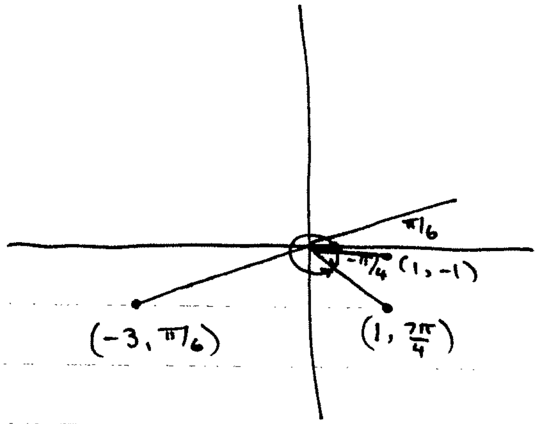
$= e^t - e^{-t} \Big|_0^3 = e^3 - e^{-3} - (1-1) = \underline{e^3 - e^{-3}}$

10.3 # 2, 3b, 3c, 5b, 6a, 12, 13, 14, 18, 26

2a. $\left(1, \frac{7\pi}{4}\right)$

b. $(-3, \pi/6)$

c. $(1, -1)$



2a. $\left(1, \frac{7\pi}{4}\right) = \left(1, 15\pi/4\right) = \left(-1, \frac{19\pi}{4}\right)$

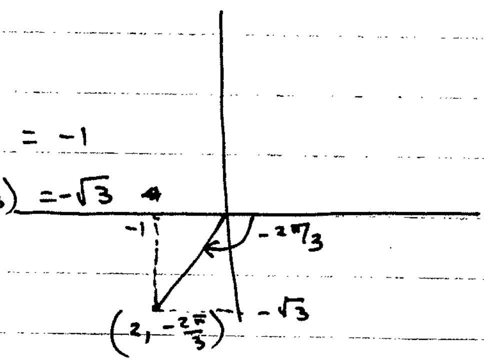
2b. $\left(-3, \pi/6\right) = \left(3, \frac{7\pi}{6}\right) = \left(-3, \frac{13\pi}{6}\right)$ $\quad (r, \theta + 2n\pi) = (-r, \theta + (2n+1)$

2c. $(1, -1) = (1, 2\pi - 1) = (-1, 3\pi - 1)$

3b. $(2, -2\pi/3)$

$x = 2\cos\left(-\frac{2\pi}{3}\right) = 2\cos\left(\frac{2\pi}{3}\right) = -1$

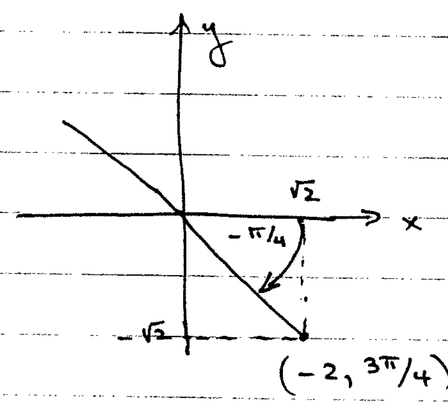
$y = 2\sin\left(-\frac{2\pi}{3}\right) = -2\sin(2\pi/3) = -\sqrt{3}$



3c. $(-2, 3\pi/4)$

$x = -2\cos(3\pi/4) = \sqrt{2}$

$y = -2\sin(3\pi/4) = -\sqrt{2}$



10.3

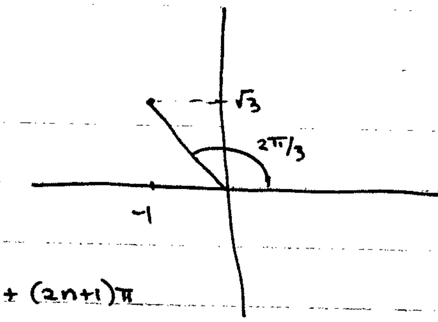
5b. $(-1, \sqrt{3})$

Find polar coordinate

$$\left.\begin{aligned} -1 &= r\cos\theta \\ \sqrt{3} &= r\sin\theta \end{aligned}\right\} \Rightarrow \tan\theta = \frac{\sqrt{3}}{-1} \qquad \boxed{\theta = \frac{2\pi}{3}}$$

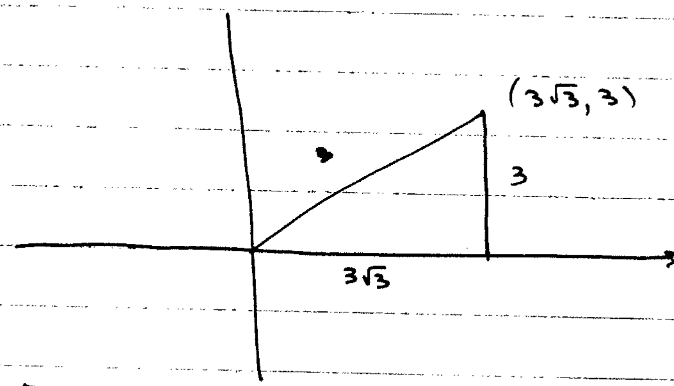
$$r^2 = (-1)^2 + (\sqrt{3})^2 = 1 + 3 = 4$$

$$\boxed{r = 2}$$



ii. $r = -2 \Rightarrow \theta = \frac{5\pi}{3} = \frac{2}{3}\pi + (2n+1)\pi$

6a. $(3\sqrt{3}, 3)$



$$\tan\theta = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}$$

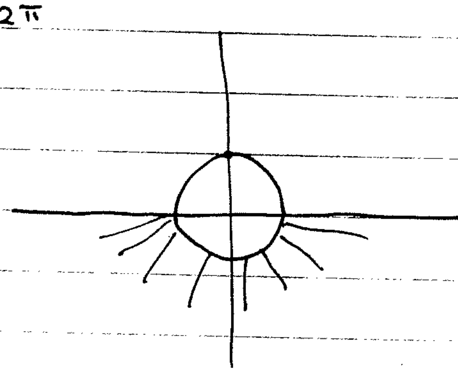
$$\theta = \pi/6$$

$$r = \sqrt{3^2 + (3\sqrt{3})^2} = \sqrt{9+27} = \sqrt{36} = 6$$

$\underline{r = 6,\ \theta = \pi/6}$

ii. $r = -6 \qquad \theta = \frac{7\pi}{6}$

12. $r \geq 1 \qquad \pi \leq \theta \leq 2\pi$



10.3

13. $P_1 \ (2, \pi/3)$

$P_2 \ (4, 2\pi/3)$

$x_1 = 2\cos\pi/3 \qquad y_1 = 2\sin\pi/3$

$x_2 = 4\cos(2\pi/3) \qquad y_2 = 4\sin 2\pi/3$

$$d = \sqrt{\left(4\cos\tfrac{2\pi}{3} - 2\cos\pi/3\right)^2 + \left(4\sin 2\pi/3 - 2\sin\pi/3\right)^2}$$

$$= \sqrt{(-2-1)^2 + (2\sqrt{3} - \sqrt{3})^2}$$

$$= \sqrt{9+3} = \sqrt{12}$$

14. $$d^2 = (r_2\cos\theta_2 - r_1\cos\theta_1)^2 + (r_2\sin\theta_2 - r_1\sin\theta_1)^2$$

$$= r_2^2\cos^2\theta_2 - 2r_1r_2\cos\theta_1\cos\theta_2 + r_2^2\sin^2\theta_2$$
$$+ r_1^2\cos^2\theta_1 - 2r_1r_2\sin\theta_1\sin\theta_2 + r_1^2\sin^2\theta_1$$

$$d^2 = r_2^2 + r_1^2 - 2r_1r_2\cos(\theta_2 - \theta_1)$$

18. $\theta = \pi/3$

Note: $\tan\theta = \dfrac{y}{x}$

$$\Rightarrow \quad \theta = \arctan\frac{y}{x}$$

$$\arctan\frac{y}{x} = \pi/3$$

$$\frac{y}{x} = \tan(\pi/3)$$

$$y = \tan\left(\tfrac{\pi}{3}\right)\cdot x$$

a straight line thru the origin!

26 $\quad xy = 4 \implies y = \frac{4}{x}$ if $x \neq 0$

$$\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \implies r\sin\theta = \frac{4}{r\cos\theta}$$

or $\underline{r^2 \sin\theta \cos\theta = 4}$

10.5 # 4, 6, 11, 16, 22, 23, 41

4. $3x^2+8y=0$

$x^2=-\frac{8}{3}y$ compare to $x^2=4py$

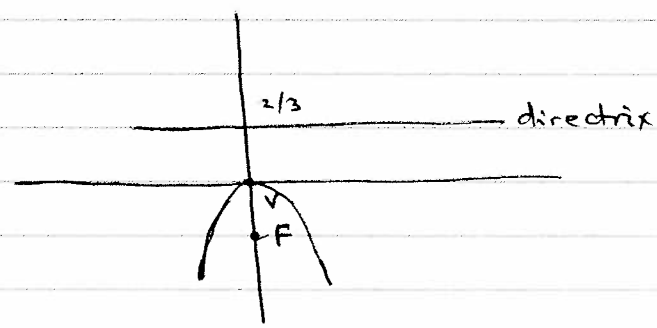
$4p=-\frac{8}{3}$

$p=-2/3$

Focus $(0,-\frac{2}{3})$

Vertex $(0,0)$ no shift

directrix $y=\frac{2}{3}$



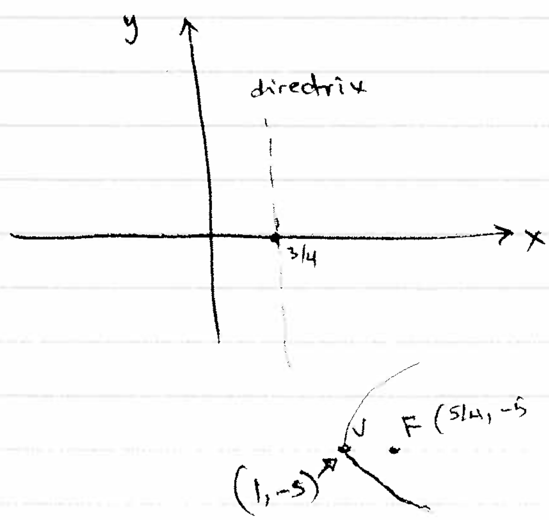
6. $x-1=(y+5)^2$ Vertex $(1,-5)$

$4p=1$ $p=1/4$

Focus $(\frac{5}{4},-5)$

directrix : $x-1=-\frac{1}{4}$

$\underline{x=\frac{3}{4}}$



11. $\frac{x^2}{2} + \frac{y^2}{4} = 1$

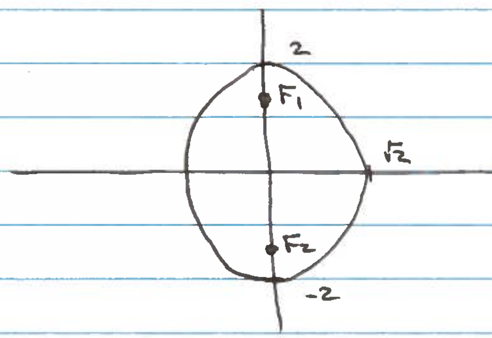
$a^2 = 4$ $b^2 = 2$

$a = 2$ $b = \sqrt{2}$

$c^2 = 4 - 2 = 2 \Rightarrow c = \pm\sqrt{2}$

Vertices $(0, \pm 2)$

Focal points $(0, \pm\sqrt{2})$



16. $x^2 + 3y^2 + 2x - 12y + 10 = 0$

$x^2 + 2x \quad + 3(y^2 - 4y \quad) = -10$

$x^2 + 2x + 1 \quad + 3(y^2 - 4y + 4) = -10 + 1 + 12$

$(x+1)^2 + 3(y-2)^2 = 3$

$\frac{(x+1)^2}{3} + \frac{(y-2)^2}{1} = 1$

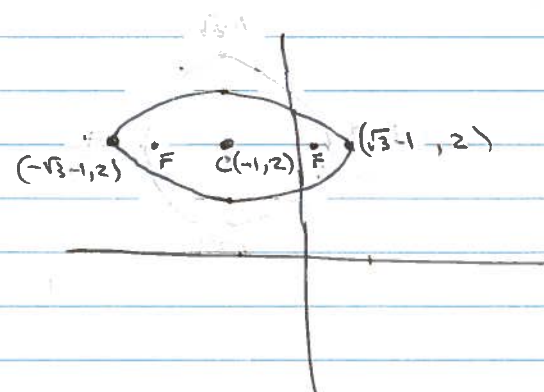
$a^2 = 3$ $b^2 = 1$ $c^2 = 3 - 1 = 2$

$a = \pm\sqrt{3}$ $b = \pm 1$ $c = \pm\sqrt{2}$

center $(-1, 2)$

Vertex $(\pm\sqrt{3} - 1, 2)$

Focal points $(\pm\sqrt{2} - 1, 2)$



22. $y^2 - 16x^2 = 16$

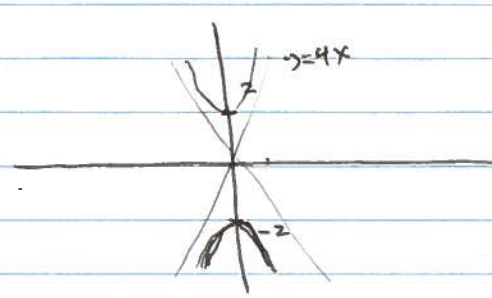
$\frac{y^2}{16} - \frac{x^2}{1} = 1$ Hyperbola

$a = 4$ $b = 1$ $c^2 = 16 + 1 = 17$

$V(0, \pm 4)$

$F(0, \pm\sqrt{17})$

$y = \pm 4x$ asymptotes



23. $4x^2 - y^2 - 24x - 4y + 28 = 0$

$4(x^2 - 6x + 9) - (y^2 + 4y + 4) = -28 + 36 - 4$

$4(x-3)^2 - (y+2)^2 = 4$

$$\frac{(x-3)^2}{1} - \frac{(y+2)^2}{4} = 1$$

Hyperbola

$a = 1$

$b = 2$

$c^2 = 1 + 4 = 5 \Rightarrow c = \sqrt{5}$

$V = (\pm 1 + 3, -2) = \begin{matrix}(4, -2)\\(2, -2)\end{matrix}$

$y + 2 = \pm 2(x-3)$ asymptotes

$F = (\pm\sqrt{5} + 3, -2) = \begin{matrix}(\sqrt{5} + 3, -2)\\(-\sqrt{5} + 3, -2)\end{matrix}$

10.5

41 Ellipse Center $(-1, 4)$
vertex $(-1, 0)$
focus $(-1, 6)$

$$\frac{(x+1)^2}{b^2} + \frac{(y-4)^2}{a^2} = 1$$

F $(-1, 6)$

C $(-1, 4)$

V $(-1, 0)$

$$c^2 = a^2 - b^2$$

$$V = (-1, \pm a + 4) = (-1, 0) \Rightarrow a = 4$$

$$F\ (-1, \pm c + 4) = (-1, 6) \Rightarrow c = 2$$

$$4 = 16 - b^2$$
$$b^2 = 12$$

$$\underline{\frac{(x+1)^2}{12} + \frac{(y-4)^2}{16} = 1}$$

12.1 #1, 3, 7, 20, 28, 39

1. $(4, 0, -3)$

3. $A(-4, 0, -1)$
$B(3, 1, -5)$
$C(2, 4, 6)$
closest to yz plane $\Rightarrow |x|$ is smallest $\Rightarrow C$
lies on xz plane $\Rightarrow y = 0 \Rightarrow A$.

7. $P(3, -2, -3)$
$Q(7, 0, 1)$
$R(1, 2, 1)$

$$d(P,Q) = \sqrt{(7-3)^2 + (0+2)^2 + (1+3)^2} = \sqrt{16+4+16} = \sqrt{36} = 6$$
$$d(Q,R) = \sqrt{(1-7)^2 + (2-0)^2 + (1-1)^2} = \sqrt{36+4} = \sqrt{40}$$
$$d(R,P) = \sqrt{(3-1)^2 + (-2-2)^2 + (-3-1)^2} = \sqrt{4+16+16} = \sqrt{36} = 6$$

Isosceles triangle. Not right angle.

20. $(2, 1, 4)$ & $(4, 3, 10)$ are endpoints of diameter

$$\text{diameter} = \sqrt{(4-2)^2 + (3-1)^2 + (10-4)^2}$$
$$= \sqrt{4+4+36} = \sqrt{44} \Rightarrow \text{radius} = \frac{\sqrt{44}}{2} = \sqrt{\frac{44}{4}} = \sqrt{11}$$

The center is the midpt

$$C\left(\frac{4+2}{2}, \frac{3+1}{2} + \frac{10+4}{2}\right)$$
$$C(3, 2, 7)$$
$$(x-3)^2 + (y-2)^2 + (z-7)^2 = 11$$

28. $z^2 = 1 \Rightarrow z = \pm 1$ any x, y

$\Rightarrow$ 2 planes parallel to the xy coordinate plane, one is a unit higher and the other a unit lower

41 ~~30~~. $P(x,y,z)$ equidistant from $A(-1,5,3)$ & $(6,2,-2) = B$

$$d^2(P,A) = (x+1)^2 + (y-5)^2 + (z-3)^2 = (x-6)^2 + (y-2)^2 + (z+2)^2 = d^2(P,B)$$

$$\cancel{x^2} + 2x + 1 + \cancel{y^2} - 10y + 25 + \cancel{z^2} - 6z + 9 =$$

$$\cancel{x^2} - 12x + 36 + \cancel{y^2} - 4y + 4 + \cancel{z^2} + 4z + 4$$

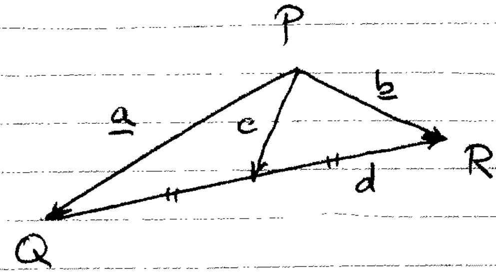
$$14x - 6y - 10z = -1 - 25 - 9 + 36 + 4 + 4$$

$$14x - 6y - 10z = 9$$

plane !

12-2 #7, 21, 29, 32, 35

7.



$\underline{c} + \frac{1}{2}\overrightarrow{QR} = \underline{b}$ or $\underline{c} + \underline{d} = \underline{b}$

also: $\underline{a} + 2\underline{d} = \underline{b}$

$\underline{d} = \underline{b} - \underline{c}$

$\underline{a} + 2\underline{b} - 2\underline{c} = \underline{b} \Rightarrow 2\underline{c} = \underline{a} + \underline{b}$

$\underline{c} = \frac{1}{2}(\underline{a} + \underline{b})$

and $\underline{d} = \underline{b} - \frac{1}{2}(\underline{a} + \underline{b}) = \frac{1}{2}(\underline{b} - \underline{a})$

21. $a = i + 2j - 3k$

$b = -2i - j + 5k$

$a + b = -i + j + 2k$

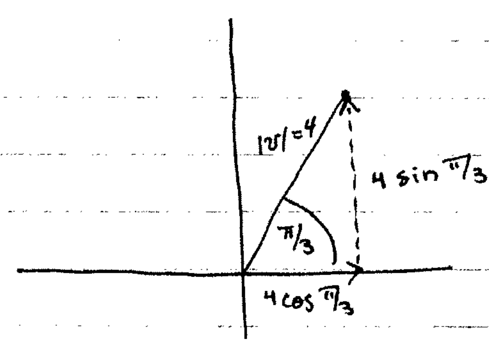
$2a + 3b = 2i + 4j - 6k + (-6i - 3j + 15k)$

$= -4i + j + 9k$

$|a| = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{1+4+9} = \sqrt{14}$

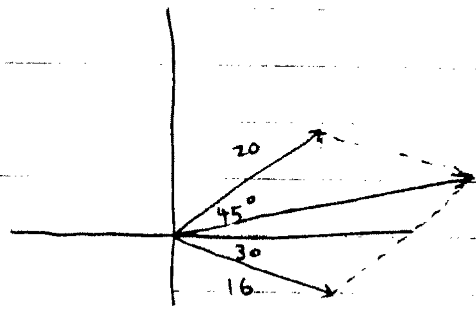
$|a - b| = \sqrt{3^2 + 3^2 + (-8)^2} = \sqrt{9 + 9 + 64} = \sqrt{82}$

29.



