BLOTTO POLITICS

Alan Washburn
Operations Research Department, Naval Postgraduate School, Monterey, CA 93943, awashburn@nps.edu

This paper considers abstract election games motivated by the United States Electoral College (USEC). There are two political parties, and the electoral votes in each state go to the party that spends the most money there, with an adjustment for a "head start" that one party or the other may have in that state. The states have unequal numbers of electoral votes, and elections are decided by majority rules. Each party has a known budget, and much depends on the information that informs how that budget is spent. Three situations are considered: (1) one party's spending plan is known to the other, (2) spending is gradually revealed as the parties spend continuously in time, and (3) neither side knows anything about the other's spending. The last situation resembles a Blotto game, hence the title.

Key words: political campaign spending, Blotto, presidential, game theory

1. Introduction

Before the advent of the secret ballot in the nineteenth century, it was common for citizens to be bribed to vote as instructed by the briber (Anderson and Tollison (1990)). While outright bribery of voters is no longer common, spending money to win popular votes seems to be an enduring and increasingly important part of politics. According to the United States Federal Election Commission (FEC, 2012), the total cost of presidential campaigns has been increasing at about 10%/year over the last several decades, considerably faster than inflation. Presidential elections cost billions of dollars, and the spending is effective. Levitt(1994) estimates that the cost of swinging 1/3 of 1% of the popular votes in a house election is about $100,000.

The demonstrable influence of money on politics is not generally thought to be a good thing. However, the impulse to optimize is irresistible. Given that money is an important determinant of politics, what follows about how given budgets should be distributed over the states? This paper is devoted to that question. We will examine only an extreme form of election where politics is reduced to a two-person zero-sum game where the party that spends the most money in a state wins that state's electoral votes. Individual voters will not even be represented, nor will any aspect of politics other than money. However, we will consider games that differ in the information available to the two sides when spending decisions are made, and will find that the influence of this information is just as important as budget levels.

In reading this paper, the reader may wish to consult a Microsoft Excel ™ workbook named *ECollege.xlsm*, which can be found at the Downloads link at http://faculty.nps.edu/awashburn/ .

2. The general model

Although we will approach the problem abstractly, we will continue to refer to the national voting entities as "states" because of the application to the USEC. We will also refer to the single critical resource as money, even though our analysis would apply equally well to any other scalar resource. The candidate's time, for example, is a resource that must also be divided among the states.

Let the number of states be n, a positive integer, and let the budgets of the two parties be b for the "Blue" party and r for the "Red" party. Each state has a prescribed positive number of electoral votes v_i, and $\mathbf{v} = (v_i)$ is the corresponding vector. The votes $\mathbf{v}$ are not necessarily integers. All electoral votes in each state go to the party that spends the most money there, and the party with the larger number of electoral votes overall wins the national election. Equivalently, let $V = (\sum_{i=1}^{n} v_i)/2$. Then any party that can capture more than V electoral votes will win the election.

We also include a bias vector $\mathbf{z} = (z_i)$, where z_i is a possibly negative monetary bias in favor of Blue, who will consistently be the maximizer in the games formulated below. We can reasonably expect that the parties will contest mainly in states where $|z_i|$ is not too large. These are the states that are "in play" (Merolla, Munger, and Tofias, 2005). Our goal is to investigate how the chances of success depend on $\mathbf{z}$, $\mathbf{v}$, b, and r, as well as the information available to the two sides as the campaign progresses.

Let x_i and y_i be Blue's spending and Red's spending in state i, respectively, and let $\mathbf{x}$ and $\mathbf{y}$ be the corresponding vectors. Also let $I()$ be a scalar function of one real variable that is 1 for positive arguments, 0 for negative arguments, or 0.5 if the argument is 0. The number of electoral votes won by Blue is then

$$A(\mathbf{x},\mathbf{y},\mathbf{v},\mathbf{z}) = \sum_{i=1}^{n} v_i I(x_i + z_i - y_i) \tag{1.1}$$

We are assuming in (1.1) that Blue and Red exactly split a state's electoral votes should it happen that the argument of $I()$ is 0. Blue wishes to maximize (1.1), and Red wishes to minimize the same quantity, so we have a zero-sum game.

The above definitions do not completely define the contest. Much depends on what we assume about who knows what and when as the two budgets are spent during the political campaign. We distinguish three cases:

1. One party is for some reason compelled to reveal its budget allocations before the other party acts. Without loss of generality, we will assume the former party to be Blue. This is clearly a worst-case assumption for Blue, so we call it "Worst Case Blue". It is covered in section 3.
2. The parties continuously observe each other during the campaign, with each spending money in the knowledge of what the other party has done in the past. This case is "Continuous Public Spending". It is covered in section 4.
3. Both parties secretly determine all budget allocations without any knowledge of the other's allocations, and reveal them all at once. This case is "Secret Spending". It is covered in section 5.

With respect to the political process in the United States, case 2 comes closest to capturing the information flow that occurs during an actual campaign. The FEC (FEC, 2012) aids this process by making state-by-state spending data continuously available. Case 1 is included because it bounds the effect of money, and case 3 is included to explore the effects of secrecy.

3. Worst Case Blue

Here we consider only the bias-free situation ($\mathbf{z} = 0$). The payoff to Blue in terms of electoral votes won will be

$$\max_{\mathbf{x}} \inf_{\mathbf{y}} A(\mathbf{x},\mathbf{y},\mathbf{v},\mathbf{0}) \tag{1.2}$$

The notation means that **y** is chosen knowing **x**. The "infimum" is necessary because there will typically be no minimizing **y** — Red should either make y_i equal to 0 or to a number slightly exceeding x_i. For any given **x**, define the function $R(\mathbf{x})$ to be the minimized objective function of the following binary integer program with variables **u**:

$$\begin{aligned} &\text{minimize} \sum_{i=1}^{n} u_i x_i, \\ &\text{subject to} \sum_{i=1}^{n} u_i v_i > V;\ u_i = 0 \text{ or } 1 \end{aligned} \tag{1.3}$$

This function represents the smallest Red budget that will win the national election when Blue uses **x**. Variable u_i is 1 if Red spends enough to capture state i, otherwise 0, and the constraint is to the effect that Red must capture more than half the electoral votes in order to win. Red's budget must actually slightly exceed $R(\mathbf{x})$ in order for him to capture all of his selected states.

There is also a minimal tying budget $r(\mathbf{x})$ that can be found by replacing ">" in (1.3) by "≥". The winning and tying budgets are equal if national ties are impossible, but in general we can only say that $r(\mathbf{x}) \le R(\mathbf{x})$.

Red's problem of computing $R(\mathbf{x})$ is in essence a knapsack problem, a fundamentally difficult type of optimization (NP-complete, to be precise, see Ahuja, Magnanti and Orlin (1993)). Blue has the additional problem of maximizing $R(\mathbf{x})$, so we might call Blue's problem a "worst case knapsack problem (WCKP)". We will later outline a procedure for solving the WCKP, but first consider some attractive strategies for Blue and Red that, while not always optimal, are simple and effective.

The Blue strategy of making x_i proportional to v_i (call it $\mathbf{x}^*$) is attractive in the sense that it is simple, does not require that Red's budget be known, and forces Red to make an investment proportional to the state's value in order to win any state. The strategy also ensures Blue victory as long as Red has less than half of Blue's budget. To prove this, let S be any set of states, and let $val(S) \equiv \sum_{i \in S} v_i$ and $budget(\mathbf{x}, S) \equiv \sum_{i \in S} x_i$. Then $budget(\mathbf{x}^*, S) = b\ val(S)/(2V)$ — the proportionality factor $2V$ makes the overall Blue budget for all states be b. If $budget(\mathbf{x}^*, S) < b/2$, then it follows that $val(S) < V$; that is, Red will lose the election unless his budget is at least half of Blue's.

In situations where national ties are possible, Red also has an easily described, widely effective strategy. Since ties are possible, Red can always find a collection of states S for which $val(S) = V$ and also $val(S') = V$, where S' is the complement of S. Since $budget(\mathbf{x}; S) + budget(\mathbf{x}; S') = b$ regardless of **x,** Red can purchase either S or S' as long as $r \ge b/2$, thus assuring at least a tie. Furthermore, the state with the smallest value of x_i must be in one set or the other, and for that state $x_i \le b/n$. If $r \ge b/2 + b/n$, then Red can guarantee victory by buying the small state plus the set that does not include it.

The above results mean that, roughly speaking, Red needs only about half of Blue's budget in order to win. This factor of two is shared with what we might call the gerrymandering problem. In the gerrymandering problem you are given a fixed number of districts within a state, all of which must have the same number of voters. Each district is won by majority rules, and each voter's preference for Red or Blue is known. If Red is able to define the district boundaries so that he just barely wins each district or puts 0 Red voters in it, he can take half of the districts with just slightly more than half of Blue's budget. If the districts do not have to be equal-sized in terms of total voters, thus permitting very small "rotten boroughs", Red can do even better.

To find the exact tipping value for Red's budget, we must solve the WCKP. In general, a strategy for Red in the WCKP can be expressed as a collection of winning subsets S_1, …, S_m, for each of which $val(S_k) > V$. Red observes **x** and chooses the subset for which $budget(\mathbf{x}, S_k)$ is minimal, winning the election if and only if that budget is r or smaller. Given that collection of m subsets, Blue can maximize the budget required for Red victory by solving the following linear program with variables **x**:

$$\begin{aligned}&\text{maximize } r\\&\text{subject to } budget(\mathbf{x}, S_k) \geq r;\ k = 1, ..., m\\&\sum_{i=1}^{n} x_i \leq b \text{ and } x_i \geq 0;\ i = 1, ..., n\end{aligned} \tag{1.4}$$

Our algorithm for solving WCKP is to alternate solving (1.3) and (1.4) until convergence occurs, beginning with **x*** in (1.3) to find the first winning subset S_1. After each solution of (1.4), the optimal **x** is input to (1.3), and a new winning subset S is determined as the set of states for which $u_i = 1$. If S is already in the collection S_1, …, S_m, then the algorithm terminates with the collection being optimal for Red and the last **x** being optimal for Blue. Otherwise S is added to the collection and (1.4) is solved again. There are only finitely many winning subsets, so the algorithm must eventually terminate. Let $\mathbf{x_R}$ be the final value of **x**, and let R* be the final objective function of (1.4). Red will win if $r > R^*$.

If Red's budget is insufficient for victory, it may still be sufficient to avoid loss. Red's strategy will again be a collection of subsets, but now we only require $val(S_k) \geq V$ for each subset in the collection. Using the same column generation algorithm, we can now find $\mathbf{x_r}$ and r* by consistently using the modified version of (1.3) where ">" is replaced by "≥". Blue will win using $\mathbf{x_r}$ if $r < r^*$. A tie will result if $r^* < r < R^*$, with one optimal strategy for Blue being $\mathbf{x_R}$.

Example 1: Consider a four-state example where **v** = (3,2,1,1) and $b = 1$. In spite of the relatively large value of v_2, state 2 is tactically equivalent to states 3 and 4. We find that r(**x***) = 4/7 = 0.57, but $r(\mathbf{x_r}) = r(0.4, 0.2, 0.2, 0.2) = 0.6$ a larger value. We see that **x*** does not necessarily maximize r(**x**). Ties are not possible, so r* = R* = 0.6.