$\underline{v} = 4\cos\frac{\pi}{3}\, i + 4\sin\frac{\pi}{3}\, j$

32.



$\underline{a} = 20\cos 45°\, i + 20\sin 45°\, j$

$\underline{b} = 16\cos 30°\, i - 16\sin 30°\, j$

$\underline{a}+\underline{b} = \left(20\frac{\sqrt{2}}{2} + 16\frac{\sqrt{3}}{2}\right)i + \left(20\frac{\sqrt{2}}{2} - 16\cdot\frac{1}{2}\right)j$

$= (10\sqrt{2}+8\sqrt{3})i + (10\sqrt{2}-8)j$

$|a+b|^2 = (10\sqrt{2}+8\sqrt{3})^2 + (10\sqrt{2}-8)^2 =$

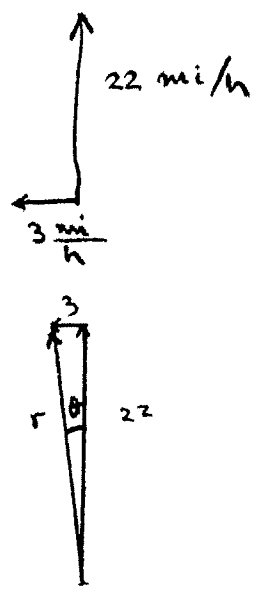
$= 200 + 160\sqrt{6} + 192 + 200 - 160\sqrt{2} + 64$

$= 656 + 160(\sqrt{6}-\sqrt{2})$

$|a+b| = \sqrt{656+160(\sqrt{6}-\sqrt{2})}$

$\tan\theta = \dfrac{10\sqrt{2}-8}{10\sqrt{2}+8\sqrt{3}}$ both positive $\Rightarrow$ first quadrant.

35.



$r = \sqrt{22^2+3^2} = \sqrt{484+9} = \sqrt{493}$

$\tan\theta = \frac{3}{22} \Rightarrow \theta = 8°$

12.3 #1, 14, 20, 23d, 25, 27, 49

1. a. no
   b. yes
   c. yes
   d. yes
   e. no
   f. no

14.

$$\underbrace{\begin{matrix} a & \text{hamburgers} \\ b & \text{hot dogs} \\ c & \text{soft drink} \end{matrix}}_{\text{vector } \vec{A}} \qquad \underbrace{\begin{matrix} \$2 & \text{each} \\ \$1.50 & \text{each} \\ \$1 & \text{each} \end{matrix}}_{\text{vector } \vec{P}}$$

total revenue is $\vec{A} \cdot \vec{P}$

20. $\vec{a} = i + 2j - 2k$
    $\vec{b} = 4i - 3k$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{4+6}{\sqrt{1+4+4}\,\sqrt{16+9}} = \frac{10}{3\cdot 5} = \frac{2}{3}$$

$$\theta = \arccos(2/3)$$

23d. $\vec{a} = 2i + 6j - 4k$
     $\vec{b} = -3i - 9j + 6k$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{-6-54-24}{\sqrt{4+36+16}\,\sqrt{9+81+36}} = \frac{\overset{\overset{14}{\cancel{42}}}{\cancel{-84}}}{\underbrace{\sqrt{56}}_{\cancel{2}\sqrt{14}}\;\underbrace{\sqrt{126}}_{\cancel{3}\cdot\sqrt{14}}} = -1$$

$\Rightarrow \theta = \pi$ parallel

25. $P(+1,-3,-2)$
$Q(2,0,-4)$
$R(6,-2,-5)$

$\vec{PQ} = \langle 1, 3, -2 \rangle$
$\vec{PR} = \langle 5, 1, -3 \rangle$
$\vec{QR} = \langle 4, -2, -1 \rangle$

Is $\vec{PQ} \perp \vec{PR}$?

$\vec{PQ} \cdot \vec{PR} = 5 + 3 + 6$ no
$\vec{PQ} \cdot \vec{QR} = 4 - 6 + 2 = 0$
$\Downarrow$
$\vec{PQ} \perp \vec{QR}$

right angle triangle

27. orthogonal to $i+j$ & $i+k$
$v = \langle a, b, c \rangle$

$\Rightarrow a + b = 0$
$a + c = 0$

$\Rightarrow v = \langle a, -a, -a \rangle$

to make it of length 1 $|\vec{v}| = a\sqrt{3}$

$\Rightarrow \underline{\frac{1}{\sqrt{3}} \langle 1, -1, -1 \rangle}$

49. $\vec{D} = (6-0)i + (12-10)j + (20-8)k = 6i + 2j + 12k$
$\vec{F} = 8i - 6j + 9k$
$W = \vec{F} \cdot \vec{D} = 8 \cdot 6 - 6 \cdot 2 + 9 \cdot 12 = 48 - 12 + 108 = \underline{144}$

12.4 #5, 6, 13, 15, 17, 19, 29, 38

5. $a = i - j - k$

$b = \frac{1}{2} i + j + \frac{1}{2} k$

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & -1 & -1 \\ \frac{1}{2} & 1 & \frac{1}{2} \end{vmatrix} = i \begin{vmatrix} -1 & -1 \\ 1 & \frac{1}{2} \end{vmatrix} - j \begin{vmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} + k \begin{vmatrix} 1 & -1 \\ \frac{1}{2} & 1 \end{vmatrix}$$

$$= i\left(-\frac{1}{2} + 1\right) - j\left(\frac{1}{2} + \frac{1}{2}\right) + k\left(1 + \frac{1}{2}\right)$$

$$a + b = \underline{\frac{1}{2} i - j + \frac{3}{2} k}$$

prove that $a \times b$ is orthogonal to a & to b.

$\underline{a} \cdot (a \times b) = \frac{1}{2} - 1 \cdot (-1) - 1 \cdot \frac{3}{2} = \frac{1}{2} + 1 - \frac{3}{2} = 0$ ✓

$\underline{b} \cdot (a \times b) = \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot (-1) + \frac{1}{2} \cdot \frac{3}{2} = \frac{1}{4} - 1 + \frac{3}{4} = 0$ ✓

6. $\underline{a} = t i + \cos t\, j + \sin t\, k$

$\underline{b}\ i - \sin t\, j + \cos t\, k$

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ t & \cos t & \sin t \\ 1 & -\sin t & \cos t \end{vmatrix} = i(\cos^2 t + \sin^2 t) - j(t \cos t - \sin t) + k(-t \sin t - \cos t)$$

$$\underline{a} \times \underline{b} = i - j(t \cos t - \sin t) - k(t \sin t + \cos t)$$

Check $\underline{a} \times \underline{b} \perp \underline{a}$

$$(\underline{a} \times \underline{b}) \cdot \underline{a} = t - (t \cos t - \sin t)\cos t - (t \sin t + \cos t) \sin t$$

$$= t - t\cos^2 t + t \cancel{\sin t} \cos t - t \sin^2 t - \cancel{t \sin t} \cos t$$

$$= t - t(\cos^2 t + \sin^2 t) = t - t = \underline{0}$$

$$(\underline{a} \times \underline{b}) \cdot \underline{b} = 1 + \sin t(t \cos t - \sin t) - \cos t(t \sin t + \cos t)$$

$$= 1 + t \cancel{\sin t} \cos t - \sin^2 t - t \cancel{\sin t} \cos t - \cos^2 t$$

$$= 1 - (\sin^2 t + \cos^2 t) = 1 - 1 = 0$$

13. a) $a \cdot (b \times c)$ yes

b) $a \times (b \cdot c)$ no $b \cdot c$ is scalar

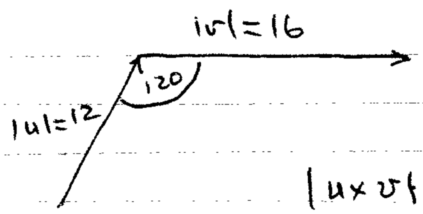
c) $a \times (b \times c)$ yes

d) $(a \cdot b) \times c$ no see b.

e) $(a \cdot b) \times (c \cdot d)$ no crossing 2 scalars

f) $(a \times b) \cdot (c \times d)$ yes

15.



$|u \times v| = |u|\,|v| \sin\theta = 12 \cdot 16 \cdot \sin 120$

directed into the page.

Actually

$v = 16i$ $\quad u = 12\cos 60 + 12 \sin 60$

$$u \times v = \begin{vmatrix} i & j & k \\ 12\cos 60 & 12 \sin 60 & 0 \\ 16 & 0 & 0 \end{vmatrix} = k(-12 \cdot 16 \sin 60)$$

17. $\underline{a} = \langle 2, -1, 3 \rangle \quad \underline{b} = \langle 4, 2, 1 \rangle$

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ 2 & -1 & 3 \\ 4 & 2 & 1 \end{vmatrix} = i(-1-6) - j(2-12) + k(4+4)$$

$$= \underline{-7i + 10j + 8k}$$

$$\underline{b} \times \underline{a} = \begin{vmatrix} i & j & k \\ 4 & 2 & 1 \\ 2 & -1 & 3 \end{vmatrix} = i(6+1) - j(12-2) + k(-4-4)$$

$$= \underline{7i - 10j - 8k} = -(a \times \underline{b})$$

19. $\underline{a} = \langle 3, 2, 1\rangle \quad \underline{b} = \langle -1, 1, 0\rangle$

$\underline{a}\times\underline{b}$ is orthogonal to both $= \begin{vmatrix} i & j & k \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{vmatrix} = i(-1) - j(1) + k(3+2)$

$\underline{v} = \underline{a}\times\underline{b} = -i - j + 5k$

$$\pm \frac{v}{|v|} = \pm \frac{-i-j+5k}{\sqrt{1+1+25}} = \mp \frac{1}{\sqrt{27}}(i+j-5k)$$

29. a. $P(1,0,1) \quad Q(-2,1,3) \quad R(4,2,5)$

$\overrightarrow{PQ} = \langle -3, 1, 2\rangle \qquad \overrightarrow{PR} = \langle 3, 2, 4\rangle$

$$\vec{n} = \overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -3 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = i(4-4) - j(-12-6) + k(-6-3)$$

$$= \underline{18j - 9k}$$

$n = 18j - 9k$

b. area of the triangle $= \frac{1}{2}|\overrightarrow{PQ}\times\overrightarrow{PR}|$

$$= \frac{1}{2}\sqrt{18^2+9^2} = \frac{1}{2}\sqrt{9^2\cdot 4 + 9^2} = \underline{\frac{1}{2}\cdot 9\sqrt{5}}$$

38. $\underline{a} = AB = (3-1)i + (-1-3)j + (6-2)k = 2i - 4j + 4k$

$\underline{b} = BC = (5-3)i + (2+1)j + (0-6)k = 2i + 3j - 6k$

$\underline{c} = CD = (3-5)i + (6-2)j + (-4-0)k = -2i + 4j - 4k$

$\underline{a}\cdot(b\times c) = \begin{vmatrix} 2 & -4 & 4 \\ 2 & 3 & -6 \\ -2 & 4 & -4 \end{vmatrix} = 0$ since first row is a multiple of the third.

check by computing:

$$= 2\begin{vmatrix} 3 & -6 \\ 4 & -4 \end{vmatrix} - (-4)\begin{vmatrix} 2 & -6 \\ -2 & -4 \end{vmatrix} + 4\begin{vmatrix} 2 & 3 \\ -2 & 4 \end{vmatrix}$$

$$= 2(-12+24) + 4(-8-12) + 4(8+6)$$

$$= 2\cdot 12 - 4\cdot 20 + 4\cdot 14 = 4(6-20+14) = 4\cdot 0 = 0 \quad \checkmark$$

12.5 #2, 5, 12, 13, 17, 21, 30, 37, 65, 69

2. $\underline{v} = \langle 1, 3, -\frac{2}{3} \rangle$

$P_0 = (6, -5, 2)$

$\frac{x-6}{1} = \frac{y+5}{3} = \frac{z-2}{-2/3}$ or $\underline{x-6 = \frac{y+5}{3} = -\frac{3}{2}(z-2)}$

or
$x = 6+t$
$y = -5+3t$
$z = 2-\frac{2}{3}t$

5. $P_0\,(1, 0, 6)$

Perpendicular to $x+3y+z=5$

$\underline{n} = i+3j+k$

This $\underline{n}$ is in the direction of the line thru P_0:

$\frac{x-1}{1} = \frac{y}{3} = \frac{z-6}{1}$ or $\underline{x-1 = \frac{y}{3} = z-6}$

or
$x = 1+t$
$y = 3t$
$z = 6+t$

12. The line of intersection of $x+2y+3z=1$ and
$x-y+z=1$

$n_1 = i+2j+3k$ $\qquad n_2 = i-j+k$

$n_1 \times n_2$ is in the direction of the line

$$\underline{v} = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & -1 & 1 \end{vmatrix} = i(2+3) - j(1-3) + k(-1-2)$$

$\underline{v} = 5i+2j-3k$

12. Cont.

A point on the line: choose $z=0$

$$\left.\begin{aligned} x+2y&=1 \\ x-y&=1 \end{aligned}\right\} -$$

$$3y=0 \Rightarrow y=0$$
$$\Rightarrow x=1$$

$P(1,0,0)$ is on the line of intersection

$$\frac{x-1}{5} = \frac{y-0}{2} = \frac{z-0}{-3}$$

13. line thru $(-4,-6,1)$
$(-2,0,-3)$

$$\vec{v}_1 = (-2+4)i + (0+6)j + (-3-1)k$$
$$= 2i+6j-4k$$

line thru $(10,18,4)$
$(5,3,14)$

$$\vec{v}_2 = -5i-15j+10k$$

Are $\vec{v}_1, \vec{v}_2$ parallel?

$$\vec{v}_1 \cdot \vec{v}_2 = -10-90-40 = -140$$
$$|\vec{v}_1| = \sqrt{4+36+16} = \sqrt{56}$$
$$|\vec{v}_2| = \sqrt{25+225+100} = \sqrt{350}$$

$$\cos\theta = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1||\vec{v}_2|} = \frac{-140}{\underbrace{\sqrt{56}}_{2\sqrt{14}}\,\underbrace{\sqrt{350}}_{5\sqrt{14}}} = \frac{-140}{10\cdot 14} = -1$$

yes

17. line segment from $(2,-1,4)$ to $(4,6,1)$

$$\vec{v} = (4-2)i + (6+1)j + (1-4)k$$

$$\vec{v} = 2i + 7j - 3k$$

$$\frac{x-2}{2} = \frac{y+1}{7} = \frac{z-4}{-3}$$ Equations of line

$x - 2 = 2t$
$y + 1 = 7t$ $\quad 0 \le t \le 1$ is the line segment.
$z - 4 = -3t$

21. $L_1: \quad x - 2 = \frac{y-3}{-2} = \frac{z-1}{-3}$

$L_2: \quad x - 3 = \frac{y+4}{3} = \frac{z-2}{-7}$

$\underline{v}_1 = i - 2j - 3k \qquad \underline{v}_2 = i + 3j - 7k$

parallel? $\quad \underline{v}_1 \neq \alpha \underline{v}_2 \quad$ no!

or: $$\cos\theta = \frac{v_1 \cdot v_2}{|v_1||v_2|} = \frac{1 - 6 + 21}{\sqrt{1+4+9}\sqrt{1+9+49}} = \frac{16}{\sqrt{14}\sqrt{59}} \neq \pm 1$$

Is there a point of intersection?

$x - 2 = t$	$x - 3 = \tau$
$y - 3 = -2t$	$y + 4 = 3\tau$
$z - 1 = -3t$	$z - 2 = -7\tau$

compare x & y on both lines $\quad x - 2 = \frac{y-3}{-2} \Rightarrow -2x + 4 = y - 3$

$\quad x - 3 = \frac{y+4}{3} \Rightarrow 3x - 9 = y + 4$

$-5x + 13 = -7$

$x = -\frac{20}{-5} = 4$

If $x = 4$ then $\frac{y-3}{-2} = 2 \Rightarrow y - 3 = -4 \Rightarrow \underline{y = -1}$

For $x = 4$ $\quad t = 2 \quad \tau = 1 \quad$ for these values $\quad y - 3 = -4 \Rightarrow y = -1$

$y + 4 = 3 \cdot 1 \Rightarrow y = -1$

21. Cont.

$$z = 1 - 3t = 1 - 6 = -5$$

$$z = 2 - 7\tau = 2 - 7 \cdot 1 = -5$$

Since the z values agree the point is $(4, -1, -5)$

Lines intersect!

30. The plane containing $x = 1 + t$, $y = 2 - t$, $z = 4 - 3t$ $\Rightarrow v = i - j - 3k$

$\underline{v}$ is in the plane!

Parallel to the plane $5x + 2y + z = 1 \Rightarrow n = 5i + 2j + k$

$\underline{n}$ is normal to the plane of interest.

Plane of interest $\quad 5(x - x_0) + 2(y - y_0) + (z - z_0) = 0$

The point (x_0, y_0, z_0) is on the plane <u>and</u> on the line

$\Rightarrow (1, 2, 4)$ is such a point $(t = 0!)$

$$\underline{5(x-1) + 2(y-2) + (z-4) = 0}$$

37. Plane thru $(-1,2,1)$ and contains line of intersection of the planes $\begin{cases} x+y-z=2 \\ 2x-y+3z=1 \end{cases}$

$x-2y+4z=1$

$\vec{n}_1 = i+j-k$ normal to plane 1
$\vec{n}_2 = 2i-j+3k$ normal to plane 2
$\vec{n}_1 \times \vec{n}_2$ is normal to both $\Rightarrow$ in the direction of line of intersection.