Example 2: Assume **v** = (3,2,1) and $b = 1$. Ties are possible, so r(**x**) and R(**x**) are different functions. To maximize R(**x**), Blue should choose $\mathbf{x_R}$ = (1,0,0), which forces Red to have a budget of more than R* = 1 to win. If r = 1, Red can tie, but cannot win. To maximize $r(\mathbf{x})$, Blue might choose $\mathbf{x}^*$. The strategy $\mathbf{x}^*$ will achieve a tie as long as $r \leq 2/3$, but will lose against Red budgets larger than 2/3. The strategy $\mathbf{x_r} = (1/2, 1/4, 1/4)$ also maximizes $r(\mathbf{x})$, and $\mathbf{x_r}$ dominates $\mathbf{x}^*$ in the sense that it guarantees at least a tie as long as $r \leq 0.75$.

Example 3: The USEC. For simplicity assume that b = 538, the total number of electoral votes. The 51 states (Washington D.C. is a state for our purposes) have a subset S (indeed, many such subsets) for which $val(S)$ = 269, so ties are possible. It turns out $\mathbf{x_r}$ and $\mathbf{x_R}$ are both equal to $\mathbf{x}^*$, with r^* = 269 and R^*= 270, as can be demonstrated using the column generation scheme outlined above. Sheets "BlueSheet" and "RedSheet" of *ECollege.xlsm* do this. The 20 winning subsets that constitute an optimal strategy for Red have the property that, for every $\mathbf{x}$, at least one of them has a cost that does not exceed 270. Blue will win using $\mathbf{x}^*$ if $r \leq 269$, or at least tie if $269 < r \leq 270$. Blue will lose if $r > 270$.

We could modify (1.2)-(1.4) to deal with nonzero biases, but Worst Case Blue is of relatively low interest because of its extreme view of Red's information advantage. Even if there were some kind of a Watergate incident where Blue's spending plans were discovered by Red early in a political campaign, Blue could still observe Red's subsequent spending and modify his own plans accordingly. Such considerations lead us to the next case.

4. Continuous Public Spending

Here we imagine that each side can spend money in any manner consistent with obeying an overall budget constraint over a known election time period [0, T], except that the rate of spending per unit time can never exceed some large but finite number M. We assume that the two parties "continuously" observe each other's spending. To be precise, we assume that the interval [0, T] is divided into an arbitrarily large number m of small "ticks", and that spending within each tick is informed by knowledge of the other party's spending in all previous ticks, but not the current tick. During each tick, each party can spend at most MT/m, dividing that amount over the states at will based on knowledge of the other party's spending over previous ticks.

The above description is essentially that of a differential game (Isaacs, 1965). Dekel, Jackson and Wolinsky (2008) describe a different time-based political competition where the parties take turns making offers to individual voters, each of whom has his own utility function. Most other game-based treatments of political spending do not involve time, instead resembling the Secret spending case that will be treated below in section 5.

4.1 No biases

We suppose first that $\mathbf{z}$ = 0. If $b > r$, Blue can always win by first choosing any subset S for which $val(S) > V$ and then "taking" all the states in that subset. Taking a set of states is effectively the strategy of matching the other party's allocations in that set, plus a little bit more to ensure victory. In detail, Blue first allocates an amount slightly exceeding MT/m to every state in S, using several ticks to do so if necessary and meanwhile ignoring Red's allocations. The total budget required to do this is $buff = \#(S)MT/m$, where #(S) is the number of states in S. This leaves Blue a residual budget of $b' = b - buff$ which we can assume to be larger than r because m is arbitrarily large. This "buffer" is intended to prevent Red's capturing part of S in the last tick where Blue cannot react to Red's spending. Since $b' > r$, Blue can still match Red's spending within S, including any spending done by Red while Blue was accumulating the buffer, and will therefore be ahead by enough prior to the last interval to prevent Red from making any last-minute captures.

If $r > b$, then Red can similarly win by taking some winning set of states. In general, if at any time a party finds itself with a superior budget and a winning set of states in which there are no adverse biases, then that party can guarantee to win by taking those states.

Nothing is spent in states not taken. This is in contrast to guidance such as the "3/2 rule" of Brams and Davis (1974) that determines *a priori* how much money should be spent as a function of the number of electoral votes available. This same tension about whether to concentrate on a subset of the states exists in real political campaigns, witness Senator Goldwater's decision in 1964 to abandon early plans to write off the big industrial northeastern states. Although we will see below that directly "taking" a set of states may not be optimal when there are biases, the optimal strategy will still concentrate all effort in a carefully selected subset of the states. If Goldwater was playing a game like the one described here, then he should have stuck with his early plans.

In summary, the probability of Blue's winning is 1 if $b > r$ or 0 if $r > b$, and in either case an optimal strategy for the winner is to set up a buffer and then imitate the other party's spending in a winning set of states. If $r = b$, then neither side can accumulate the required buffer, and symmetry demands that each side win half the time. We defer to section 5 consideration of the mixed strategies required to deal with that possibility.

4.2 Biases, but no ties

We next consider the possibility that the bias vector $\mathbf{z}$ is not 0. We will show that the introduction of a bias vector is equivalent to adding a certain amount B, possibly negative, to Blue's budget. Define the set function $bias(S) \equiv \sum_{i \in S} z_i$. For simplicity, we temporarily assume that a tie in the national vote is impossible. A set of states S will be called a winner if $val(S) > V$; the only other possibility is that S is a loser.

The states can be partitioned into three sets P, Z, and N in which the bias is initially positive, zero, or negative. We deal first with the possibility that P is a winner. In that case consider the following knapsack problem KP for Red, with variables $\mathbf{u}$:

$$\begin{aligned} &\text{maximize } \sum_{i \in P} u_i z_i \\ &\text{subject to } \sum_{i \in P} u_i v_i \le V \qquad (1.5) \\ &u_i = 0 \text{ or } 1 \text{ for } i \in P \end{aligned}$$

The binary variables u_i can be viewed as indicating which states remain in P; the rest are moved from P to N because Red spends enough in each of them to cancel the bias. If P' is the set of remaining states where the bias is still positive, then the objective function of KP is $bias(P')$, and Red's objective is to maximize that quantity subject to making sure that P' is not a winner. Let $B = bias(P) - \text{bias}(P')$, a positive quantity that represents the aforementioned equivalent budget adjustment.