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 2 & -1 & 3 \end{vmatrix} = (3-1)i - (3+2)j + (-1-2)k = \boxed{2i-5j-3k}$$

We need a point on the 2 planes. Since we have 2 equations with 3 unknowns, we choose one say

$x=0 \Rightarrow \left.\begin{matrix} y-z=2 \\ -y+3z=1 \end{matrix}\right\} +$

$2z=3 \Rightarrow z=3/2 \Rightarrow y=2+3/2=7/2$

$$\begin{cases} x = t \\ y = \frac{7}{2} - \frac{5}{2}t \\ z = \frac{3}{2} - \frac{3}{2}t \end{cases}$$ equations of the line

Take 2 points on the lines
$Q\,(0,\ 7/2,\ 3/2)$
$R\,(1,\ 1,\ 0)$

Using $P(-1,2,1)$ and those 2 we can find the equation of a plane

$\vec{QR} = (1, -5/2, -3/2)$
$\vec{PR} = (-2, 1, 1)$

$\vec{QR} \times \vec{PR}$ is normal to the plane

37. cont.

$$\vec{QR} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 1 & -5/2 & -3/2 \\ -2 & 1 & 1 \end{vmatrix} = i\left(-5/2 + 3/2\right) - j\left(1-3\right) + k\left(1-5\right)$$

$$= -i + 2j - 4k$$

$-(x+1) + 2(y-2) - 4(z-1) = 0$ (I used P)

$-x \underline{-1} + 2y - \cancel{4} - 4z + \cancel{4} = 0$

$-x + 2y - 4z = 1$

$\boxed{x - 2y + 4z = -1}$

Another way

$x=0$

$$\left.\begin{aligned} y - z &= 2 \\ -y + 3z &= 1 \end{aligned}\right\} +$$

$2z = 3$

$z = 3/2 \qquad y = 2 + 3/2 = 7/2$

$Q\left(0, 7/2, 3/2\right)$

$z=0$

$$\left.\begin{aligned} x + y &= 2 \\ 2x - y &= 1 \end{aligned}\right\} +$$

$3x = 3$

$x = 1$

$y = 2 - 1 = 1$

$R(1, 1, 0)$

These are the same points we had before
we only need $\vec{PR} \times \vec{QR}$ and the rest as before.

12.5

65. line thru $P(0,1,2)$ parallel to $x+y+z=2$
& perpendicular to line

$$\begin{cases} x = 1+t \\ y = 1-t \\ z = 2t \end{cases}$$

Clearly P is <u>not</u> on the plane

$n = i+j+k$ is normal to plane
$v = i-j+2k$ is in the direction of the line
we want a line normal to v and to n (parallel to plane)

$$n \times v = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = i\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} - j\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + k\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$= \underline{3i - j - 2k}$ in the direction of line we want

$$\underline{\frac{x-0}{3} = \frac{y-1}{-1} = \frac{z-2}{-2}}$$

or
$$\begin{cases} x = 3t \\ y = 1-t \\ z = 2-2t \end{cases}$$

69. $P(4,1,-2)$ $\quad a = \overrightarrow{QR} = v$ for our line
$= i - 2j - 3k$
$b = \overrightarrow{QP}$ from Q on the line to P
Say $Q = (1,3,4)$ on the line

$$\begin{cases} x = 1+t \\ y = 3-2t \\ z = 4-3t \end{cases}$$

$b = \overrightarrow{QP} = -3i + 2j + 6k$

12.5
69 cont.

$$d = \frac{|a \times b|}{|a|}$$

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & -2 & -3 \\ -3 & 2 & 6 \end{vmatrix} = i(-12+6) - j(6-9) + k(2-6)$$

$$= -6i + 3j - 4k$$

$$|a \times b| = \sqrt{36+9+16} = \sqrt{61}$$

$$|a| = \sqrt{1+4+9} = \sqrt{14}$$

$$d = \sqrt{\frac{61}{14}}$$

12.6 #21-28, 29, 33

21. ellipsoid, so either $\underline{IV}$ or $\underline{VII}$

$$\frac{x^2}{1} + \frac{y^2}{1/4} + \frac{z^2}{1/9} = 1 \Rightarrow \text{major axis in x-direction}$$

$\Rightarrow$ $\underline{VII}$

22. $9x^2 + 4y^2 + z^2 = 1$
another ellipsoid $\Rightarrow$ $\underline{IV}$

23. $x^2 - y^2 + z^2 = 1$
hyperboloid one sheet $\Rightarrow$ $\underline{II}$

24. $-x^2 + y^2 - z^2 = 1$
hyperboloid 2 sheets $\Rightarrow$ $\underline{III}$

25. $y = 2x^2 + z^2$
paraboloid $\Rightarrow$ $\underline{VI}$

26. $y^2 = x^2 + 2z^2$
cone $\Rightarrow$ I

27. $x^2 + 2z^2 = 1$
cylinder $\Rightarrow$ $\underline{VIII}$

28. $y = x^2 - z^2$ $\Rightarrow$ $\underline{V}$

29. old $z^2 = 4x^2 + 9y^2 + 36$

$4x^2 + 9y^2 - z^2 = -36$

$-\frac{x^2}{9} - \frac{y^2}{4} + z^2 = 1$

hyperboloid 2 sheet

33. $4x^2 + y^2 + 4z^2 - 4y - 24z + 36 = 0$

$4x^2 + (y^2 - 4y + 4) + 4(z^2 - 6z + 9) = -36 + 4 + 36$

$4x^2 + (y-2)^2 + 4(z-3)^2 = 4$

$x^2 + \frac{(y-2)^2}{4} + (z-3)^2 = 1$

ellipsoid centered at $(0, 2, 3)$

29 $y^2 = x^2 + \frac{1}{9}z^2$

It is a cone

31. $x^2 + 2y - 2z^2 = 0$

$2y = 2z^2 - x^2$

$y = z^2 - \frac{1}{2}x^2$

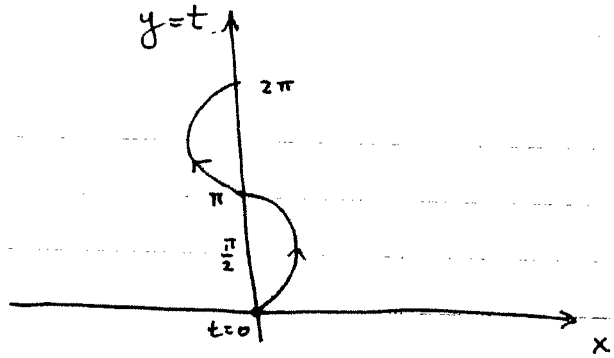
hyperbolic paraboloid

13.1 # 7, 11, 17, 39, 40, 41, 47

7. $\vec{r}(t) = \langle \sin t, t \rangle$

Sketch the curve

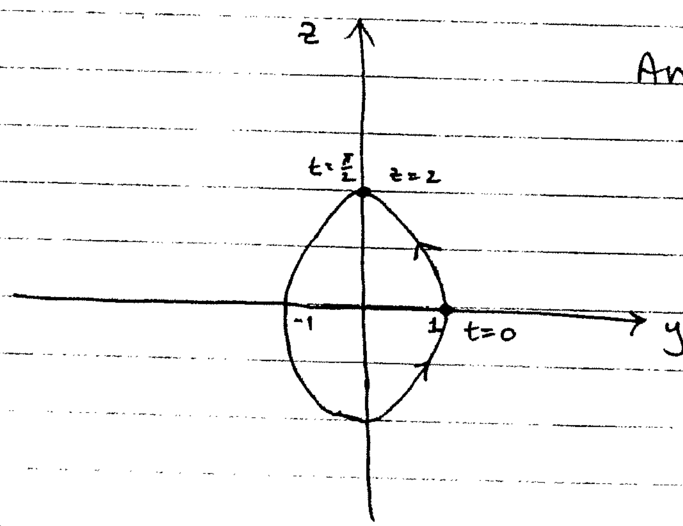
$y = t$ then $x = \sin y$



11. $r(t) = \langle 1, \cos t, 2\sin t \rangle$

$x \equiv 1$
$y = \cos t$
$z = 2\sin t$

curve is in a plane parallel to yz plane, one unit to the right



An ellipse in the plane.

17. Find vector eq. & parametric eqs for the line segment from P to Q

$P(2,0,0) \qquad Q(6,2,-2)$

$\vec{v}$ - vector $= \vec{PQ} = 4i + 2j - 2k$

$$\left.\begin{aligned} x-2 &= 4t \\ y &= 2t \\ z &= -2t \end{aligned}\right\} \qquad 0 \le t \le 1$$

$\vec{r} = \langle 4t+2,\ 2t,\ -2t \rangle$

39. $x = t^2$

$y = 1-3t$ pass thru $(1,4,0)$ & $(9,-8,28)$

$z = 1+t^3$

$x=1 \Rightarrow t = \pm 1$

to get $y = 4 = 1-3t \Rightarrow t = -1 \Rightarrow z = 0$ ✓

$y = 1-3t = -8$

$1+8 = 3t$

$3 = t$

$\Rightarrow x = 3^2 = 9$ ✓

$z = 1+3^3 = 1+27 = 28$ ✓

Not thru $(4,7,-6)$

$1-3t = 7 \Rightarrow 3t = 1-7 = -6$

$t = -2 \Rightarrow x = (-2)^2 = 4$ ✓

$z = 1+(-2)^3 = 1-8 = -7 \neq -6$

40. Find a vector function that represents the curve of intersection of

1. $x^2+y^2=4$ cylinder
2. $z=xy$

$x=\frac{z}{y}$ (assuming $y\neq 0$)

$\frac{z^2}{y^2}+y^2=4$

$z^2+y^4=4y^2$ or $y^4-4y^2+z^2=0$

$y^2_{1,2}=\frac{4\pm\sqrt{16-4z^2}}{2}=2\pm\sqrt{4-z^2}$

messy.

Try cylindrical coordinates:

1. is $r^2=4$ or $r=2$
2. is $z=r^2\cos\theta\sin\theta$

$\Downarrow$

$z=4\cos\theta\sin\theta$

$$\vec{r}(t)=\langle \underbrace{2\cos\theta}_{x}, \underbrace{2\sin\theta}_{y}, \underbrace{4\cos\theta\sin\theta}_{=2\sin 2\theta} \rangle \quad 0\le\theta\le 2\pi$$

41.

$z=\sqrt{x^2+y^2}$ cone

$z=1+y$ plane

$x^2+y^2=(1+y)^2$

$x^2+\cancel{y^2}=\cancel{y^2}+2y+1$

$x^2=2y+1$ parabola

$z=1+y=1+\frac{x^2-1}{2}=\frac{1}{2}x^2+\frac{1}{2}$

$$\vec{r}=\langle \underbrace{t}_{=x}, \underbrace{\frac{t^2-1}{2}}_{=y}, \underbrace{\frac{t^2+1}{2}}_{z=1+y} \rangle \quad \text{any } t.$$

check that $x^2+y^2=z^2$.

47. Suppose

$\vec{r}_1(t) = \langle t^2, 7t-12, t^2 \rangle$

$\vec{r}_2(t) = \langle 4t-3, t^2, 5t-6 \rangle$ $\quad t \geq 0$

Do the particles collide

$t^2 = 4t-3$ $\qquad 7t-12 = t^2$ $\qquad t^2 = 5t-6$

$t^2 - 4t + 3 = 0$

$t = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm 2}{2} \begin{matrix} \nearrow 3 \\ \searrow 1 \end{matrix}$

$7 \cdot 3 - 12 = 9 = 3^2 \checkmark$

$7 \cdot 1 - 12 = -5 \neq 1^2$

So only $t=3$ is possible.

How about z at that time?

$3^2 = 9 \qquad 5 \cdot 3 - 6 = 15 - 6 = 9$

YES

At $t=3$

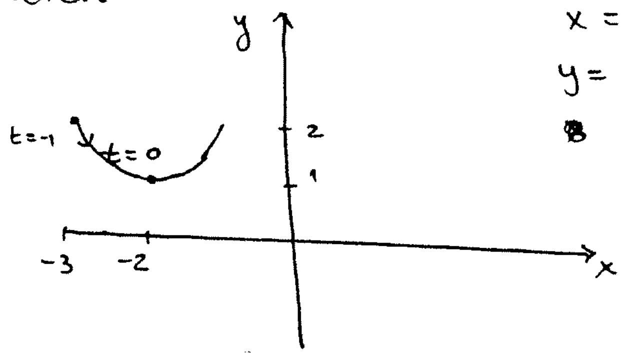
$x = t^2 = 9 \qquad y = 7 \cdot 3 - 12 = 9 \qquad z = t^2 = 9$

The point of collision $(9, 9, 9)$

13.2 # 3, 6, 10, 15, 17, 19, 21, 23, 33, 36

#3 $\vec{r}(t) = \langle t-2, t^2+1 \rangle$ $t = -1$

a. sketch



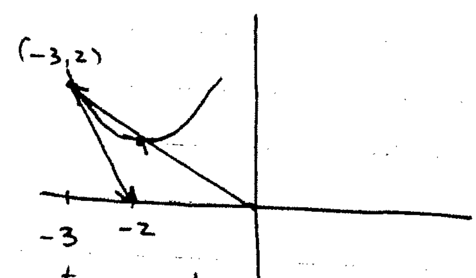
$\left.\begin{array}{l} x = t-2 \\ y = t^2+1 \end{array}\right\} \Rightarrow t = x+2$

$y = (x+2)^2+1$

Parabola

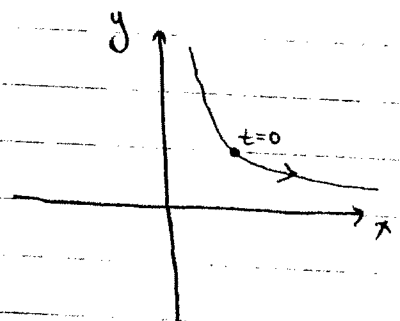
b. $\vec{r}\,'(t) = \langle 1, 2t \rangle$

c.



6. $\vec{r}(t) = e^t i + e^{-t} j$ $t=0$

a. sketch

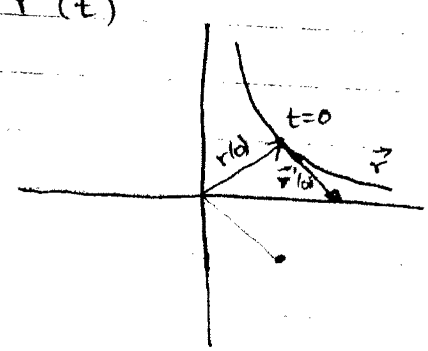


$x = e^t$ $y = e^{-t} = \frac{1}{x}$

$\Rightarrow y = \frac{1}{x}$ but both $x, y > 0$

b. $\vec{r}\,'(t) = e^t i - e^{-t} j$

c. sketch $\vec{r}\,'(t)$



13.2

10. $\vec{r}(t) = \langle \tan t, \sec t, 1/t^2 \rangle$

$$\vec{r}\,'(t) = \left\langle \left(\frac{\sin t}{\cos t}\right)', \left(\frac{1}{\cos t}\right)', (t^{-2})' \right\rangle$$

$$= \left\langle \frac{1}{\cos^2 t}, \frac{+\sin t}{\cos^2 t}, -2t^{-3} \right\rangle$$

$$= \langle \sec^2 t, \tan t \sec t, -2t^{-3} \rangle$$

15. $\vec{r}(t) = \vec{a} + t\vec{b} + t^2\vec{c}$

$\vec{r}\,'(t) = \vec{b} + 2t\vec{c}$ (assuming $\vec{a}, \vec{b}, \vec{c}$ indep. of t)

17. $\vec{r}(t) = \langle te^{-t}, 2\arctan t, 2e^t \rangle$ $t=0$

To find $T(t)$ we need $\vec{r}\,'(t)$

$$\vec{r}\,'(t) = \left\langle e^{-t} - te^{-t}, \frac{2}{1+t^2}, 2e^t \right\rangle$$

$$|\vec{r}\,'(t)|^2 = \left[e^{-t}(1-t)\right]^2 + \frac{4}{(1+t^2)^2} + 4e^{2t}$$

$$= e^{-2t}(1-2t+t^2) + \frac{4}{(1+t^2)^2} + 4e^{2t}$$

$$|\vec{r}\,'(0)|^2 = 1\cdot 1 + \frac{4}{1} + 4 = 9$$

$$\vec{r}\,'(0) = \left\langle 1-0, \frac{2}{1}, 2 \right\rangle = \langle 1, 2, 2 \rangle$$

$$\vec{T}(0) = \frac{\langle 1, 2, 2 \rangle}{\sqrt{9}} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$$

13.2

19. $\vec{r}(t) = \cos t\ i + 3t\,j + 2\sin 2t\ k$

$r'(t) = -\sin t\ i + 3j + 4\cos 2t\ k$

$r'(0) = 3j + 4k$

$|r'(0)|^2 = 9 + 16 = 25 \Rightarrow |\vec{r}'(0)| = 5$

$\vec{T}(0) = \frac{3}{5}j + \frac{4}{5}k$

21. $\vec{r}(t) = \langle t, t^2, t^3 \rangle$

$\underline{\vec{r}'(t) = \langle 1, 2t, 3t^2 \rangle}$

$\underline{\vec{r}''(t) = \langle 0, 2, 6t \rangle}$

$T(1) = \dfrac{\vec{r}'(1)}{|\vec{r}'(1)|} = \dfrac{\langle 1, 2, 3 \rangle}{\sqrt{1+4+9}} = \underline{\left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle}$

$\vec{r}'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = i \begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix} - j \begin{vmatrix} 1 & 3t^2 \\ 0 & 6t \end{vmatrix} + k \begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix}$

$= i\,(12t^2 - 6t^2) - j \cdot 6t + k \cdot 2$

$= \underline{6t^2 i - 6t j + 2k}$

23. $x = 1 + 2\sqrt{t}$

$y = t^3 - t \qquad (3, 0, 2) \Rightarrow \underline{t = 1}$

$z = t^3 + t$

$x' = \frac{1}{\sqrt{t}} \qquad x'(1) = 1$

$y' = 3t^2 - 1 \Rightarrow y'(1) = 2 \Rightarrow \dfrac{x-3}{1} = \dfrac{y}{2} = \dfrac{z-2}{4}$

$z' = 3t^2 + 1 \qquad z'(1) = 4$

or

$$\begin{cases} x = 3 + t \\ y = 2t \\ z = 2 + 4t \end{cases}$$

13.2

33. $\vec{r}_1 = \langle t, t^2, t^3 \rangle$
$\vec{r}_2 = \langle \sin t, \sin 2t, t \rangle$

The origin is when $t=0$

$\vec{r}_1\,' = \langle 1, 2t, 3t^2 \rangle$
$\vec{r}_2\,' = \langle \cos t, 2\cos 2t, 1 \rangle$

$$\cos\theta = \left.\frac{\vec{r}_1\,' \cdot \vec{r}_2\,'}{|\vec{r}_1\,'|\,|\vec{r}_2\,'|}\right|_{t=0} = \frac{\langle 1,0,0\rangle \cdot \langle 1,2,1\rangle}{\sqrt{1^2}\,\sqrt{1^2+2^2+1^2}} = \frac{1}{\sqrt{6}}$$

$\Rightarrow \quad \theta \sim 66°$

36. $$\int_0^1 \left(\frac{4}{1+t^2}\,j + \frac{2t}{1+t^2}\,k\right) dt$$

$$\int_0^1 \frac{4}{1+t^2}\,dt = 4 \arctan t \Big|_0^1 = 4\left(\frac{\pi}{4} - 0\right)$$

$$\int_0^1 \frac{2t}{1+t^2}\,dt = \int_1^2 \frac{du}{u} = \ln|u| \Big|_1^2 = \ln 2 - 0$$

$u = 1+t^2 \qquad du = 2t\,dt$

$$\int_0^1 \left(\frac{4}{1+t^2}\,j + \frac{2t}{1+t^2}\,k\right) dt = \pi j + \ln 2\, k$$

13.3 #1, 9, 13, 18, 21, 23, 27

1. $\vec{r}(t) = \langle t, 3\cos t, 3\sin t \rangle \quad -5 \le t \le 5$

$r' = \langle 1, -3\sin t, 3\cos t \rangle$

$|r'| = \sqrt{1 + 9\sin^2 t + 9\cos^2 t} = \sqrt{1+9} = \sqrt{10}$

$$\int_{-5}^{5} \sqrt{10}\, dt = \sqrt{10}\, t \Big|_{-5}^{5} = 5\sqrt{10} + 5\sqrt{10} = \underline{10\sqrt{10}}$$

9. $\vec{r} = \langle \sin t, \cos t, \tan t \rangle, \quad 0 \le t \le \pi/4$

$\vec{r}\,' = \langle \cos t, -\sin t, \sec^2 t \rangle$

$|\vec{r}\,'| = \sqrt{\cos^2 t + \sin^2 t + \sec^4 t} = \sqrt{1 + \sec^4 t}$

$$\int_0^{\pi/4} \sqrt{1 + \sec^4 t}\, dt$$ Approximate the integral.