Theorem: If P is a winner and if B is as defined above, then Red (Blue) can win the game if r is greater than (less than) $b + B$.
Proof: We first show that Red can win the game if $r > b + B$. Red can assure this by first spending B to reduce P to P'. Since P' is not a winner, it follows that the rest of the states are a winner. The rest of the states have no positive biases, and Red still has more money left than Blue, so Red can take them.

Blue's strategy for winning if $r < b + B$ is slightly more complicated. Blue initially waits until Red has spent so much money in P that the remaining positively-biased states P' are no longer a winner. By the design of program KP, Red will have to spend at least B in order to accomplish this (if Red does not do so, then Blue can win by taking P'). Once P' is no longer a winner, Red will have at most $r - B - C$ left, where C is $-bias(N')$ and N' is the set of states that were originally in P, but which Red has effectively given a negative bias by spending more than necessary to remove them from P (C is a nonnegative number). Blue now spends C to neutralize the negative bias of all states in N', leaving him with $b - C$. At this point all the states originally in P have a nonnegative bias, and Blue still has more money than Red, so he can take P. This kind of tactic for Blue will be called "waiting for B in P". QED

If instead N is a winner, consider the knapsack problem KN with variables $\mathbf{u}$:

$$\begin{aligned} &\text{minimize } \sum_{i \in N} u_i z_i \\ &\text{subject to } \sum_{i \in N} u_i v_i \le V \\ &u_i = 0 \text{ or } 1 \text{ for } i \in N \end{aligned} \tag{1.6}$$

It is now Blue that is compelled to reduce N to the point where N' is not a winner, where N' is the set of states retaining a negative bias. The cost of this to blue is $B = bias(N) - \text{bias}(N')$, a negative quantity. The theorem remains valid, and can be proved similarly. If Blue can win, he does it by taking N. If Red can win, he does it by waiting for B in N.

The final case is where neither P nor N is a winner. In that case B is 0. The theorem remains valid, but the proof changes somewhat. Since N is not a winner, it follows that $P + Z$ is a winner, and all states in that set have a nonnegative bias. If $b > r$, Blue can therefore win by taking $P + Z$. Likewise Red can win if $b < r$ by taking $N + Z$. This completes the proof that introducing biases is equivalent to adjusting Blue's budget, at least when ties are impossible.

4.3 The general case

For completeness, we next consider the possibility where national ties are possible, as in the USEC. The reader may wish to skip this section—a quick summary is that the strategies of taking and waiting continue to suffice.

A given set of states S can now be either a winner ($val(S) > V$), neutral ($val(S) = V$) or a loser ($val(S) < V$). Table 1 displays a matrix of the possibilities, together with the subsections that deal with them.

	N is a loser	N is neutral	N is a winner
P is a loser	4.3.1	4.3.3	4.3.3
P is neutral	4.3.2	4.3.4	N/A
P is a winner	4.3.2	N/A	N/A

Table 1: Applicability of the four subsections

4.3.1 *P* and *N* are both losers

If P and N are both losers, then $B = 0$ and the argument is as in the case where ties are impossible. Whoever has the most money wins the election.

4.3.2 *N* is a loser, but not *P*.

In addition to program KP, we must also consider problem KPP where the symbol "≤" is replaced by "<". The objective function of KPP is $bias(P'')$, where P'' is a loser. Let $B = bias(P) - bias(P')$ and let $B_P = bias(P) - bias(P'')$. Necessarily $B \le B_P$ because KP is a relaxation of KPP. If $r > b + B_P$, then Red can win by taking all the states that are not in P''. If $b + B < r < b + B_P$, then Red can tie by taking all the states not in P'. Blue can also tie when $b + B < r < b + B_P$ by waiting for Red to spend B_P in P; if Red spends less than that, then the remaining positive states are not a loser, and if Red spends that amount or more, then Blue can at least tie by taking P. Finally, if $r < b + B$, Blue can win by waiting for Red to spend B in P and then taking P+Z ($P + Z$ is a winner because N is a loser).

4.3.3 *P* is a loser, but not *N*

Let KNN be similar to KN, except that the symbol "≤" is replaced by "<". The objective function of KNN is $val(N'')$, where N'' is a loser. Let $B = bias(N) - bias(N')$ and let $B_N = bias(N) - bias(N'')$, with $B \ge B_N$ because KP is a relaxation of KPP. Arguing as in 4.3.2, we can conclude that Blue can win if $r < b + B_N$, that either side can force a tie if $b + B_N < r < b + B$, and that Red can win if $b + B < r$.

4.3.4 Neither *P* nor *N* is a loser

In this case we necessarily have $val(P) = val(N) = V$. Take $B = 0$ and let B_P and B_N be computed as in 4.3.2 and 4.3.3. Following the same line of argument, we conclude that either side can force a tie if $b + B_N < r < b + B_P$, and that Red (Blue) can win if r is larger than the upper limit (smaller than the lower limit).

Example 4: If $\mathbf{z} = (2, -1, -2)$ and $\mathbf{v} = (2, 1, 1)$, then section 4.3.4 applies and $B_P = 2$, $B_N = -1$. The game value is "tie" if both sides start with the same budget.

Example 5: The USEC with standard bias

Let Blue be the Democrats and Red be the Republicans, and assume that one party or the other will win any presidential election. It is well known that certain states are biased in favor of one party or another. To quantify the bias, we update the methodology of Owen, Lindner, and Steffen (2008). There are three steps:

1. For each state k, determine the latent fraction f_k of the two-party popular vote that goes to the Republicans. This is done by discounting the measured fractions in the last 5 presidential elections with a discount factor of 0.8. The discount factor and the measured fractions for years preceding 2008 are taken from Owen, Lindner, and Steffen (2008). The measured fractions for 2008 come from the Federal Elections Commission (FEC, 2010).
2. Let d_k be the latent number of excess Democratic popular votes. We set $d_k = c_k(1-2f_k)$, where c_k is the size of the two-party popular vote in the 2008 election.
3. Let $z_k = ad_k$,where a is half the cost of switching a popular vote from one party to the other, assumed in this example to be \$10. The resulting vector **z** is the standard bias in the Democrats' favor, positive in Blue states and negative in Red states. It is shown in millions of dollars (\$M) on sheet "Bias" of the workbook *ECollege.xlsm*.

With standard bias, $val(P) = 281$, so the states in which Blue has an advantage are a winner and section 4.3.2 applies. Program KP has the solution $B = \$0.213\text{M}$, and $B_P = B$. The budget offset B is the cost to the Republicans for winning Ohio, a Blue state with 18 electoral votes, leaving P' a loser with only 263 votes. Blue has an advantage, but it is a small one relative to most state biases. The Blue bias in California alone, for example, is \$19.5M.

5. Secret Spending

Section 4 assumes that the number of time ticks is arbitrarily large, whereas this section, in essence, deals with the possibility that the whole election time interval consists of a single tick. In that case both parties must allocate budgets entirely in the dark about the other party's allocations. It would be equivalent to dispense with time altogether. The election is reduced to a two-person zero-sum game where each side secretly allocates its money to the states.

We can distinguish two different TPZS games that we will call BG(**v**,**z**,*b,r*) and EG(**v**,**z**,*b,r*), depending on the payoff function. In BG (which is short for Blotto Game), the payoff function when **x** and **y** are employed is A(**x**,**y**,**v**,**z**); that is, Blue's utility is the number of electoral votes in his favor. The payoff in EG(**v**,**z**,*b*,*r*) is $I(A(\mathbf{x},\mathbf{y},\mathbf{v},\mathbf{z}) - V)$, which is either 0 (Red wins), 1 (Blue wins) or 0.5 (tie). Laslier and Picard (2002) refer to BG as the plurality game and to EG as the tournament game. It is important to distinguish between the two because the payoff in BG is a simple sum, whereas the payoff in EG is a nonlinear function of that sum. The general idea in the following is to first solve BG, and then try to find a way to use BG to approximate EG.

5.1 Blotto games and relaxations

There has been considerable previous work on games like BG, most of it in a military context (see Eckler and Burr (1972)). The name comes from the fictional Colonel Blotto, who must solve the problem of dividing his four units among four fortresses when he doesn't know the defensive allocations of his opponent (Phillips, 1933). Although Blotto's name was not applied to the class until after World War Two (the first use of the term known to the author is Tukey (1949)), an analysis by Borel (1921) is already a solution of a Blotto game. Borel deals with continuous allocations and majority-rules decisions in each of three areas. Subsequent papers along this line include Gross and Wagner (1950), Galiano (1969), Roberson (2006) and the present work. Most previous work assumes that all areas (states) are identical, whereas the present work does not. Notable BG-like politico-economic publications are those of Snyder(1989), Myerson (1993), Laslier and Picard (2002), Laslier (2002) and Kvasov (2007). Hortala-Vallve and Llorente-Saguer (2011) consider a nonzero-sum game where the areas are not identical, specifying payoff functions for which optimal strategies are pure, rather than mixed. With the payoff function of either BG or EG, we expect optimal strategies to be mixed.

Let random variables X_i and Y_i stand for the two players' allocations to area *i*, and also define the random variable $V_i \equiv v_i \operatorname{sgn}(X_i - z_i - Y_i)$, so that Blue's expected utility is $E(\sum_{i=1}^{n} V_i) = \sum_{i=1}^{n} E(V_i)$. The equality is justified because finite sums and expected values commute, and the equality is important because computing $E(V_i)$ requires only the marginal distributions of X_i and Y_i. However, the decoupling of states is imperfect in BG(**v**,**z**,*b*,*r*) because we still have the requirement that the allocations must have fixed sums over all the states. We will approach BG by first considering a "very relaxed" Lagrangian relaxation VRBG(**v**,**z**,λ,μ) where the two players are not required to meet fixed total budgets, but only to pay for their allocations to the various states at the rates λ (Blue) or μ (Red). The analytical beauty of VRBG(**v**,**z**,λ,μ) is that the states decouple completely, and can therefore be considered separately. Myerson (1993) and Hart (2008) consider similar relaxations of Blotto games, presumably for the same reason.

5.1.1 The very relaxed game VRBG(v,z,λ,μ)

It suffices to consider a single generic state with value *v* and bias *z*. If Blue allocates *X* and Red allocates *Y*, then the payoff to Blue is $vI(X + z - Y) - \lambda X + \mu Y$. The allocations are required to be nonnegative, but are otherwise unconstrained. Blue, for example, will keep *X* smaller than v/λ not because there is a constraint to that effect, but because he would otherwise be spending more than the state is worth.

The strategy spaces are not compact and the payoff function is not continuous, so there is no theorem guaranteeing that a value for VRBG even exists. A solution does exist, however, as is proved by exhibiting it in the appendix. Here we record only the results needed in the continuation. With optimal distributions for *X* and *Y*, define

$$\begin{aligned} EV(v,z,\lambda,\mu) &\equiv vE(I(X+z-Y)) = v - EV(v,-z,\mu,\lambda) \\ EX(v,z,\lambda,\mu) &\equiv E(X) = EY(v,-z,\mu,\lambda) \\ EY(v,z,\lambda,\mu) &\equiv E(Y) = EX(v,-z,\mu,\lambda) \end{aligned} \tag{1.7}$$

In each of the three formulas constituting (1.7), the first equality is by definition, and the second is an observation of symmetry where the roles of Red and Blue are reversed and the sign of *z* is changed. We will not prove the symmetry observations.