Trapezoidal rule:

$$\frac{\sqrt{1+\sec^4 0} + \sqrt{1+\sec^4 \frac{\pi}{4}}}{2} \cdot \frac{\pi}{4} = \frac{\sqrt{1+1^4} + \sqrt{1+(\sqrt{2})^4}}{2} \frac{\pi}{4} =$$

$$= \frac{\pi}{8}\left(\sqrt{2} + \sqrt{5}\right) \sim 1.433$$

actually we should take more points to get higher accuracy.
With these 2 end points we get ~ 1.433

Using Maple

```
with(student[Calculus1]):
ApproximateInt(sqrt(1+(sec(t))^4), t=0..Pi/4, method=trapezoid):
evalf(%);
```

we get 1.279812861

13. $\vec{r} = \langle 2t,\ 1-3t,\ 5+4t \rangle$

$$\vec{r}\,' = \langle 2, -3, 4 \rangle$$

$$|\vec{r}\,'| = \sqrt{4+9+16} = \sqrt{29}$$

$$s = \int_0^t \sqrt{29}\,dt = \sqrt{29}\,t$$

$$t = \frac{s}{\sqrt{29}}$$

$$\vec{r} = \left\langle \frac{2}{\sqrt{29}}s,\ 1-\frac{3}{\sqrt{29}}s,\ 5+\frac{4}{\sqrt{29}}s \right\rangle$$

18. $\vec{r} = \langle t^2, \sin t - t\cos t, \cos t + t\sin t \rangle, \quad t > 0$

a. $\vec{r}\,' = \langle 2t, \cancel{\cos t} - \cancel{\cos t} + t\sin t, -\cancel{\sin t} + \cancel{\sin t} + t\cos t$

$\vec{r}\,' = \langle 2t, t\sin t, t\cos t \rangle$

$|\vec{r}\,'|^2 = 4t^2 + t^2\sin^2 t + t^2\cos^2 t =$

$= 4t^2 + t^2(\underbrace{\sin^2 t + \cos^2 t}_{=1}) = 5t^2$

$|\vec{r}\,'| = t\sqrt{5}$

$\vec{T} = \langle \frac{2t}{t\sqrt{5}}, \frac{t\sin t}{t\sqrt{5}}, \frac{t\cos t}{t\sqrt{5}} \rangle$

$\vec{T} = \langle \frac{2}{\sqrt{5}}, \frac{\sin t}{\sqrt{5}}, \frac{\cos t}{\sqrt{5}} \rangle$

$T' = \langle 0, \frac{\cos t}{\sqrt{5}}, \frac{-\sin t}{\sqrt{5}} \rangle$

$|T'|^2 = \frac{\cos^2 t}{5} + \frac{(-\sin t)^2}{5} = \frac{1}{5}$

$|T'| = \frac{1}{\sqrt{5}}$

$N = \frac{T'}{|T'|} = \langle 0, \cos t, -\sin t \rangle$

b. $\kappa(t) = \frac{|T'|}{|r'|} = \frac{1/\sqrt{5}}{t\sqrt{5}} = \frac{1}{5t}$

13.3

21. $\vec{r} = t^3 j + t^2 k$

$$\kappa = \frac{|\vec{r}\,' \times r''|}{|r'|^3}$$

$$\vec{r}\,' = 3t^2 j + 2tk$$
$$\vec{r}\,'' = 6tj + 2k$$

$$\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} i & j & k \\ 0 & 3t^2 & 2t \\ 0 & 6t & 2 \end{vmatrix} = i(6t^2 - 12t^2) = \underline{-6t^2 i}$$

$$|\vec{r}\,' \times \vec{r}\,''| = -6t^2 i$$
$$|\vec{r}\,'| = \sqrt{9t^4 + 4t^2} = \pm\sqrt{9t^2+4}$$

$$K = \frac{6t^2}{t^3(9t^2+4)^{3/2}}$$

23. $\vec{r} = 3t\,i + 4\sin t\,j + 4\cos t\,k$

$$\vec{r}\,' = 3i + 4\cos t\,j - 4\sin t\,k$$
$$\vec{r}\,'' = \quad -4\sin t\,j - 4\cos t\,k$$

$$\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} i & j & k \\ 3 & 4\cos t & -4\sin t \\ 0 & -4\sin t & -4\cos t \end{vmatrix} = i(\underbrace{-16\cos^2 t - 16\sin^2 t}_{=-16}) - j(-12\cos t) + k(-12\sin t)$$

$$|\vec{r}\,' \times \vec{r}\,''|^2 = (-16)^2 + (12\cos t)^2 + (-12\sin t)^2$$
$$= 16^2 + 12^2 = \begin{array}{r} 256 \\ 144 \\ \hline 400 \end{array} = 400$$

$$|\vec{r}\,' \times \vec{r}\,''| = 20$$
$$|r'|^2 = 9 + 16 = 25$$

$$K = \frac{20}{5^3} = \frac{4}{25}$$

13.3

27. $y = x^4$

$f(x) = x^4$

$f'(x) = 4x^3$

$f''(x) = 12x^2$

$$K(x) = \frac{|f''(x)|}{[1 + f'(x)^2]^{3/2}} = \frac{12x^2}{[1 + 16x^6]^{3/2}}$$

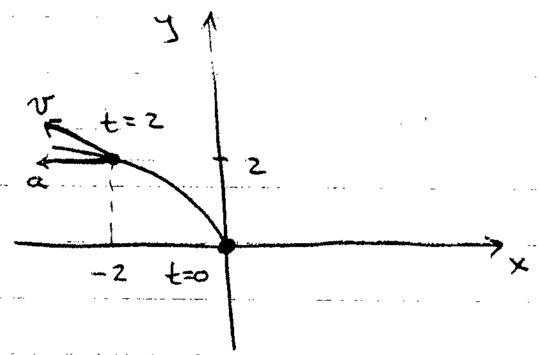
13.4 #3, 10, 15, 19, 23, 25

3. $\vec{r} = \langle -\frac{1}{2}t^2, t \rangle \qquad t = 2$

$\vec{v} = \vec{r}\,' = \langle -t, 1 \rangle$ velocity

$\vec{a} = \vec{r}\,'' = \langle -1, 0 \rangle$ acceleration

$|v| = \sqrt{1+t^2}$ speed



10. $\vec{r}(t) = \langle 2\cos t, 3t, 2\sin t \rangle$

$\vec{v}, \vec{a}, |\vec{v}|$

$\vec{v} = \langle -2\sin t, 3, 2\cos t \rangle$

$\vec{a} = \langle -2\cos t, 0, -2\sin t \rangle$

$|\vec{v}| = \sqrt{4\sin^2 t + 9 + 4\cos^2 t} = \sqrt{4+9} = \sqrt{13}$

13.4

15. $\vec{a} = i + 2j \qquad v(0) = k \qquad r(0) = i$

$$v(t) = \int_0^t (i+2j)\,dt = ti + 2tj + C$$

$$v(0) = k \Rightarrow C = k$$

$$\underline{v(t) = ti + 2tj + k}$$

$$r(t) = \frac{t^2}{2} i + t^2 j + tk + K$$

$$r(0) = K = i$$

$$\Rightarrow \underline{r(t) = \left(1 + \frac{t^2}{2}\right) i + t^2 j + tk}$$

19. $\vec{r} = \langle t^2, 5t, t^2 - 16t \rangle$

$$\vec{v} = \langle 2t, 5, 2t-16 \rangle$$

$$|\vec{v}| = \sqrt{4t^2 + 25 + (2t-16)^2}$$

Min $|\vec{v}|$ is the same as Min $|\vec{v}|^2$

To minimize $|\vec{v}|^2$, we find $\frac{d|\vec{v}|^2}{dt} = 0$

$$8t + 2(2t-16)\cdot 2 = 0$$

$$8t + 8t - 4\cdot 16 = 0$$

$$16t = 4\cdot 16$$

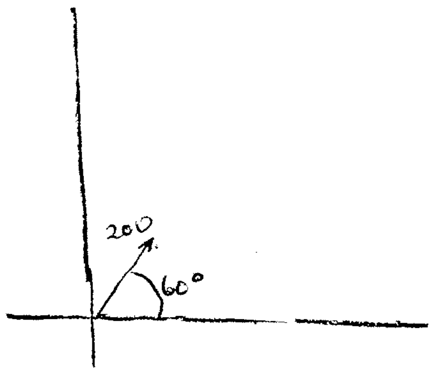
$$t = 4 \quad \text{at } t=4 \text{ we have a minimum}$$

$$|\vec{v}(4)| = \sqrt{64 + 25 + (-8)^2} = \sqrt{64 + 25 + 64} = \sqrt{153}$$

23. $|\vec{v}(0)| = 200 \frac{m}{s}$

angle $= 60^\circ$

Find $\begin{cases} \text{range} \\ \text{max height} \\ \text{speed at impact} \end{cases}$



$\vec{a} = -gj = -9.8 \frac{m}{s^2} j$

$\vec{v}(t) = -gtj + \vec{C}$

$\vec{C} = \vec{v}(0) = 200 \cos 60^\circ i + 200 \sin 60^\circ j$

$\vec{v}(t) = 200 \underbrace{\cos 60^\circ}_{1/2} i + (200 \underbrace{\sin 60^\circ}_{\sqrt{3}/2} - 9.8t) j$

$\vec{r}(t) = 100\ ti + (100\sqrt{3}t - \frac{9.8}{2}t^2) j + \vec{K}$

$\vec{K} = \vec{r}(0) = 0$

$\Rightarrow \quad x(t) = 100t \qquad y(t) = 100\sqrt{3}t - 4.9t^2$

range = horizontal distance is when $y=0 \Rightarrow t(100\sqrt{3} - 4.9t) = 0$

$\Rightarrow$ range $= x\left(\frac{100\sqrt{3}}{4.9}\right) = 100 \frac{100\sqrt{3}}{4.9} \cong \underline{3534.8\ m} \qquad \Rightarrow t = \frac{100\sqrt{3}}{4.9}$

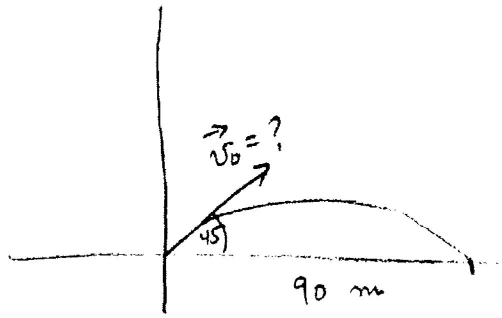
max height = when $y'(t) = 0 \Rightarrow 100\sqrt{3} - 9.8t = 0 \Rightarrow t = \frac{100\sqrt{3}}{9.8}$

$y\left(\frac{100\sqrt{3}}{9.8}\right) = 100\sqrt{3}\frac{100\sqrt{3}}{9.8} - 4.9\left(\frac{100\sqrt{3}}{9.8}\right)^2 \cong \underline{1530.61\ m}$

$\left|\vec{v}\left(\frac{100\sqrt{3}}{4.9}\right)\right|$ is speed at impact $= \sqrt{100^2 + \left(100\sqrt{3} - \cancel{9.8}^{2} \frac{100\sqrt{3}}{\cancel{4.9}}\right)^2}$

$= \sqrt{10000 + 10000 \cdot 3} = \underline{\underline{200}}$

25.



$$\vec{a} = -g j$$

$$\vec{v} = -gt j + \vec{v}_0$$

$$\vec{r}(t) = -\tfrac{1}{2} g t^2 j + t\vec{v}_0 + \underbrace{\vec{D}}_{=0}$$

$$\downarrow$$

$$\vec{v}_0 = |\vec{v}_0| \cos\alpha \, i + |\vec{v}_0| \sin\alpha \, j$$

$$x = |\vec{v}_0| \cos\alpha \; t \qquad y = \left(|\vec{v}_0| \sin\alpha - \tfrac{1}{2} g t\right) t$$

$x = 90$ at impact, i.e. when $y = 0$

$$|\vec{v}_0| \sin 45° - \tfrac{1}{2} g t = 0$$

$$t = \frac{2 |\vec{v}|_0 \sin 45}{g}$$

$$x = |\vec{v}_0| \cos 45° \frac{2 |\vec{v}_0| \sin 45°}{g} = \frac{\overbrace{2 \sin 45° \cos 45°}^{= \sin 90° = 1} |v_0|^2}{g}$$

$$x = \frac{|\vec{v}_0|^2}{g} = 90 \Rightarrow |\vec{v}_0|^2 = 90 \cdot g$$

$$|\vec{v}_0| = \sqrt{90 g} \sim 30 \tfrac{m}{s}$$

14.1 # 10, 32, 36, 64

10. $F(x,y) = 1 + \sqrt{4-y^2}$

$F(3,1) = 1 + \sqrt{4-1^2} = 1+\sqrt{3}$

Domain of F: $4-y^2 \geq 0$

or: $y^2 \leq 4$ or $-2 \leq y \leq 2$

Range of F: when $y=\pm 2$ we have 1 (lowest value)

when $y=0$ we have $1+2$ (highest — " —)

$\Rightarrow$ Range of F: $[1,3]$

32. Match

a. $f(x,y) = |x|+|y|$ always ≥ 0 linear in each — VI

b. $f(x,y) = |xy|$ always ≥ 0 quadratic — V

c. $f(x,y) = \frac{1}{1+x^2+y^2}$ alway ≥ 0 and below 1 — I

d. $f(x,y) = (x^2-y^2)^2$ vanishes when $x=\pm y$ — IV

e. $f(x,y) = (x-y)^2$ vanishes when $y=x$ — II

f. $f(x,y) = \sin(|x|+|y|)$ oscillatory — III

36. Cone is on 2 — contours are equidistant

64. $f(x,y) = z = \frac{x-y}{1+x^2+y^2}$

contours $x-y = K(1+x^2+y^2)$

$$Kx^2 - x + Ky^2 + y = -K$$

$$K\left(x^2 - \frac{x}{K} + \frac{1}{4K^2}\right) + K\left(y^2 + \frac{y}{K} + \frac{1}{4K^2}\right) = -K + \frac{1}{4K} + \frac{1}{4K}$$

$$\left(x - \frac{1}{2K}\right)^2 + \left(y + \frac{1}{2K}\right)^2 = \frac{-K + \frac{1}{2K}}{K} = \frac{-2K^2+1}{2K^2}$$

ellipses (if $2K^2-1<0$)

14.2 #7, 9, 13, 20, 29

7. $$\lim_{(x,y)\to(2,1)} \frac{4-xy}{x^2+3y^2} = \frac{4-2}{4+3} = \frac{2}{7}$$

9. $$\lim_{(x,y)\to(0,0)} \frac{x^4-4y^2}{x^2+2y^2}$$

If we go along y axis then the limit is -2

If we go along x axis then the $\lim_{x\to 0} x^2 = 0$

$\Rightarrow$ limit does not exist.

13. $$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0 \quad \text{on any curve}$$

Note that $0 \le \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \le |x|$ since $|y| < \sqrt{x^2+y^2}$ using Pythagorean Thm.

So when $(x,y)\to(0,0)$ $|x|\to 0$ and we sandwiched the function.

14.2

20. $$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz}{x^2+y^2+z^2} =$$

Along any of the axes the limit is zero
(say x axis then $y=z=0$)

Along the line $y=x$ in the xy plane then $z=0$

we get $$\lim_{x,y\to(0,0)} \frac{x^2}{x^2+x^2} = \frac{1}{2} \neq 0$$

$\Rightarrow$ Does not exist

29. $$f = \frac{xy}{1+e^{x-y}}$$

The problem with continuity only when denominator vanishes, i.e. $1+e^{x-y}=0$ which is not possible.

Therefore the function is always continuous.

14.3. # 13, 21, 25, 27, 33, 35, 41, 43, 47, 51

13. $f(x,y) = x^2 y^3$ Domain = all x and all y = $\mathbb{R}^2$

$f_x = 2xy^3$ $f_y = 3x^2y^2$

See Maple Solution14-3-13.mws

21. $f = \frac{x}{y}$

$f_x = \underline{\frac{1}{y}}$ $f_y = (xy^{-1})_y = -xy^{-2} = \underline{-\frac{x}{y^2}}$

25. $g = (u^2 v - v^3)^5$

$g_u = 5(u^2v - v^3)^4 \cdot 2uv$

$g_v = 5(u^2v - v^3)^4 \cdot (u^2 - 3v^2)$

27. $R = \arctan(pq^2)$

$R_p = \frac{1}{1+(pq^2)^2} \cdot q^2$

$R_q = \frac{1}{1+(pq^2)^2} \cdot 2pq$

33. $w = \ln(x + 2y + 3z)$

$w_x = \frac{1}{x+2y+3z} \cdot 1$

$w_y = \frac{1}{x+2y+3z} \cdot 2$

$w_z = \frac{1}{x+2y+3z} \cdot 3$

35. $u = xy \arcsin(yz)$

$u_x = y \arcsin(yz)$

$u_y = x \arcsin(yz) + xy \dfrac{1}{\sqrt{1-(yz)^2}} \cdot z$

$u_z = xy \dfrac{1}{\sqrt{1-(yz)^2}}\, y$

Recall $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$

41. $f = \ln\left(x + \sqrt{x^2+y^2}\right)$

$f_x = \dfrac{1}{x + \sqrt{x^2+y^2}} \cdot \left(1 + \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x\right)$

$f_x(3,4) = \dfrac{1}{3+\sqrt{9+16}} \left(1 + \frac{1}{2}(9+16)^{-1/2} \cdot 6\right)$

$= \dfrac{1}{3+5} \left(1 + \frac{1}{\cancel{2}} \frac{1}{5} \cdot \overset{3}{\cancel{6}}\right)$

$= \dfrac{1 + 3/5}{8} = \dfrac{8}{5 \cdot 8} = \dfrac{1}{5}$

43. $f = \dfrac{y}{x+y+z}$

$f_y = \dfrac{1 \cdot (x+y+z) - y \cdot 1}{(x+y+z)^2} = \dfrac{x+z}{(x+y+z)^2}$

$f_y(2,1,-1) = \dfrac{2-1}{(2+1-1)^2} = \dfrac{1}{4}$

47. $x^2 + 2y^2 + 3z^2 = 1$

$$\frac{\partial z}{\partial x} = ?$$

$$2x + 0 + 6z \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{2x}{6z} = -\underline{\frac{x}{3z}}$$

$$\frac{\partial z}{\partial y} = ?$$

$$0 + 4y + 6z \frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{4y}{6z} = \underline{-\frac{2}{3}\frac{y}{z}}$$

51. a. $z = f(x) + g(y)$

$$\underline{\frac{\partial z}{\partial z} = f'(x)} \qquad \underline{\frac{\partial z}{\partial y} = g'(y)}$$

b. $z = f(x+y)$

$$\frac{\partial z}{\partial x} = \frac{\partial f(x+y)}{\partial (x+y)} \cdot \frac{\partial (x+y)}{\partial x} = f'(x+y) \cdot 1 = \underline{f'(x+y)}$$

$$\frac{\partial z}{\partial y} = \underline{f'(x+y) \cdot 1}$$

14.4 #1, 4, 11, 25, 33

1. $z = 3y^2 - 2x^2 + x$; $P(2,-1,-3)$

$z_x = -4x+1$ $\qquad z_x\big|_P = -8+1 = -7$

$z_y = 6y$ $\qquad z_y\big|_P = -6$

$z-(-3) = -7(x-2) - 6(y+1)$

$z+3+7x-14+6y+6=0$

$\underline{7x+6y+z=5}$

4. $z = x e^{xy}$ $\qquad (2,0,2)$

$z_x = e^{xy} + xye^{xy}$ $\qquad z_x\big|_{2,0,2} = e^0 + 0 = 1$

$z_y = x^2 e^{xy}$ $\qquad z_y\big|_{2,0,2} = 4$

$z-2 = 1(x-2) + 4(y-0)$

$z-x-4y = 2-2 = 0$

$\underline{z-x-4y=0}$

11. $f = 1 + x\ln(xy-5)$ $\qquad (2,3)$

$f_x = \ln(xy-5) + x\dfrac{1}{xy-5}y$

$f_x\big|_{2,3} = \ln(6-5) + \dfrac{2}{6-5}\cdot 3 = \ln 1 + 6 = \underline{6}$

$f_y = x\dfrac{1}{xy-5}x$

$f_y\big|_{2,3} = 2\dfrac{1}{6-5}2 = \underline{4}$

$L(x,y) = f(2,3) + f_x(2,3)(x-2) + f_y(2,3)(y-3)$

$L(x,y) = [1 + \underbrace{2\ln(6-5)}_{=0}] + 6(x-2) + 4(y-3)$

$L(x,y) = \underline{1 + 6(x-2) + 4(y-3)}$

25. $z = e^{-2x}\cos(2\pi t)$

$z_x = -2e^{-2x}\cos(2\pi t)$

$z_t = -2\pi e^{-2x}\sin(2\pi t)$

$dz = -2e^{-2x}\cos(2\pi t)\,dx - 2\pi e^{-2x}\sin(2\pi t)\,dt$

33. $L = 30\text{ cm}$ $\quad \Delta L = \Delta W = .1\text{ cm}$
$W = 24\text{ cm}$

$\text{Area} = L \cdot W$

$d(\text{Area}) = W \cdot dL + L \cdot dW = 24 \cdot 0.1 + 30 \cdot 0.1 = \underline{5.4}$

14.5 #3, 7, 21, 27, 31, 38

3. $z = \sqrt{1+x^2+y^2}$ $\quad x = \ln t \quad y = \cos t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt}$$

$$= \frac{1}{\cancel{2}\sqrt{1+x^2+y^2}}\,\cancel{2}x \cdot \frac{1}{t} + \frac{1}{\cancel{2}\sqrt{1+x^2+y^2}}\,\cancel{2}y\,(-\sin t)$$

$$\frac{dz}{dt} = \frac{x/t - y\sin t}{\sqrt{1+x^2+y^2}}$$

7. $z = x^2y^3 \quad x = s\cos t \quad y = s\sin t$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial s} = \underline{2xy^3 \cdot \cos t + 3x^2y^2 \sin t}$$

$$\frac{\partial z}{\partial t} = \underline{2xy^3(-s\sin t) + 3x^2y^2(s\cos t)}$$

21. $z = x^4 + x^2y \quad x = s + 2t - u \quad y = stu^2$

when $s=4,\ t=2,\ u=1$

then $x = 4+4-1 = 7$

$y = 4\cdot 2\cdot 1^2 = 8$

$$\frac{\partial z}{\partial s} = (4x^3+2xy)\cdot \underbrace{\frac{\partial x}{\partial s}}_{=1} + x^2 \cdot \underbrace{\frac{\partial y}{\partial s}}_{=tu^2} = \underbrace{(4\cdot 7^3 + 2\cdot 7\cdot 8)}_{1484} + \underbrace{7^2\cdot 2\cdot 1^2}_{98} = \underline{1582}$$

$=$

$$\frac{\partial z}{\partial t} = (4x^3+2xy)\cdot 2 + x^2\cdot su^2 = \underbrace{(4\cdot 7^3+2\cdot 7\cdot 8)}_{1484}\cdot 2 + 7^2\cdot 4\cdot 1^2 = \underline{3164}$$

$$\frac{\partial z}{\partial u} = (4x^3+2xy)(-1) + x^2\,2st = \underbrace{(4\cdot 7^3+2\cdot 7\cdot 8)}_{1484}(-1) + \underbrace{7^2\cdot 2\cdot 4\cdot 2}_{784} = -700$$

27. $y\cos x = x^2 + y^2$

$$\frac{dy}{dx} = ?$$

$$\frac{dy}{dx}\cos x + y(-\sin x) = 2x + 2y\frac{dy}{dx}$$

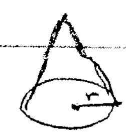
$$\frac{dy}{dx}(\cos x - 2y) = 2x + y\sin x$$

$$\frac{dy}{dx} = \frac{2x + y\sin x}{\cos x - 2y}$$

31. $x^2 + 2y^2 + 3z^2 = 1$

$$\frac{\partial z}{\partial x} = ? \qquad 2x + 0 + 6z\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{2x}{6z} = \underline{-\frac{x}{3z}}$$

$$\frac{\partial z}{\partial y} = ? \qquad 0 + 4y + 6z\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{4y}{6z} = \underline{-\frac{2}{3}\frac{y}{z}}$$



38. $\frac{dr}{dt} = 1.8\ \frac{in}{s}$

$$\frac{dh}{dt} = -2.5\ \frac{in}{s}$$

$$V = \frac{\pi r^2 h}{3}$$

$$\frac{dV}{dt} = \frac{2}{3}\pi r h\frac{dr}{dt} + \frac{\pi r^2}{3}\frac{dh}{dt}$$

$r = 120$ in , $h = 140$ in

$$\frac{dV}{dt} = \frac{2}{3}\pi \cdot 120 \cdot 140 \cdot 1.8 - \frac{\pi}{3}120^2 \cdot 2.5 = 8160\pi$$

14.6 # 5, 6, 7, 11, 19, 21, 25, 28, 34, 41

5. $f(x,y) = ye^{-x}$ $(0,4)$ $u = \cos\frac{2\pi}{3}\, i + \sin\frac{2\pi}{3}\, j$

$$D_u f = \underbrace{-ye^{-x}}_{\frac{\partial f}{\partial x}} \cdot \cos\frac{2\pi}{3} + \underbrace{e^{-x}}_{\frac{\partial f}{\partial y}} \sin\frac{2\pi}{3}$$

$$D_u f(0,4) = -4e^{-0}\cos\frac{2\pi}{3} + e^{-0}\sin\frac{2\pi}{3}$$
$$= 2 + \sqrt{3}/2$$

6. $f = e^x \cos(y)$ $(0,0)$ $\theta = \pi/4$

$\nabla f = \langle e^x \cos y, \ -e^x \sin y \rangle$

$\nabla f \big|_{(0,0)} = \langle 1, 0 \rangle$

$$D_u f \big|_{(0,0)} = \langle 1, 0 \rangle \cdot \langle \cos\pi/4, \ \sin\pi/4 \rangle = \cos\pi/4 = \sqrt{2}/2$$

7. $f = \sin(2x+3y)$ $P(-6,4)$ $\underline{u} = \frac{1}{2}(\sqrt{3}\, i - j)$

a. $\nabla f = \langle 2\cos(2x+3y), \ 3\cos(2x+3y) \rangle$

b. $\nabla f \big|_P = \langle 2\cos(-12+12), \ 3\cos(-12+12) \rangle = \langle 2, 3 \rangle$

c. $D_u f \big|_P = \langle 2, 3 \rangle \cdot \frac{1}{2}\langle \sqrt{3}, -1 \rangle = 2 \cdot \frac{\sqrt{3}}{2} - 3 \cdot \frac{1}{2} = \underline{\sqrt{3} - \frac{3}{2}}$

11. $f = e^x \sin y \quad (0, \frac{\pi}{3}) \quad \vec{v} = \langle -6, 8 \rangle$

$$\Downarrow$$

$$\vec{u} = \frac{\langle -6, 8 \rangle}{\sqrt{36+64}} = \langle \frac{-6}{10}, \frac{8}{10} \rangle = -\frac{3}{5}i + \frac{4}{5}j$$

$$\nabla f = \langle e^x \sin y, e^x \cos y \rangle$$

$$\nabla f\Big|_p \cdot \vec{u} = \langle \sin\frac{\pi}{3}, \cos\frac{\pi}{3} \rangle \cdot \langle -\frac{3}{5}, \frac{4}{5} \rangle$$

$$= -\frac{3}{5} \cdot \frac{\sqrt{3}}{2} + \frac{4}{5} \cdot \frac{1}{2} = \frac{4 - 3\sqrt{3}}{10} \sim -.1196$$

19. $f = \sqrt{xy} \quad P(2, 8) \quad \vec{v} = \vec{PQ} = \langle 3, -4 \rangle$

$$\vec{u} = \frac{\langle 3, -4 \rangle}{5}$$

$$\nabla f = \langle \frac{y}{2\sqrt{xy}}, \frac{x}{2\sqrt{xy}} \rangle$$

$$\nabla f\Big|_p = \langle \frac{8}{2\sqrt{16}}, \frac{2}{2\sqrt{16}} \rangle = \langle 1, \frac{1}{4} \rangle$$

$$\nabla f\Big|_p \cdot \vec{u} = \frac{3}{5} - \frac{1}{4} \cdot \frac{4}{5} = \underline{\underline{\frac{2}{5}}}$$

21. $f = 4y\sqrt{x}$ max rate of change at $P(4,1)$ and the direction

$$\nabla f = \langle 4y \frac{1}{2\sqrt{x}}, 4\sqrt{x} \rangle$$

$$\nabla f\Big|_p = \langle \frac{2}{\sqrt{4}}, 4\sqrt{4} \rangle = \langle 1, 8 \rangle$$

$$\left| \nabla f\big|_p \right| = \sqrt{1+64} = \sqrt{65}$$

The direction is $\frac{\nabla f}{|\nabla f|} = \langle \frac{1}{\sqrt{65}}, \frac{8}{\sqrt{65}} \rangle$

14.6

25. $f = \sqrt{x^2+y^2+z^2}$ $\quad (3,6,-2)$

max rate of change & direction

$$\nabla f = \frac{x}{\sqrt{x^2+y^2+z^2}}\, i + \frac{y}{\sqrt{x^2+y^2+z^2}}\, j + \frac{z}{\sqrt{x^2+y^2+z^2}}\, k$$

$$\nabla f\Big|_{(3,6,-2)} = \frac{3}{\sqrt{9+36+4}}\, i + \frac{6}{\sqrt{49}}\, j - \frac{2}{\sqrt{49}}\, k$$

$$= \frac{3}{7} i + \frac{6}{7} j - \frac{2}{7} k$$

$$|\nabla f| = \sqrt{\frac{9+36+4}{49}} = 1 \quad \text{max rate}$$

Direction is given by ∇f since it is a unit vector.

28. $f = y e^{-xy}$ $\quad (0,2)$

$$\nabla f = \langle -y^2 e^{-xy},\ e^{-xy} - x e^{-xy} \rangle$$

$$\nabla f\Big|_{(0,2)} = \langle -4,\ 1-0 \rangle = \langle -4, 1 \rangle$$

$$D_u f\Big|_{(0,2)} = 1 \Rightarrow \left.\begin{array}{r} -4u_1 + u_2 = 1 \\ u_1^2 + u_2^2 = 1 \end{array}\right\}$$

$\underline{u} = u_1 i + u_2 j$ is the direction

$$u_2 = 1 + 4u_1$$

$$u_1^2 + (1+4u_1)^2 = 1$$

$$u_1^2 + \cancel{1} + 8u_1 + 16u_1^2 = \cancel{1}$$

$$17u_1^2 + 8u_1 = 0$$

$$u_1(17u_1 + 8) = 0 \Rightarrow u_1 = 0 \quad u_2 = 1$$

or $u_1 = -\frac{8}{17}$, $u_2 = 1 - 4\cdot\frac{8}{17} = \frac{17-32}{17}$

$u_2 = \frac{-15}{17}$

$\underline{u} = \langle 0, 1 \rangle$ or $u = \langle -\frac{8}{17}, \frac{-15}{17} \rangle$

14.6

34. $z = 1000 - 0.005x^2 - 0.01y^2$

$P = (60, 40, 966)$

$x =$ East

$y =$ north

a. Walk South (y decreases, x constant)

are you descending/ascending? at what rate?

$\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = -0.02y$, since y decreases $\frac{\partial f}{\partial y}$ increases

ascending

The rate is $|-0.02 \cdot 40| = \underline{0.8}$

b. Walk northwest (y increases, x decreases)

$$\nabla f\Big|_P \cdot \underline{u} = \langle -0.01x, -0.02y \rangle\Big|_P \cdot \underbrace{\langle -1, 1 \rangle \tfrac{1}{\sqrt{2}}}_{\text{direction}}$$

$$= \langle -0.01 \cdot 60, -0.02 \cdot 40 \rangle \cdot \langle -1, 1 \rangle \tfrac{1}{\sqrt{2}}$$

$$= \langle -0.6, -0.8 \rangle \cdot \langle -1, 1 \rangle \tfrac{1}{\sqrt{2}}$$

$$= \frac{0.6 - 0.8}{\sqrt{2}} = \frac{-0.2}{\sqrt{2}}$$ descending

$$\left|\nabla f\big|_P \cdot u\right| = \frac{0.2}{\sqrt{2}}$$ rate

c. slope largest at direction

$$\frac{\nabla f}{|\nabla f|}\Bigg|_P = \frac{\langle -0.6, -0.8 \rangle}{\sqrt{.6^2 + .8^2}} = \frac{\langle 0.6, -0.8 \rangle}{\sqrt{.36 + .64}} \cancel{= \langle -0.6, -0.8 \rangle}$$

14.6

41.

$$2(x-2)^2 + (y-1)^2 + (z-3)^2 = 10 \qquad (3,3,5)$$

$$\nabla f = \langle 4(x-2),\ 2(y-1),\ 2(z-3) \rangle$$

$$\nabla f \Big|_{(3,3,5)} = \langle 4(3-2),\ 2(3-1),\ 2(5-3) \rangle$$

$$= \langle 4, 4, 4 \rangle$$

a. Tangent plane: $\cancel{4}(x-3) + \cancel{4}(y-3) + \cancel{4}(z-5) = 0$

$$\underline{x+y+z=11}$$

b. normal line is in the direction of $\langle 4,4,4 \rangle$

$$\frac{x-3}{4} = \frac{y-3}{4} = \frac{z-5}{4}$$

14.7 #5, 9, 13, 29, 32, 39

5. $f = x^2 + xy + y^2 + y$

$f_x = 2x + y = 0$ $\qquad$ $f_y = x + 2y + 1 = 0$

$\Downarrow$

$y = -2x$

$x + 2(-2x) + 1 = 0$

$x - 4x + 1 = 0$

$3x = 1$

$x = 1/3 \Rightarrow y = -2\left(\frac{1}{3}\right) = -\frac{2}{3}$

$\left(\frac{1}{3}, -\frac{2}{3}\right)$ is a candidate

$f_{xx} = 2$ $\qquad$ $f_{xy} = 1$ $\qquad$ $f_{yy} = 2$ $\qquad$ $D = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 > 0$

$D > 0$ & $f_{xx} > 0$ local min at $\left(\frac{1}{3}, -\frac{2}{3}\right)$

9. $f = y^3 + 3x^2y - 6x^2 - 6y^2 + 2$

$f_x = 6xy - 12x = 0$ $\qquad$ $f_y = 3y^2 + 3x^2 - 12y = 0$

$\Downarrow$ $\qquad$ $\downarrow$

$6x(y-2) = 0$ $\qquad$ If $x = 0$ then $3y^2 - 12y = 0$

$x = 0$ OR $y = 2$ $\qquad$ $3y(y-4) = 0$

$\downarrow$

$y = 0$, $y = 4$

(0,0) $\quad$ (0,4)

If $y = 2$ then

$f_y = 3 \cdot 4 + 3x^2 - 12 \cdot 2 = 0$

$3x^2 = 12$

$x^2 = 4$

$x = \pm 2$

(2,2) $\quad$ (-2,2)

9. Cont.

$f_{xx} = 6y - 12 \qquad f_{xy} = 6x \qquad f_{yy} = 6y - 12$

1. $(0,0)$: $f_{xx} = -12 < 0$; $f_{xy} = 0$; $f_{yy} = -12$; $D = \begin{vmatrix} -12 & 0 \\ 0 & -12 \end{vmatrix} = 144 > 0$

   local max

2. $(0,4)$ $f_{xx} = 24 - 12 = 12 > 0$; $f_{xy} = 0$; $f_{yy} = 12$; $D = \begin{vmatrix} 12 & 0 \\ 0 & 12 \end{vmatrix} = 144 > 0$

   local min

3. $(2,2)$ $f_{xx} = 0 \quad f_{xy} = 12 \quad f_{yy} = 0 \quad D = \begin{vmatrix} 0 & 12 \\ 12 & 0 \end{vmatrix} < 0$

   saddle point

4. $(-2,2)$ $f_{xx} = 0 \quad f_{xy} = -12 \quad f_{yy} = 0 \quad D = \begin{vmatrix} 0 & -12 \\ -12 & 0 \end{vmatrix} = -144 < 0$

   saddle point

13. $f = e^x \cos y$

$f_x = e^x \cos y = 0 \qquad f_y = -e^x \sin y = 0$

$\downarrow \qquad\qquad \downarrow$

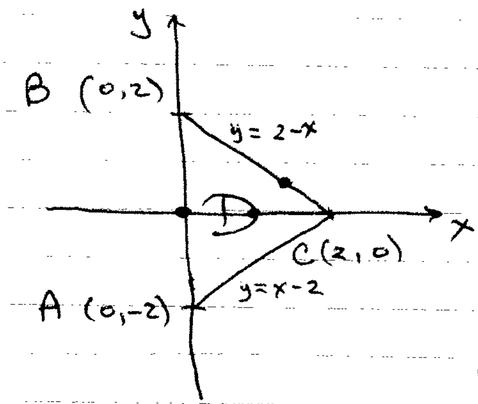
$y = \frac{\pi}{2} \pm n\pi \qquad y = m\pi$

Not possible

when $x \to -\infty$ $\quad f_x, f_y, f \to 0$

None

29. $f = x^2 + y^2 - 2x$



$f_x = 2x - 2 = 0 \Rightarrow x = 1$

$f_y = 2y = 0 \Rightarrow y = 0$

$(1,0)$ is inside

$f(1,0) = 1^2 + 0^2 - 2\cdot 1 = 1 - 2 = \underline{-1}$

Check on AB $\Rightarrow x = 0$

$f(x=0) = y^2 \Rightarrow f' = 2y = 0 \Rightarrow y = 0$

$f(0,0) = \underline{0}$

check on BC: $y = 2 - x$

$f = x^2 + (2-x)^2 - 2x$

$f' = 2x - 2(2-x) - 2 = 2x - 4 + 2x - 2 = 4x - 6$

$\Rightarrow x = \frac{3}{2}, \quad y = 2 - \frac{3}{2} = \frac{1}{2}$

$f\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{9}{4} + \frac{1}{4} - 3 = \frac{10-12}{4} = \underline{-\frac{1}{2}}$

check on AC $\quad y = x - 2$

$f = x^2 + (x-2)^2 - 2x$

$f' = 2x + 2(x-2) - 2 = 2x + 2x - 4 - 2 = 4x - 6 = 0$

$x = +\frac{3}{2} \quad y = \frac{3}{2} - 2 = -\frac{1}{2}$

$f\left(\frac{3}{2}, -\frac{1}{2}\right) = \frac{9}{4} + \frac{1}{4} - 3 = \underline{-\frac{1}{2}}$