Define the dimensionless constants $\delta = \mu z / v$ and $\rho = \mu / \lambda$. Results are trivial if $|\delta| \geq 1$, since both allocations are 0, with the state going to Blue (Red) if δ is positive (negative). Otherwise, as long as $z \geq 0$, we have (see appendix)

$$EV(v,z,\lambda,\mu) = \begin{cases} 0.5v\rho \text{ if } \delta+\rho \leq 1 \\ v(1-\dfrac{1-\delta^2}{2\rho}) \text{ if } \delta+\rho \geq 1 \end{cases} \tag{1.8}$$

$$\lambda EX(v,z,\lambda,\mu) = \begin{cases} 0.5v\rho \text{ if } \delta+\rho \leq 1 \\ v\dfrac{(1-\delta)^2}{2\rho} \text{if } \delta+\rho \geq 1 \end{cases} \tag{1.9}$$

$$\mu EY(v,z,\lambda,\mu) = \begin{cases} v(\delta+0.5\rho) \text{ if } \delta+\rho \leq 1 \\ v\dfrac{1-\delta^2}{2\rho} \text{if } \delta+\rho \geq 1 \end{cases} \tag{1.10}$$

All three functions are continuous across the boundary where $\delta + \rho = 1$. If $z < 0$, the symmetry observations determine all three functions, or see the appendix for explicit expressions. The value of the game is $EV - \lambda EX + \mu EY$.

The solution of VRBG($\mathbf{v}$,$\mathbf{z}$,λ,μ) is simply a matter of summation over the states. Slightly overloading the notation, define

$$\begin{aligned} EV(\mathbf{v},\mathbf{z},\lambda,\mu) &\equiv \sum_{i=1}^{n} EV(v_i,z_i,\lambda,\mu) \\ EX(\mathbf{v},\mathbf{z},\lambda,\mu) &\equiv \sum_{i=1}^{n} EX(v_i,z_i,\lambda,\mu) \\ EY(\mathbf{v},\mathbf{z},\lambda,\mu) &\equiv \sum_{i=1}^{n} EY(v_i,z_i,\lambda,\mu) \end{aligned} \tag{1.11}$$

Equations (1.11) complete the solution of VRBG($\mathbf{v}$,$\mathbf{z}$,λ,μ).

5.1.2. The relaxed game RBG(v,z,*b*,*r*)

We next consider a game RBG($\mathbf{v}$,$\mathbf{z}$,b,r) where the players are required to observe the constraints b and r, respectively, but only on the average. We can solve this game as long as we can find a pair (λ,μ) such that equations (1.12) are satisfied:

$$\begin{aligned} EX(\mathbf{v},\mathbf{z},\lambda,\mu) &= b \\ EY(\mathbf{v},\mathbf{z},\lambda,\mu) &= r \end{aligned} \tag{1.12}$$

If we can find such a pair, then the strategies optimal in VRBG($\mathbf{v}$,$\mathbf{z}$,λ,μ) are also optimal in RBG($\mathbf{v}$,$\mathbf{z}$,b,r), and the value of RBG($\mathbf{v}$,$\mathbf{z}$,b,r) is EV($\mathbf{v}$,$\mathbf{z}$,λ,μ). The proof that it suffices to solve (1.12) is the proof that Lagrangian relaxation is a fail-safe technique (Everett, 1963). A proof in a game theory context can be found in Penn (1967) or Washburn (2003), along with a technique for constructing bounds out of near misses.

Example 6: Suppose $\mathbf{v}$ = (4,2,1) and $\mathbf{z}$ = (3,0,0). We solve the game VRBG($\mathbf{v}$,$\mathbf{z}$,2,1), finding from (1.11) that $EV = 2.25 + 0.5 + 0.25 = 3$, $EX = 0.125 + 0.25 + 0.125 = 0.5$, and $EY = 1.75 + 0.5 + 0.25 = 2.5$. Should it happen that (b, r) = (0.5, 2.5), the solution of the game is at hand. Blue's advantage in state 1 is outweighed by the fact that Red has a budget that is five times as large, resulting in Blue taking only 3 out of seven total electoral votes, on the average. If (b, r) is not (0.5, 2.5), then we have completed only the first step of finding the right pair (λ, μ), and will have to continue the search.

Example 7: **The USEC with standard bias and equal budgets.** We take μ = 0.2, and search for the λ that makes the budgets equal. The result is that (λ, μ) = (0.1957, 0.2000) and b = r = \$1282M. On the average Blue wins 275.3 electoral votes, on the average. As in example 5, the standard bias favors the Democrats. For details see sheet "LamMu" of *Ecollege.xlsm*.

If $\mathbf{z}$ = 0, RBG($\mathbf{v}$,$\mathbf{z}$,b,r) can be solved directly without dealing with (1.12). For both sides, the average allocation to any state should be proportional to the state's value, and the resulting probability that Blue wins the state is

$$BW(b,r) = \begin{cases} 0.5b/r \text{ if } b \le r \\ 1-0.5r/b \text{ if } b \ge r \end{cases}. \tag{1.13}$$

Since this is the same in every state, the expected number of electoral votes won by Blue is $BW(b,r)\sum_{i=1}^{n} v_i$.

In summary, we have reduced the problem of solving RBG($\mathbf{v}$,$\mathbf{z}$,b,r) to a problem of solving, at worst, two simultaneous equations in two unknowns.