$f(0,2) = \underline{4}$

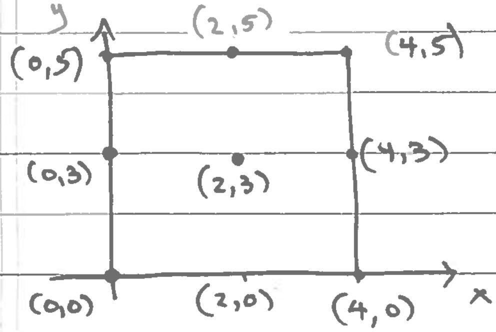
$f(2,0) = 4 - 4 = \underline{0}$

$f(0,-2) = \underline{4}$

Highest value is 4 at $(0,2)$ & $(0,-2)$
lowest values is -1 at $(1,0)$

32. $f = 4x + 6y - x^2 - y^2$

$D = \{(x,y) \mid 0 \le x \le 4,\ 0 \le y \le 5\}$



$f_x = 4 - 2x = 0 \qquad x = 2$

$f_y = 6 - 2y = 0 \qquad y = 3 \qquad \Rightarrow \underline{(2,3)}$

on $x=0$ (left)

$f(x=0,y) = 6y - y^2$

$f' = 6 - 2y = 0 \qquad \underline{y=3} \quad \Rightarrow \underline{(0,3)}$

on $y=0$ (bottom)

$f(x, y=0) = 4x - x^2$

$f' = 4 - 2x = 0 \qquad x = 2 \quad \Rightarrow \underline{(2,0)}$

on top $y=5$

$f(x, y=5) = 4x + 30 - x^2 - 25$

$f' = 4 - 2x = 0 \qquad x = 2 \Rightarrow \underline{(2,5)}$

on right $x=4$

$f(x=4, y) = 16 + 6y - 16 - y^2$

$f' = 6 - 2y = 0 \qquad y = 3 \quad \Rightarrow \underline{(4,3)}$

Now evaluate $f(x,y)$ for each of the 9 points:

$f(0,0) = \underline{0}$

$f(2,0) = 8 - 4 = \underline{4}$

$f(4,0) = 16 - 16 = \underline{0}$

$f(0,3) = 18 - 9 = \underline{9}$

$f(2,3) = 8 + 18 - 4 - 9 = \underline{13}$

$f(4,3) = 16 + 18 - 16 - 9 = \underline{9}$

$f(0,5) = 30 - 25 = \underline{5}$

$f(2,5) = 8 + 30 - 4 - 25 = \underline{9}$

$f(4,5) = 16 + 30 - 16 - 25 = \underline{5}$

Highest value is 13 at (2,3)

Lowest value is 0 at (0,0) & (4,0)

39. Shortest distance from $(2, 0, -3)$ to $x+y+z=1$

$D = (\text{distance})^2 = (x-2)^2 + (y-0)^2 + (z+3)^2$

where $x+y+z=1$ or $z = 1-x-y$

$\Rightarrow \quad D = (x-2)^2 + (y\ )^2 + (4-x-y)^2$

$D = \underline{x^2} - 4x + 4 + \underline{y^2} + (4-x-y)^2$

$D_x = 2x - 4 + 2(4-x-y)(-1)$

$D_y = 2y + 2(4-x-y)(-1)$

$D_x = 2x - 4 - 8 + 2x + 2y = 0$

$D_y = 2y - 8 + 2x + 2y = 0$

$\begin{aligned} 4x+2y &= 12 \\ 2x+4y &= 8 \end{aligned} \quad \Rightarrow \quad \left.\begin{aligned} 2x+y &= 6 \\ 2x+4y &= 8 \end{aligned}\right\} -$

$-3y = -2$

$y = 2/3 \Rightarrow 2x = 6 - \frac{2}{3} = \frac{16}{3} \Rightarrow x = \frac{8}{3}$

$D\left(\frac{8}{3}, \frac{2}{3}\right) = \left(\frac{8}{3} - 2\right)^2 + \left(\frac{2}{3}\right)^2 + \left(4 - \frac{8}{3} - \frac{2}{3}\right)^2 = \frac{4}{9} + \frac{4}{9} + \frac{4}{9} = \frac{12}{9} = \frac{4}{3}$

distance $= \frac{2}{\sqrt{3}}$

$\underline{D_{xx}\left(\frac{8}{3}, \frac{2}{3}\right)} = 2 + 2 = 4$; $D_{xy} = 2$; $D_{yy} = 4$

$D_{xx} D_{yy} - D_{xy}^2 = 4 \cdot 4 - 2^2 = \underline{16 - 4 = 12 > 0}$

local min so distance is shortest

The point on the plane closest to the given pt is $x = \frac{8}{3}$ $y = \frac{2}{3}$ $z = -\frac{7}{3}$.

14.8 # 3, 7, 15, 21

3. $f = x^2 + y^2$ $\quad xy = 1$

$F = x^2 + y^2 + \lambda(xy - 1)$

$F_x = 2x + \lambda y = 0 \Rightarrow x = -\frac{\lambda}{2} y$

$F_y = 2y + \lambda x = 0$ $\quad 2y + \lambda(-\frac{\lambda}{2} y) = 0$

$F_\lambda = xy - 1 = 0$ $\quad y(2 - \frac{1}{2}\lambda^2) = 0$

$\Downarrow$

$y = 0$ or $\lambda^2 = 4 \Rightarrow \lambda = \pm 2$

$\downarrow$

$x = 0$

but in this case

$F_\lambda \neq 0$

not possible.

$\lambda = \pm 2 \Rightarrow x = \mp y$ & $F_\lambda = xy - 1 = 0$

$\mp y^2 - 1 = 0$

$\pm y^2 + 1 = 0$

$y^2 = 1$ or $\cancel{y^2 = -1}$

$y = \pm 1$

$x = \mp 1$

$f = (\mp 1)^2 + (\pm 1)^2 = 1 + 1 = \underline{2}$

~~This is a minimum.~~

No max: f increases with x & y.

7. $f = 2x + 2y + z$; $x^2+y^2+z^2 = 9$

$F = 2x+2y+z + \lambda(x^2+y^2+z^2-9)$

$F_x = 2 + 2\lambda x = 0 \quad \rightarrow \quad x = -\frac{1}{\lambda}$

$F_y = 2 + 2\lambda y = 0 \quad \rightarrow \quad y = -\frac{1}{\lambda}$

$F_z = 1 + 2\lambda z = 0 \quad \rightarrow \quad z = -\frac{1}{2\lambda}$

$F_\lambda = x^2+y^2+z^2-9 = 0$

$\Rightarrow \left(-\frac{1}{\lambda}\right)^2 + \left(-\frac{1}{\lambda}\right)^2 + \left(-\frac{1}{2\lambda}\right)^2 = 9$

$1 + 1 + \frac{1}{4} = 9\lambda^2$

$\lambda^2 = \frac{1}{4} \qquad \lambda = \pm \frac{1}{2}$

$x = \mp 2$; $y = \mp 2$; $z = \mp 1$

$f(-2,-2,-1) = -4-4-1 = -9$ min
$f(2,2,1) = 4+4+1 = 9$ max

15. $f = x + 2y$
$x+y+z = 1$
$y^2+z^2 = 4$

$F = x+2y+\lambda(x+y+z-1) + \mu(y^2+z^2-4)$

$F_x = 1 + \lambda = 0 \quad \Rightarrow \boxed{\lambda = -1}$

$F_y = 2 + \lambda + 2\mu y = 0 \qquad F_y = 1 + 2\mu y = 0 \Rightarrow y = -\frac{1}{2\mu}$

$F_z = \lambda + 2\mu z = 0 \qquad F_z = -1 + 2\mu z = 0 \Rightarrow z = \frac{1}{2\mu}$

$F_\lambda = x+y+z-1 = 0$

$\Downarrow$

$F_\mu = y^2+z^2-4 = 0 \qquad F_\mu = \left(-\frac{1}{2\mu}\right)^2 + \left(\frac{1}{2\mu}\right)^2 - 4 = 0$

$\frac{2}{4\mu^2} = 4$

$\mu^2 = \frac{2}{16} = \frac{1}{8}$

$\boxed{\mu = \pm \frac{1}{2\sqrt{2}}}$

15. cont. $y = -\frac{1}{2\mu} = \mp \frac{1}{2 \cdot 2\sqrt{2}} = \mp \frac{1}{4\sqrt{2}}$

$$z = \frac{1}{2\mu} = \pm \frac{1}{4\sqrt{2}}$$

F_λ is the only unused $\Rightarrow$

$$x = 1 - y - z = 1 \pm \frac{1}{4\sqrt{2}} \mp \frac{1}{4\sqrt{2}} = 1$$

$f\left(1, -\frac{1}{4\sqrt{2}}, \frac{1}{4\sqrt{2}}\right) = 1 - \frac{1}{2\sqrt{2}}$ min

$f\left(1, \frac{1}{4\sqrt{2}}, -\frac{1}{4\sqrt{2}}\right) = 1 + \frac{1}{2\sqrt{2}}$ max

21. $f = e^{-xy}$ $\quad x^2+4y^2 \le 1$

First we look inside the ellipse

$$\left.\begin{aligned} f_x &= -ye^{-xy} = 0 \\ f_y &= -xe^{-xy} = 0 \end{aligned}\right. \Rightarrow (0,0)$$

Note that $f(0,0)$ is 1. The exponential grows in the 2nd & 4th quadrants. The exponential decays in the 1st & 3rd quadrants. Therefore $(0,0)$ is a saddle point.

$$F = e^{-xy} + \lambda(x^2+4y^2-1)$$

$$F_x = -ye^{-xy} + 2\lambda x = 0 \Rightarrow -xy\,e^{-xy} + 2\lambda x^2 = 0$$

$$F_y = -xe^{-xy} + 8\lambda y = 0 \Rightarrow -xy\,e^{-xy} + 8\lambda y^2 = 0 \Big\} -$$

$$F_\lambda = x^2+4y^2-1 = 0 \qquad 2\lambda(x^2-4y^2) = 0$$

$$\Rightarrow \lambda = 0 \text{ or } x = \pm 2y$$

$\lambda = 0 \Rightarrow (0,0)$ which we had before

$x = \pm 2y \Rightarrow F_\lambda = (\pm 2y)^2 + 4y^2 - 1 = 0 \Rightarrow 8y^2 = 1$

$y = \pm \frac{1}{2\sqrt{2}} \Rightarrow x = \pm 2 \frac{1}{2\sqrt{2}} = \pm \underline{\frac{1}{\sqrt{2}}}$

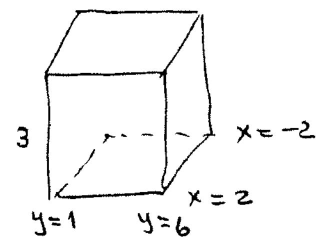
21. cont.

$$\left.\begin{aligned} f\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) &= e^{-\frac{1}{4}} \\ f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right) &= e^{-\frac{1}{4}} \end{aligned}\right\} \text{min}$$

$$\left.\begin{aligned} f\left(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right) &= e^{1/4} \\ f\left(-\frac{1}{\sqrt{2}}, +\frac{1}{2\sqrt{2}}\right) &= e^{1/4} \end{aligned}\right\} \text{max}$$

15.1 11, 12

11.



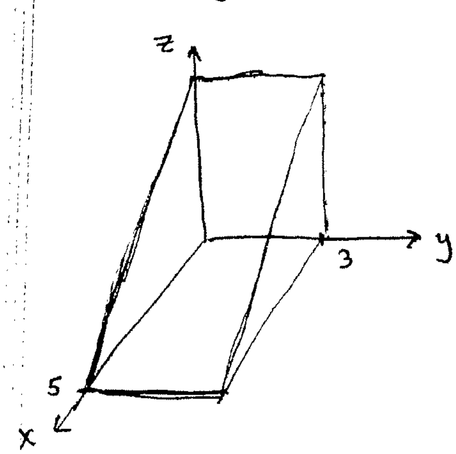
Volume $= 5 \cdot 4 \cdot 3 = 60$

$$\iint_R 3\, dA = \int_1^6 \int_{-2}^{2} 3\, dx\, dy = \int_1^6 3x \Big|_{-2}^{2} dy$$

$$= \int_1^6 (6-(-6))\, dy = 12y \Big|_1^6 = 72 - 12 = 60$$

12. $\displaystyle\int_0^3 \int_0^5 (5-x)\, dx\, dy$ slanted top

$$= \int_0^3 \left(5x - \tfrac{1}{2}x^2\right)\Big|_0^5 dy = \int_0^3 \left(25 - \tfrac{1}{2}\cdot 25\right) dy = \tfrac{25}{2}\, y \Big|_0^3 = \underline{\underline{\tfrac{75}{2}}}$$



15.2 #2, 5, 7, 11, 15, 19, 25, 27

2. a. $\int_0^5 (y + x e^y)\, dx$

$$= \int_0^5 y\, dx + e^y \int_0^5 x\, dx = xy \Big|_{x=0}^{x=5} + \tfrac{1}{2} x^2 e^y \Big|_{x=0}^{x=5}$$

$$= 5y - 0 + \frac{25}{2} e^y - 0 = \underline{5y - \frac{25}{2} e^y}$$

b. $\int_0^1 (y + x e^y)\, dy$

$$= \int_0^1 y\, dy + x \int_0^1 e^y\, dy = \tfrac{1}{2} y^2 + x e^y \Big|_{y=0}^{y=1}$$

$$= \tfrac{1}{2} + x e^1 - 0 - x = \underline{\tfrac{1}{2} + x(e-1)}$$

5. $\int_0^2 \int_0^4 y^3 e^{2x}\, dy\, dx$

$$= \int_0^2 e^{2x} \left(\int_0^4 y^3\, dy \right) dx = \int_0^2 e^{2x} \frac{y^4}{4} \Big|_0^4 dx$$

$$= 64 \frac{e^{2x}}{2} \Big|_0^2 = \underline{32(e^4 - 1)}$$

7. $\int_{-3}^{3} \int_0^{\pi/2} (y + y^2 \cos x)\, dx\, dy = \int_{-3}^{3} (xy + y^2 \sin x) \Big|_{x=0}^{x=\pi/2} dy$

$$= \int_{-3}^{3} \left(\tfrac{\pi}{2} y + y^2 - 0 - 0 \right) dy = \frac{\pi}{2} \frac{y^2}{2} + \frac{y^3}{3} \Big|_{-3}^{3}$$

$$= \cancel{\frac{9\pi}{4}} + 9 - \left(\cancel{\frac{9\pi}{4}} - 9 \right) = \underline{18}$$

11. $$\int_0^1 \int_0^1 v(u+v^2)^4\, du\, dv$$

$$= \underbrace{\int_0^1 \int_0^1 v(u+v^2)^4\, dv\, du}$$

substitute $w = u+v^2$
$dw = 2v\, dv$

$$= \int_0^1 \int_u^{u+1} w^4 \frac{dw}{2}\, du = \frac{1}{2}\int_0^1 \frac{w^5}{5}\Big|_u^{u+1} du$$

$$= \frac{1}{10}\int_0^1 \left((u+1)^5 - u^5\right) du = \frac{1}{10}\left[\frac{(u+1)^6}{6} - \frac{u^6}{6}\right]_0^1$$

$$= \frac{1}{10}\left[\left(\frac{2^6}{6} - \frac{1}{6}\right) - \frac{1}{6} + 0\right] = \frac{1}{10}\left(\frac{64}{6} - \frac{2}{6}\right)$$

$$= \underline{\frac{62}{60}}$$

15. $$\int_0^{\pi/2}\int_0^{\pi/2} \sin(x-y)\, dx\, dy = \int_0^{\pi/2} -\cos(x-y)\Big|_{x=0}^{x=\pi/2} dy$$

$$= \int_0^{\pi/2} \left[-\cos\left(\tfrac{\pi}{2}-y\right) + \cos(-y)\right] dy$$

$$= \sin\left(\tfrac{\pi}{2}-y\right) + \sin y\,\Big|_0^{\pi/2} = 0 + \sin\tfrac{\pi}{2} - \left(\sin\tfrac{\pi}{2} - 0\right)$$

$$= \underline{0}$$

19. $$\int_0^{\pi/6}\int_0^{\pi/3} x \sin(x+y)\, dy\, dx =$$

$$\int_0^{\pi/6} x \left[-\cos(x+y) \Big|_{y=0}^{\pi/3} \right] dx$$

$$= -\int_0^{\pi/6} \left(x\cos(x+\pi/3) - x\cos x \right) dx$$

Need integration by parts

$$\int \underbrace{x}_{u} \underbrace{\cos x\, dx}_{dv} = x\sin x - \int \sin x\, dx = x\sin x + \cos x$$

$$= -\left[x\sin(x+\pi/3) + \cos(x+\pi/3) \right]_0^{\pi/6} + \left[x\sin x + \cos x \right]_0^{\pi/6}$$

$$= -\pi/6 \underbrace{\sin \tfrac{\pi}{2}}_{1} - \underbrace{\cos \pi/2}_{=0} + 0 + \cos \pi/3 + \pi/6 \sin \pi/6 + \cos \pi/6 - 1$$

$$= -\pi/6 + \underbrace{\cos \pi/3}_{1/2} + \pi/6 \underbrace{\sin \pi/6}_{1/2} + \underbrace{\cos \pi/6}_{\frac{\sqrt{3}}{2}} - 1 \quad = -\pi/6 + \pi/12 + 1/2 - 1 + \frac{\sqrt{3}}{2}$$

$$= \underline{-\pi/12 - 1/2 + \frac{\sqrt{3}}{2}}$$

25. Under $4x + 6y - 2z + 15 = 0 \Rightarrow z = 2x + 3y + 15/2$

$$\int_{-1}^{2}\int_{-1}^{1} \left(2x + 3y + \frac{15}{2} \right) dy\, dx$$

$$= \int_{-1}^{2} \left(2xy + 3\frac{y^2}{2} + \frac{15}{2}y \right) \Big|_{y=-1}^{y=1} = \int_{-1} \left[\left(2x + \cancel{\tfrac{3}{2}} + \tfrac{15}{2} \right) - \left(-2x + \cancel{\tfrac{3}{2}} - \tfrac{15}{2} \right) \right] dx$$

$$= \int_{-1}^{2} \left(2x + \frac{15}{2} + 2x + \frac{15}{2} \right) dx = \int_{-1}^{2} (4x + 15)\, dx$$

$$= 2x^2 + 15x \Big|_{-1}^{2} = 8 + 30 - (2 - 15) = 38 + 13 = \underline{51}$$

27. $z = 1 - \frac{x^2}{4} - \frac{y^2}{9}$

$$V = \int_{-1}^{1}\int_{-2}^{2}\left(1 - \frac{x^2}{4} - \frac{y^2}{9}\right)dy\,dx$$

$$= \int_{-1}^{1}\left[y - \frac{yx^2}{4} - \frac{y^3}{27}\right]_{y=-2}^{y=2} dx = \int_{-1}^{1}\left[2 - \frac{x^2}{2} - \frac{8}{27} - \left(-2 + \frac{x^2}{2} + \frac{8}{27}\right)\right]dx$$

$$= \int_{-1}^{1}\left(4 - x^2 - \frac{16}{27}\right)dx = 4x - \frac{x^3}{3} - \frac{16}{27}x\,\Big|_{-1}^{1}$$

$$= 4 - \frac{1}{3} - \frac{16}{27} - \left(-4 + \frac{1}{3} + \frac{16}{27}\right) = 2\left(4 - \frac{1}{3} - \frac{16}{27}\right)$$

$$= 2\left(\frac{108 - 9 - 16}{27}\right) = \frac{2 \cdot 83}{27} = \frac{166}{27}$$

15.3 # 1, 3, 7, 9, 14, 19, 31, 49

1. $$\int_0^4 \int_0^{\sqrt{y}} x y^2 \, dx \, dy = \int_0^4 \left[\frac{x^2}{2} y^2 \right]_{x=0}^{x=\sqrt{y}} dy$$

$$= \int_0^4 \left(\frac{y^3}{2} - 0 \right) dy = \left. \frac{y^4}{8} \right|_0^4 = \frac{256}{8} = \underline{32}$$

3. $$\int_0^1 \int_{x^2}^{x} (1+2y) \, dy \, dx = \int_0^1 \left. (y + y^2) \right|_{y=x^2}^{y=x} dx$$

$$= \int_0^1 (x + \cancel{x^2} - \cancel{x^2} - x^4) \, dx = \left. \frac{x^2}{2} - \frac{x^5}{5} \right|_0^1 = \frac{1}{2} - \frac{1}{5} = \underline{\frac{3}{10}}$$

7. $$\int_{-1}^{1} \int_{-y-2}^{y} y^2 \, dx \, dy = \int_{-1}^{1} \left(\left. x y^2 \right|_{x=-y-2}^{x=y} \right) dy$$

$$= \int_{-1}^{1} [y^3 - y^2(-y-2)] \, dy = \int_{-1}^{1} (y^3 + y^3 + 2y^2) \, dy$$

$$= \left. \frac{2y^4}{4} + \frac{2y^3}{3} \right|_{-1}^{1} = \cancel{\tfrac{1}{2}} + \frac{2}{3} - \left(\cancel{\tfrac{1}{2}} - \frac{2}{3} \right) = \underline{\frac{4}{3}}$$

9. $$\int_0^{\pi} \int_0^{\sin x} x \, dy \, dx = \int_0^{\pi} \left(\left. x y \right|_{y=0}^{y=\sin x} \right) dx$$

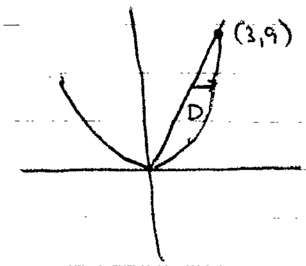
$$= \int_0^{\pi} \overset{u}{x} \underbrace{\sin x \, dx}_{dv} = -x \cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x \, dx = -\pi \underbrace{\cos \pi}_{=-1} + 0 + \sin x \Big|_0^{\pi}$$

Integration by parts

$du = dx \quad v = -\cos x$

$= \underline{\pi}$

14. $\iint_D xy\, dA$ D is enclosed by $y = x^2$ $y = 3x$



$$y = 3x,\quad y = x^2 \Rightarrow x^2 = 3x$$

$$x(x-3) = 0$$

$$\Rightarrow x = 3 \Rightarrow y = 9$$

$$\int_0^9 \underbrace{\left(\int_{y/3}^{\sqrt{y}} xy\, dx \right)}\, dy \quad \text{or} \quad \underline{\underline{\ }}\quad \int_0^3 \left(\int_{x^2}^{3x} xy\, dy \right) dx$$

$$\int_0^9 \left\{ \frac{x^2}{2} y \Big|_{x=y/3}^{x=\sqrt{y}} \right\} dy$$

$$= \int_0^9 \left(\frac{y^2}{2} - \frac{y^3}{18} \right) dy$$

$$= \frac{y^3}{6} - \frac{y^4}{4\cdot 18} \Big|_0^9$$

$$= \frac{\overset{3}{\cancel{9}}\cdot 81}{\underset{2}{\cancel{6}}} - \frac{81\cdot \overset{9}{\cancel{81}}}{4\cdot \underset{2}{\cancel{18}}}$$

$$= \frac{4\cdot 3\cdot 81}{4\cdot 2} - \frac{9\cdot 81}{4\cdot 2}$$

$$= \frac{12\cdot 81 - 9\cdot 81}{8}$$

$$= \frac{3\cdot 81}{8} = \underline{\frac{243}{8}}$$

$$\int_0^3 \left(\frac{x y^2}{2} \Big|_{y=x^2}^{y=3x} \right) dx$$

$$\int_0^3 \left(\frac{9x^3}{2} - \frac{x^5}{2} \right) dx$$

$$\frac{9x^4}{8} - \frac{x^6}{12} \Big|_0^3$$

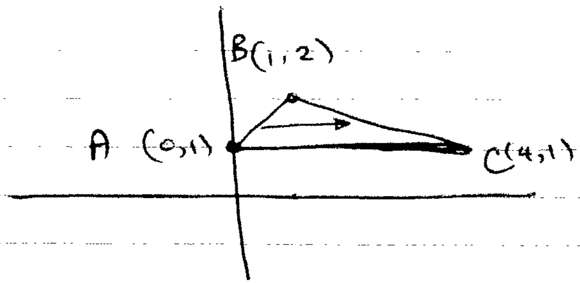
$$\frac{9\cdot 9\cdot 9}{8} - \frac{\cancel{9}\cdot\cancel{9}\cdot\overset{3}{\cancel{9}}}{\underset{4}{\cancel{12}}}$$

$$\frac{9\cdot 9\cdot 9 - 9\cdot 9\cdot 6}{8}$$

$$\frac{9\cdot 9\cdot 3}{8} = \frac{3\cdot 81}{8} = \frac{243}{8}$$

Same answer of course!

19. $\displaystyle\iint_D y^2 \, dA$



$$= \int_1^2 \left(\int_{y-1}^{-3y+7} y^2 \, dx \right) dy$$

$y = x+1$ is line AB

$y = -\frac{1}{3}x + \frac{7}{3}$ line BC

$\Downarrow$

$x = -3y + 7$

$$= \int_1^2 \left(y^2 x \Big|_{x=y-1}^{x=-3y+7} \right) dy$$

$$= \int_1^2 \left[(7-3y)\, y^2 - (y-1) y^2 \right] dy$$

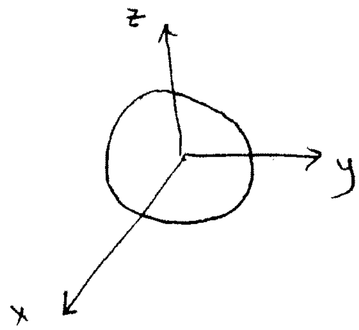
$$= \int_1^2 \left(7y^2 - 3y^3 - y^3 + y^2 \right) dy = \int_1^2 \left(8y^2 - 4y^3 \right) dy$$

$$= 8\frac{y^3}{3} - y^4 \Big|_1^2 = \frac{64}{3} - 16 - \left(\frac{8}{3} - 1 \right)$$

$$= \frac{56}{3} - 15 = \frac{56-45}{3} = \underline{\frac{11}{3}}$$

31.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y \, dy \, dx$$

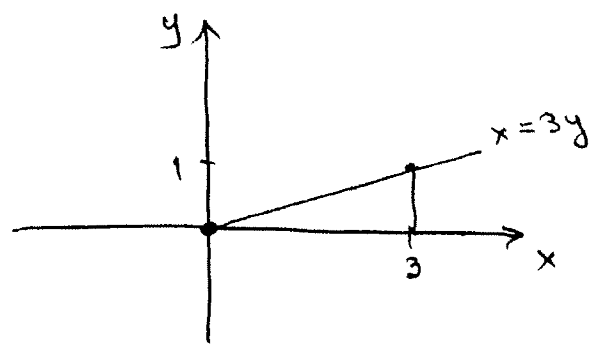


$$= \int_0^1 \left. \frac{y^2}{2} \right|_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \int_0^1 (1-x^2) \, dx = \frac{1}{2} \left(x - \frac{x^3}{3} \right) \Big|_0^1 = \frac{1}{2} \left(1 - \frac{1}{3} \right) = \frac{1}{2} \cdot \frac{2}{3} =$$

49.

$$\int_0^1 \int_{3y}^{3} e^{x^2} \, dx \, dy$$



$$= \int_0^3 \int_0^{x/3} e^{x^2} \, dy \, dx$$

$$= \int_0^3 \left. y e^{x^2} \right|_0^{x/3} dx = \frac{1}{3} \int_0^3 x e^{x^2} \, dx = \frac{1}{3} \left(\frac{1}{2} e^{x^2} \right) \Big|_0^3$$

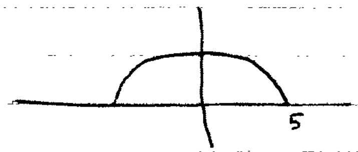
$$= \underline{\frac{1}{6} e^9 - \frac{1}{6}}$$

15.4 #7, 9, 21, 29

7. $\iint_D x^2 y \, dA$ top half of disk $c(0,0)$ $r=5$

$x = r\cos\theta$
$y = r\sin\theta$ $\quad 0 \le \theta \le \pi$

$$= \int_0^{\pi}\int_0^{5} r^3 \cos^2\theta \sin\theta \, r \, dr \, d\theta$$



$$= \int_0^{\pi} \left(\frac{r^5}{5} \Big|_0^5 \right) \cos^2\theta \sin\theta \, d\theta$$

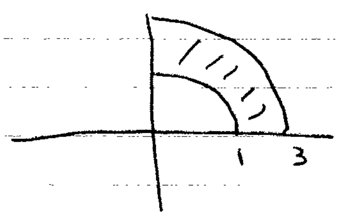
$$= 625 \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta$$

$u = \cos\theta \qquad du = -\sin\theta \, d\theta$

$$= -625 \int u^2 \, du = -625 \frac{\cos^3\theta}{3} \Big|_0^{\pi}$$

$$= \frac{625}{3} + \frac{625}{3} = \underline{\frac{1250}{3}}$$

9. $\iint_D \sin(x^2+y^2) \, dA =$



$$= \int_0^{\pi/2} \int_1^3 \sin(r^2) \, r \, dr \, d\theta$$

$u = r^2 \qquad du = 2r \, dr$

$$= \int_0^{\pi/2} \int_1^9 \frac{\sin u}{2} \, du \, d\theta = \frac{1}{2}(-\cos u)\Big|_1^9 \cdot \frac{\pi}{2}$$

$$= \underline{\frac{\pi}{4}(-\cos 9 + \cos 1)}$$

15.4

21.

$$\iint_D \left(2-\sqrt{1+x^2+y^2}\right)\,dx\,dy$$

Plane is higher!

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} \left(2-\sqrt{1+r^2}\right) r\,dr\,d\theta = 2\pi\left[\int_0^{\sqrt{3}} 2r\,dr - \int_0^{\sqrt{3}} \sqrt{1+r^2}\,r\,dr\right]$$

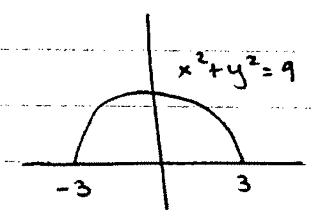
$u = 1+r^2$
$du = 2r\,dr$
$r=0 \quad u=1$
$r=\sqrt{3} \quad u=4$

$$= 2\pi\left[r^2\Big|_0^{\sqrt{3}} - \frac{1}{2}\int_1^4 \sqrt{u}\,du\right]$$

$$= 2\pi\left[3 - \cancel{\tfrac{1}{2}}\,\frac{u^{3/2}}{3/\cancel{2}}\Big|_1^4\right] = 2\pi\left[3-\left(\frac{8}{3}-\frac{1}{3}\right)\right] = 2\pi\left(3-\frac{7}{3}\right) = 2\pi\cdot\frac{2}{3} = \frac{4\pi}{3}$$

29.

$$\int_{-3}^{3}\int_0^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dy\,dx$$



$$= \int_0^{\pi}\int_0^{3} \sin(r^2)\,r\,dr\,d\theta = \pi\int_0^9 \sin(u)\,\frac{1}{2}\,du$$

$u = r^2$ $\quad$ $r=0 \to u=0$
$du = 2r\,dr$ $\quad$ $r=3 \to u=9$

$$= \frac{\pi}{2}\left(-\cos u\right)\Big|_0^9 = \underline{\frac{\pi}{2}\left(1-\cos 9\right)}$$

15.7 #2, 5, 9, 13, 19

2. (a) $\int_0^3 \int_0^1 \int_0^2 (xy + z^2)\, d\cancel{x}\, dy\, dz$

$$= \int_0^3 \int_0^1 \left(\frac{x^2}{2}y + xz^2 \Big|_{x=0}^{x=2} \right) dy\, dz$$

$$= \int_0^3 \int_0^1 (2y + 2z^2)\, dy\, dz = \int_0^3 (y^2 + 2yz^2)\Big|_{y=0}^{y=1} dz$$

$$= \int_0^3 (1 + 2z^2)\, dz = z + \frac{2}{3}z^3 \Big|_0^3 = 3 + 18 = \underline{21}$$

(b) $\int_0^2 \int_0^1 \int_0^3 (xy + z^2)\, dz\, dy\, dx$

$$\int_0^2 \int_0^1 \left(xyz + \frac{z^3}{3}\right)\Big|_{z=0}^{z=3} dy\, dx = \int_0^2 \int_0^1 (3xy + 9)\, dy\, dx$$

$$= \int_0^2 \left(3x\frac{y^2}{2} + 9y\right)\Big|_{y=0}^{y=1} dx = \int_0^2 \left(\frac{3}{2}x + 9\right) dx$$

$$= \frac{3}{2}\frac{x^2}{2} + 9x \Big|_0^2 = 3 + 18 - 0 - 0 = \underline{21}$$

(c) $\int_0^2 \int_0^3 \int_0^1 (xy + z^2)\, dy\, dz\, dx$

$$= \int_0^2 \int_0^3 \left(x\frac{y^2}{2} + yz^2\right)\Big|_{y=0}^{y=1} dz\, dx = \int_0^2 \int_0^3 \left(\frac{x}{2} + z^2\right) dz\, dx$$

$$= \int_0^2 \left(\frac{x}{2}z + \frac{z^3}{3}\right)\Big|_{z=0}^{z=3} dx = \int_0^2 \left(\frac{3}{2}x + 9\right) dx = \frac{3}{4}x^2 + 9x \Big|_0^2$$

$$= 3 + 18 = \underline{21}$$

5. $$\int_1^2 \int_0^{2z} \int_0^{\ln x} x e^{-y} \, dy \, dx \, dz$$

$$= \int_1^2 \int_0^{2z} \left(-x e^{-y} \Big|_{y=0}^{y=\ln x} \right) dx \, dz$$

$$= \int_1^2 \int_0^{2z} \left(-\frac{x}{x} + x \right) dx \, dz = \int_1^2 \int_0^{2z} (x-1) \, dx \, dz$$

$$= \int_1^2 \left(\frac{x^2}{2} - x \Big|_{x=0}^{x=2z} \right) dz = \int_1^2 (2z^2 - 2z) \, dz$$

$$= 2\frac{z^3}{3} - z^2 \Big|_1^2 = \frac{16}{3} - 4 - \left(\frac{2}{3} - 1 \right) = \frac{16}{3} - 4 - \frac{2}{3} + 1 = \frac{14}{3} - \frac{9}{3}$$

$$= \underline{\frac{5}{3}}$$

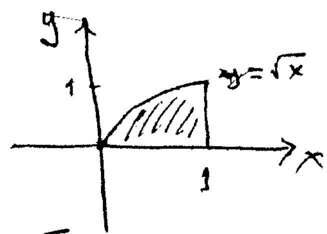
9. $$\int_0^3 \int_0^x \int_{x-y}^{x+y} y \, dz \, dy \, dx$$

$$= \int_0^3 \int_0^x \left(yz \Big|_{z=x-y}^{z=x+y} \right) dy \, dx$$

$$= \int_0^3 \int_0^x \left[(x+y)y - (x-y)y \right] dy \, dx = \int_0^3 \int_0^x 2y^2 \, dy \, dx$$

$$= \int_0^3 \frac{2}{3} y^3 \Big|_{y=0}^{y=x} dx = \int_0^3 \frac{2}{3} x^3 \, dx = \frac{2}{3} \frac{x^4}{4} \Big|_0^3 = \underline{\frac{81}{4} \cdot \frac{2}{3} = \frac{27}{2}}$$

13. $$\iiint_E 6xy\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy\,dz\,dy\,dx$$

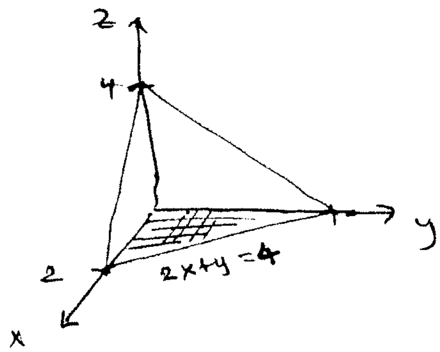


$$= \int_0^1 \int_0^{\sqrt{x}} 6xyz\Big|_{z=0}^{z=1+x+y} dy\,dx = \int_0^1 \int_0^{\sqrt{x}} 6xy(1+x+y)\,dy\,dx$$

$$= \int_0^1 \left(3xy^2+3x^2y^2+2xy^3\right)\Big|_{y=0}^{y=\sqrt{x}} dx = \int_0^1 \left(3x^2+3x^3+\underbrace{2x\cdot x^{3/2}}_{x^{5/2}}\right) dx$$

$$= x^3+\frac{3}{4}x^4+2\,\frac{x^{7/2}}{7/2}\Big|_0^1 = 1+\frac{3}{4}+\frac{4}{7} = \frac{28+21+16}{28} = \frac{65}{28}$$

19.



The volume is given by a triple integral

$$\int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz\, dy\, dx = \int_0^2 \int_0^{4-2x} (4-2x-y)\, dy\, dx$$

$$= \int_0^2 4y - 2xy - \frac{y^2}{2} \Big|_{y=0}^{y=4-2x} dx$$

$$= \int_0^2 \left[4(4-2x) - 2x(4-2x) - \frac{(4-2x)^2}{2}\right] dx$$

$$= \int_0^2 \left(16 \overset{-16x}{\cancel{-8x-8x}} + 4x^2 - \frac{16-16x+4x^2}{2}\right) dx$$

$$= \int_0^2 (8 - 8x + 2x^2)\, dx = 8x - 4x^2 + \frac{2}{3}x^3 \Big|_0^2$$

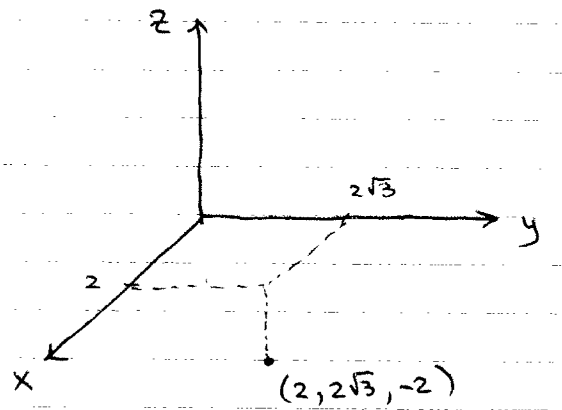
$$= \cancel{16} - \cancel{16} + \frac{16}{3} = \frac{16}{3}$$

15.8 #1, 3, 5, 8, 9, 17, 30

1. a $(4, \pi/3, -2)$ $r = 4$ $\theta = \pi/3$ $z = -2$

$x = r\cos\theta$
$y = r\sin\theta$

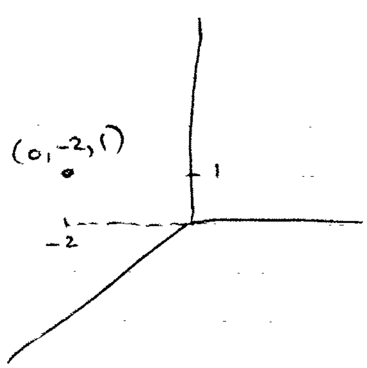
Cartesian $x = 4 \cdot \frac{1}{2} = 2$
$y = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$ $(2, 2\sqrt{3}, -2)$
$z = -2$



b. $(2, -\pi/2, 1)$ $r = 2$ $\theta = -\pi/2$ $z = 1$

$x = 2\cos(-\pi/2) = 2\cos\pi/2 = 0$
$y = 2\sin(-\pi/2) = -2\sin\pi/2 = -2$
$z = 1$

$(0, -2, 1)$
on the yz plane.