5.1.3 The Blotto game BG(v,z,*b*,*r*)

The only difference between BG($\mathbf{v}$,$\mathbf{z}$,b,r) and RBG($\mathbf{v}$,$\mathbf{z}$,b,r) is that the two players are required to restrict expenses to budgets with certainty in the former, rather than on the average. The solution of RBG($\mathbf{v}$,$\mathbf{z}$,b,r) provides the marginal distributions of the allocations for both players, so the initial question is existential: does there exist a joint distribution of allocations over all the states that has the marginal distributions known to be optimal in RBG($\mathbf{v}$,$\mathbf{z}$,b,r), and also has the property that the sum of the allocations is a constant? More briefly, we ask whether the marginal distributions of RBG($\mathbf{v}$,$\mathbf{z}$,b,r) are "playable" or not. If Blue's (Red's) marginals are playable, then the value of RBG is a lower (upper) bound on the value of BG. If both are playable, then BG has been solved.

In all three of the following examples, we consider symmetric situations where $\mathbf{z} = 0$.

Example 8: Suppose $\mathbf{v}$ = (4,2,1) and r = b = 3.5. The solution of RBG is that for each player the marginal distribution of the allocation to state i should be a uniform random variable in the interval [0, v_i], for all i. However, neither player can have an allocation to state 1 that exceeds 3.5, so the strategies of RBG are not playable in BG. The value of BG is clearly 3.5 by symmetry, but the optimal strategies are unknown.

Example 9: Suppose $\mathbf{v}$ = (1,1,1,1,1) and r = b = 2.5. In RBG, all allocations should be standard uniform random variables, so the playability question amounts to "is there any way of sampling five standard uniform random variables so that their sum is always exactly 2.5"? The answer is yes, so BG has been solved. In fact, the answer is yes as long as the number of equal-valued states exceeds 1. We will refer to such collections as "wheels", since one method of play exploits the fact that the sum of the projections of the spokes of a wagon wheel is always a constant, regardless of the wheel's orientation.

Example 10: **The USEC.** Suppose that $r = b = 269$, which is half of the total number of electoral votes. The solution of RBG is that, for both sides, every state should have a uniform distribution of spending on the interval $[0, v_i]$. There are many ways of playing these marginal distributions in BG, but perhaps the simplest consists of first partitioning the states into setpairs and wheels. A "setpair" is two disjoint sets of states for which the total number of votes in each set is the same number k. California (55) and Washington (12) might be one of the two sets, whileTexas (38), and New York (29) might be the other, with $k = 67$. A setpair can be played by selecting one standard uniform random number U. If a state in the first set has votes v, then the allocation to that state is vU. If a state in the second set has votes v, then the allocation to that state is $v(1-U)$. All allocations are uniform over the desired limits, while the total allocation for the setpair is exactly k. It is easy to partition the states into setpairs and wheels. One method of doing so is shown on sheet "NetBias" of *ECollege.xlsm*. That method attempts to maximize the number of setpairs and wheels. Since ties are possible in the USEC, a more direct method would involve only one setpair where $k = 269$, and no wheels.

In example 10, it is so easy to find ways of playing the RBG strategies that one might (and the author does) write off the playability issue even when there are biases or unequal budgets, at least when the biases are small and the budgets not too unequal, as is the case in reality. Playability is one of those difficulties that is at its worst in small, unbalanced problems such as example 8, but which goes away in larger examples such as example 10. Beale and Heselden (1962) opine that RBG may actually be more realistic than BG, since budgets are usually known only approximately in reality.

In summary, our computational strategy for solving BG($\mathbf{v}$,$\mathbf{z}$,b,r) is to first solve (1.7) for λ and μ. The optimal strategies for VRBG($\mathbf{v}$,$\mathbf{z}$,λ,μ) are then also optimal in RBG($\mathbf{v}$,$\mathbf{z}$,b,r) and (subject to playability concerns) BG($\mathbf{v}$,$\mathbf{z}$,b,r).

5.2 The game EG(v,z,b,r)

EG($\mathbf{v}$,$\mathbf{z}$,b,r) is not a Blotto game because the payoff is not a sum of payoffs from the individual states. Still, EG is so similar to BG that one hopes for some way of exploiting the relationship. There are cases where this is possible, although the USEC turns out not to be one of them.

5.2.1 BG versus EG in the USEC

Consider the USEC with no biases and equal budgets. The value of BG($\mathbf{v}$,$\mathbf{0}$,269,269) is 269, as demonstrated above, and both sides use a marginal distribution in state i that is uniform in $[0\ v_i]$ for all 51 states. By using his optimal strategy in BG($\mathbf{v}$,$\mathbf{0}$,269,269), Blue can guarantee to receive at least 269 electoral votes, on the average, as long as Red's spending does not exceed 269. If Blue uses that strategy in EG($\mathbf{v}$,$\mathbf{0}$,269,269), one might expect that Blue could guarantee to win the game half the time, or nearly so. However, Blue will almost always lose the game if he employs that strategy against a clever Red.

Here is a strategy for Red that will usually win if Blue mistakenly uses his optimal BG strategy in EG. Red can buy any state for certain by spending an amount of money equal to the state's electoral votes, since that is the upper limit of Blue's spending. With a budget of 270, Red could buy California and enough other states to make up 270 electoral votes. This would produce a winning number of electoral votes, but spends one unit too much of money. Red can correct the budgeting problem by spending only 54 in California, one less than the number of its electoral votes. Since Blue's allocation to California is uniform in [0, 55], the probability of Blue's winning California is 1/55, and this is also the probability of Blue's winning the national election. On the rare occasions when Blue wins California, he has a large surplus of electoral votes (323 to 215). On the common occasions when he loses California, he loses narrowly (268 to 270). The rare large victories compensate for the occasional small losses in BG, but not in EG. Red's strategy could be described as "lose big and win small", and the reason for California's centrality is that one can lose bigger in California than in any other state. The value of EG(**v**,**0**,269,269) is 0.5, so there must be a way for Blue (or Red) to guarantee winning half the time. However, the strategy that is optimal in BG(**v**,**0**,269,269) does not accomplish that, even approximately.