3. a. $(-1,1,1)$ $\quad x=-1 \quad y=1 \quad z=1$

$$r=\sqrt{(-1)^2+1^2}=\sqrt{2}$$

$$\tan\theta=\frac{1}{-1} \Rightarrow \theta=\frac{3\pi}{4}$$

$$\left(\sqrt{2}, \frac{3\pi}{4}, 1\right)$$

b. $(-2, 2\sqrt{3}, 3)$

$$r=\sqrt{(-2)^2+(2\sqrt{3})^2}=\sqrt{4+4\cdot 3}=\sqrt{16}=4$$

$$\tan\theta=\frac{2\sqrt{3}}{-2} \Rightarrow \theta=\frac{2\pi}{3}$$

$$\left(4, \frac{2\pi}{3}, 3\right)$$

5. $\theta=\pi/4$ half plane making an angle $\pi/4$

8. $2r^2+z^2=1 \Rightarrow 2(x^2+y^2)+z^2=1$

$$\frac{x^2}{1/2}+\frac{y^2}{1/2}+\frac{z^2}{1}=1$$

ellipsoid.

9. a) $x^2-x+y^2+z^2=1$

$$\underline{r^2-r\cos\theta+z^2=1}$$

b) $z=x^2-y^2$

$$z=r^2\cos^2\theta-r^2\sin^2\theta=r^2(\cos^2\theta-\sin^2\theta)$$

$$\Rightarrow \underline{z=r^2\cos(2\theta)}$$

17. $\iiint_E \sqrt{x^2+y^2}\, dv$

Inside a cylinder $x^2+y^2 \leq 16$ and between $z=-5,\ z=4$

$$\int_{-5}^{4}\int_{0}^{2\pi}\int_{0}^{4} r\ \underbrace{r\,dr\,d\theta\,dz}_{\text{volume}} = \int_{-5}^{4}\int_{0}^{2\pi} \frac{r^3}{3}\Big|_0^4\, d\theta\, dz$$

$$= \int_{-5}^{4}\int_{0}^{2\pi} \frac{64}{3}\, d\theta\, dz = \frac{64}{3}\cdot 2\pi\, z\Big|_{-5}^{4}$$

$$= \frac{128\pi}{3}(4-(-5)) = \frac{128\pi}{\cancel{3}}\cdot \cancel{9}^{3} = \underline{384\pi}$$

30. $$\underbrace{\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}}_{\text{top half circle}}\int_{0}^{9-x^2-y^2} \underbrace{\sqrt{x^2+y^2}}_{\equiv r}\, dz\, dy\, dx$$

change to cylindrical coordinates & evaluate

$$\int_0^{\pi}\int_0^{3}\int_0^{9-r^2} r\, dz\ r\,dr\,d\theta = \int_0^{\pi}\int_0^{3} r^2\left(z\Big|_0^{z=9-r^2}\right) dr\, d\theta$$

$$= \int_0^{\pi}\int_0^{3} r^2(9-r^2)\, dr\, d\theta = \int_0^{\pi}\left(3r^3 - \frac{r^5}{5}\right)\Big|_0^3\, d\theta$$

$$= \int_0^{\pi}\left(81 - \frac{3}{5}\cdot 81\right) d\theta = \underline{\frac{2}{5}\cdot 81\cdot \pi}$$

15.9 #1, 3, 5, 8, 10, 21, 24, 30

1. a $(6, \pi/3, \pi/6)$ $\quad \rho = 6 \quad \theta = \pi/3 \quad \varphi = \pi/6$

$$x = \rho \sin\varphi \cos\theta = 6 \sin \pi/6 \cos \pi/3 = 6 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{3}{2}$$
$$y = \rho \sin\varphi \sin\theta = 6 \cdot \tfrac{1}{2} \tfrac{\sqrt{3}}{2} = \tfrac{3\sqrt{3}}{2}$$
$$z = \rho \cos\varphi = 6 \cdot \tfrac{\sqrt{3}}{2} = 3\sqrt{3}$$

$$\underline{\left(\tfrac{3}{2}, \tfrac{3\sqrt{3}}{2}, 3\sqrt{3}\right)}$$

b. $(3, \pi/2, 3\pi/4)$

$$x = 3 \sin \tfrac{3\pi}{4} \cos \pi/2 = 3 \cdot \tfrac{\sqrt{2}}{2} \cdot 0 = 0$$
$$y = 3 \sin \tfrac{3\pi}{4} \sin \pi/2 = 3 \cdot \tfrac{\sqrt{2}}{2} \cdot 1 = \tfrac{3\sqrt{2}}{2}$$
$$z = 3 \cos \tfrac{3\pi}{4} = 3\left(-\tfrac{\sqrt{2}}{2}\right) = -\tfrac{3\sqrt{2}}{2}$$

$$\underline{\left(0, \tfrac{3\sqrt{2}}{2}, -\tfrac{3\sqrt{2}}{2}\right)}$$

3. a) $(0, -2, 0)$ $\quad x = 0 \quad y = -2 \quad z = 0$

$$\rho = \sqrt{0 + (-2)^2 + 0} = 2$$

$$\cos\varphi = \frac{z}{\rho} = 0 \Rightarrow \varphi = \pi/2$$

$$\cos\theta = \frac{x}{\rho \sin\varphi} = \frac{0}{2 \cdot 1} = 0 \Rightarrow \theta = \pi/2 \text{ or } \frac{3\pi}{2}$$

since $y < 0$

$$\underline{(2, 3\pi/2, \pi/2)}$$

b. $(-1, +1, -\sqrt{2})$

$$\rho = \sqrt{1 + 1 + 2} = 2$$
$$\cos\varphi = \frac{-\sqrt{2}}{2} \Rightarrow \varphi = 3\pi/4$$
$$\cos\theta = \frac{-1}{2\sqrt{2}/2} = -\frac{1}{\sqrt{2}} \Rightarrow \theta = 3\pi/4$$

$$\underline{\left(2, \tfrac{3\pi}{4}, \tfrac{3\pi}{4}\right)}$$

15.8

5. $\varphi = \pi/3$ cone

8. $\rho^2(\sin^2\varphi \sin^2\theta + \cos^2\varphi) = 9$

$y^2 + z^2 = 9$ cylinder

10. a $x^2 - 2x + y^2 + z^2 = 0$

$\underbrace{x^2+y^2+z^2}_{\rho^2} - 2x = 0$

$\rho^2 - 2\rho \sin\varphi \cos\theta = 0$

$\underline{\rho = 2\sin\varphi \cos\theta}$

b. $x + 2y + 3z = 1$

$\rho(\sin\varphi\cos\theta + 2\sin\varphi\sin\theta + 3\cos\varphi) = 1$

21. $\iiint_B (x^2+y^2+z^2)^2 dV$

B is inside the unit Ball

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{5} \rho^4 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta =$$

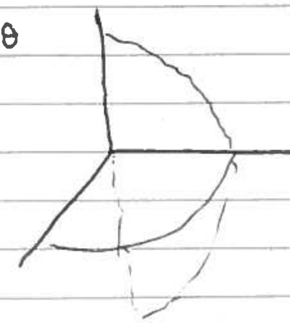
$$= \int_0^{2\pi}\int_0^{\pi} \frac{\rho^7}{7}\Big|_0^5 \sin\varphi \, d\varphi \, d\theta = \frac{5^7}{7}(-\cos\varphi)\Big|_{\varphi=0}^{\varphi=\pi} \cdot 2\pi$$

$$= \frac{2\pi}{7}(+1+1)5^7 = \frac{4\pi \cdot 5^7}{7}$$

24. $\iiint_E y^2 \, dV$

E : solid hemisphere $x^2+y^2+z^2 \le 9$
$y \ge 0$

$$\int_0^{\pi}\int_0^{\pi}\int_0^{3} (\rho \sin\varphi \sin\theta)^2 \, \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$



$$= \int_0^{\pi}\int_0^{\pi}\underbrace{\int_0^{3} \rho^4 \, d\rho}_{\frac{\rho^5}{5}\Big|_0^3 = \frac{3^5}{5}} \sin^3\varphi \, d\varphi \, \sin^2\theta \, d\theta$$

$$= \frac{3^5}{5} \int_0^{\pi} \underbrace{\sin^3\varphi}_{(1-\cos^2\varphi)\sin\varphi} d\varphi \underbrace{\int_0^{\pi} \sin^2\theta \, d\theta}$$

$$\int_0^{\pi} \frac{1-\cos 2\theta}{2} \, d\theta$$

$$-\cos\varphi\Big|_0^{\pi} - \int \cos^2\varphi \sin\varphi \, d\varphi$$

$u = \cos\varphi \qquad du = -\sin\varphi \, d\varphi$

$$1 + 1 + \frac{u^3}{3}$$

$$\frac{1}{2}\theta\Big|_0^{\pi} + \frac{1}{2}\,\frac{\sin 2\theta}{2}\Big|_0^{\pi}$$

$$2 + \frac{\cos^3\varphi}{3}\Big|_0^{\pi}$$

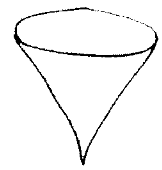
$$= \frac{\pi}{2}$$

$$2 - \frac{1}{3} - \frac{1}{3}$$

$$\frac{4}{3}$$

$$= \frac{3^{\cancel{5}\,4}}{5} \cdot \frac{\cancel{4}\,2}{\cancel{3}} \cdot \frac{\pi}{\cancel{2}} = 2\pi \cdot \frac{3^4}{5} = \frac{162\pi}{5}$$

30.



$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} dz \ dy \ dx$$

Intesection

$$x^2+y^2+z^2=4$$
$$z=\sqrt{x^2+y^2}$$
$$\Downarrow$$
$$x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2=4$$
$$2x^2+2y^2=4$$
$$\boxed{x^2+y^2=2}$$

$$= \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^2 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

outside cone

$\rho^2=4$ sphere

$$\rho\cos\varphi = \sqrt{\rho^2\sin^2\varphi} = \rho\sin\varphi$$
$$\tan\varphi = 1$$
$$\varphi = \pi/4 \ \text{cone}$$

$$= 2\pi \left(\int_{\pi/4}^{\pi/2} \sin\varphi \, d\varphi\right) \frac{\rho^3}{3}\Big|_0^2$$

$$= 2\pi \, \frac{8}{3}\left(-\cos\varphi \Big|_{\pi/4}^{\pi/2}\right) = \frac{16}{3}\pi\left(0+\frac{\sqrt{2}}{2}\right) = \frac{8\pi\sqrt{2}}{3}$$

15.10 # 3, 5, 7, 15, 17

3. $x = e^{-r}\sin\theta$

$y = e^{r}\cos\theta$

$$\frac{\partial x}{\partial r} = -e^{-r}\sin\theta \qquad \frac{\partial x}{\partial \theta} = e^{-r}\cos\theta$$

$$\frac{\partial y}{\partial r} = e^{r}\cos\theta \qquad \frac{\partial y}{\partial \theta} = -e^{r}\sin\theta$$

$$J = \begin{vmatrix} -e^{-r}\sin\theta & e^{-r}\cos\theta \\ e^{r}\cos\theta & -e^{r}\sin\theta \end{vmatrix} = +\sin^2\theta - \cos^2\theta = \underline{-\cos(2\theta)}$$

5. $x = \frac{u}{v}$

$y = \frac{v}{w}$

$z = \frac{w}{u}$

$$\frac{\partial x}{\partial u} = \frac{1}{v} \qquad \frac{\partial y}{\partial u} = 0 \qquad \frac{\partial z}{\partial u} = -\frac{w}{u^2}$$

$$\frac{\partial x}{\partial v} = -\frac{u}{v^2} \qquad \frac{\partial y}{\partial v} = \frac{1}{w} \qquad \frac{\partial z}{\partial v} = 0$$

$$\frac{\partial x}{\partial w} = 0 \qquad \frac{\partial y}{\partial w} = -\frac{v}{w^2} \qquad \frac{\partial z}{\partial w} = \frac{1}{u}$$

$$J = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} & 0 \\ 0 & \frac{1}{w} & -\frac{v}{w^2} \\ -\frac{w}{u^2} & 0 & \frac{1}{u} \end{vmatrix} = \frac{1}{v}\begin{vmatrix} \frac{1}{w} & -\frac{v}{w^2} \\ 0 & \frac{1}{u} \end{vmatrix} + \frac{u}{v^2}\begin{vmatrix} 0 & -\frac{v}{w^2} \\ -\frac{w}{u^2} & \frac{1}{u} \end{vmatrix}$$

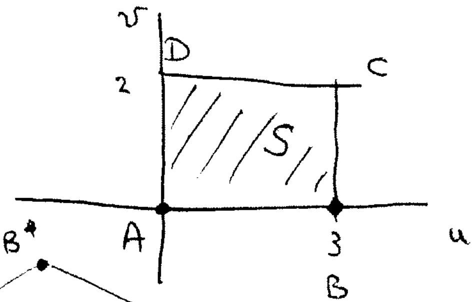
$$= \frac{1}{v}\left(\frac{1}{u}\frac{1}{w} + 0\right) + \frac{u}{v^2}\left(-\frac{vw}{u^2w^2}\right)$$

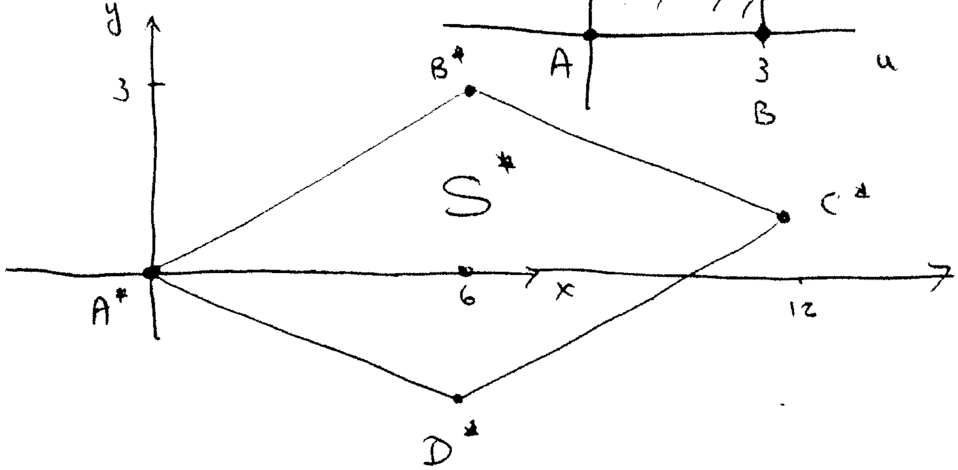
$$= \frac{1}{uvw} - \frac{1}{uvw} = 0$$

7. $S = \{(u,v) \mid 0 \le u \le 3 \ , \ 0 \le v \le 2\}$

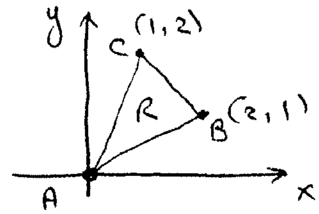
$x = 2u + 3v$

$y = u - v$





15. $\iint_R (x - 3y)\, dA$

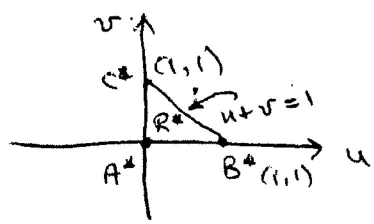


$x = 2u + v \rightarrow x = 2u + v$

$y = u + 2v \Rightarrow 2y = 2u + 4v$

$x - 2y = -3v$

$\underline{v = \dfrac{2y - x}{3}}$



$u = y - 2v = y - \frac{2}{3}(2y - x) = y - \frac{4}{3}y + \frac{2}{3}x$

$\underline{u = -\frac{1}{3}y + \frac{2}{3}x}$

$\dfrac{\partial x}{\partial u} = 2 \quad \dfrac{\partial x}{\partial v} = 1 \quad \dfrac{\partial y}{\partial u} = 1 \quad \dfrac{\partial y}{\partial v} = 2 \qquad J = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3$

$$\iint_R (x - 3y)\, dA = \iint_{R^*} \Big(\underbrace{2u + v}_{x} - 3\underbrace{(u + 2v)}_{y}\Big) \underbrace{3}_{J}\, du\, dv$$

$$= \iint_{R^*} 3(-u - 5v)\, du\, dv = -3 \int_0^1 \int_0^{1-v} (u + 5v)\, du\, dv$$

11. continue

$$= -3\int_0^1 \left(\tfrac{1}{2}u^2 + 5uv\right)\Big|_{u=0}^{u=1-v} dv = -3\int_0^1 \left[\tfrac{1}{2}(1-v)^2 + 5v(1-v)\right] dv =$$

$$= -3\int_0^1 \left(\tfrac{1}{2} - v + \tfrac{1}{2}v^2 + 5v - 5v^2\right) dv = -3\left[\tfrac{1}{2}v + 2v^2 - \tfrac{3}{2}v^3\right]\Big|_0^1 =$$

$$= -3\left(\tfrac{1}{2} + 2 - \tfrac{3}{2}\right) = -3 \cdot 1 = -3$$

17. $\iint_R x^2\, dA$ $\quad$ R is the ellipse $9x^2 + 4y^2 = 36$

$x = 2u$

$y = 3v$

$\Rightarrow \quad 9(2u)^2 + 4(3v)^2 = 36 \Rightarrow u^2 + v^2 = 1$

$$J = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

$$\underset{\substack{R \\ \text{ellipse}}}{\iint} x^2\, dA = \underset{\substack{R^* \\ \text{Circle}}}{\iint} 4u^2 \cdot 6\, du\, dv = \text{use polar coordinates}$$

$u = r\cos\theta$

$v = r\sin\theta$

$$24\int_0^{2\pi}\int_0^1 r^2\cos^2\theta \cdot \underbrace{r\,dr\,d\theta}_{du\,dv} = 24\int_0^{2\pi} \frac{r^4}{4}\cos^2\theta \Big|_{r=0}^{r=1} d\theta$$

$$= 6\int_0^{2\pi} \underbrace{\cos^2\theta}_{\frac{\cos 2\theta + 1}{2}}\, d\theta = 3\int_0^{2\pi} (\cos 2\theta + 1)\, d\theta$$

$$= 3\left(\frac{\sin 2\theta}{2} + \theta\right)\Big|_0^{2\pi} = 3\,(0 + 2\pi) = \underline{\underline{6\pi}}$$