In spite of this disappointment, it may be useful to explore the consequences of both sides playing their BG optimal strategies in EG. The BG strategies at least incorporate reasonable adaptations to biases and budgets, and, as a practical matter, it is unlikely that either side will be so confident as to employ strategies such as the one suggested for Red above. An approximation to the probability of Blue winning can be found by assuming that the BG allocations to each of the 51 states are all independent. Let p_i be the probability that Blue wins state i, and let random variable W be the number of electoral votes won by Blue. Then $E(W) \equiv m = \sum_{i=1}^{51} v_i p_i$ and $Var(W) \equiv \sigma^2 = \sum_{i=1}^{51} v_i^2 p_i (1-p_i)$. These equations are consequences of our independence assumption and the binary nature of the outcome in each state. Since there are 51 states, W should be approximately normal by the Central Limit Theorem, so the probability that Blue wins is

$$P(W > 269) \cong \Phi(\frac{m-269}{\sigma}), \tag{1.14}$$

where $\Phi()$ is the CDF of the standard normal distribution. The solution of the Blotto game determines p_i in every state, and (1.15) determines the probability that Blue wins.

Example 11. The USEC: With the same assumptions as in example 7, the probability of Blue winning is 0.55, again showing that the Democrats have an advantage. If instead of the standard bias we assume **z** = 0, then $p_i = BW(b,r)$ in every state. Figure 1 shows how Blue's win probability depends on the budget ratio when **z** = 0. See sheets "LamMu" and "PWin" of *ECollege.xlsm* for details of these calculations.

5.2.2. Small EG games

Since BG seems to be at best an imperfect guide to playing EG, we are left with little choice but strategy enumeration if we wish to solve EG games. In this section we consider some small EG games in the hope of establishing an insight into the nature of optimal strategies. We consider only games where $\mathbf{z} = 0$.

Games with only two states are of no interest, since the party with the larger budget can win (or at least avoid losing) by spending all of it in the larger state. Therefore consider the game EG((1,1,1),**0**,1,1). This game has three identical areas and a unit budget for each side. Its counterpart BG((1,1,1),**0**,1,1) has a solution that has been known since Borel solved it in 1921. Blue uses a mixed strategy where X_i is uniform in [0, 2/3] for i = 1,2,3, and Red behaves similarly. When Blue uses this strategy, he wins either 1 or 2 areas with equal probability, no matter what Red does, so he can guarantee an average payoff of 1.5. The same strategy also guarantees that he will win EG half the time, since winning EG is the same as winning two areas. We conclude that BG and EG share the same optimal strategies.

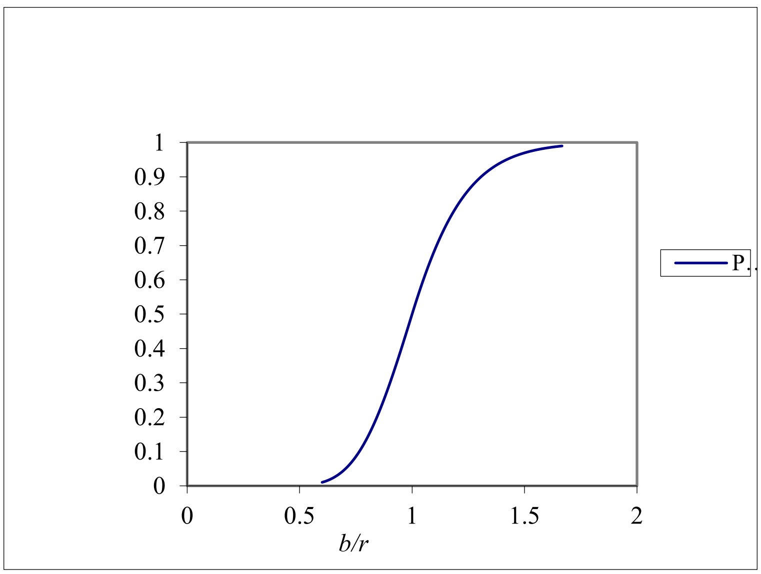


Figure 1: Probability that Blue wins the USEC as a function of the budget ratio.

Next consider the game EG((v_1,v_2,v_3),**0**,1,1), where v_1, v_2, and v_3 are such that a triangle with those sides can be drawn. This game is strategically equivalent to EG((1,1,1),**0**,1,1), since either game will be won by whichever player captures two states, and therefore has the same optimal strategies. However, we could not have found the optimal EG strategy by first solving BG and then proving that the same solution is optimal in EG. Any optimal strategy in BG will have marginal strategies proportional to electoral votes (Roberson, 2006), and is therefore not optimal in EG unless all states are equal. If no triangle with sides (v_1,v_2,v_3) is possible, then it is optimal in EG (but not in BG) for both sides to spend the entire budget in the largest state, and the result is a tie.

We finally turn to EG(2,1,1,1),**0**,5,5), a game with no biases, equal budgets, one big state and three small ones. This is perhaps the smallest interesting example where the states are not equal. The big state is sufficient with any other state, but will lose by itself. The RBG game has uniform allocations in all states, with the upper limit in state i being $2v_i$. If Blue uses any strategy with those marginal distributions, Red can win 75% of the time by buying one small state and committing the rest of his budget (3) to the large state. Therefore the BG strategy is not optimal in EG. We seek a strategy that is optimal, or at least one that will win more than 25% of the time. Our method will be to make various discrete approximations.

We consider two kinds of approximation. Both players are restricted to using integer allocations in game $G_{b,r}$, while $H_{b,r}$ is a nonsymmetric game where Blue is restricted to integers, but Red is permitted to use any allocations (integer-valued or not) that sum to r. The value of $G_{x,x}$ is of course 0.5. Our interest in $H_{x,x}$ is that any mixed strategy for Blue establishes a lower bound on the value of EG(2,1,1,1),**0**,x,x) because Red's strategy choice is not artificially restricted in $H_{x,x}$.

Table 2 shows $G_{5,5}$ and its solution when both budgets are 5, with the first row and column being labels for the pure strategies of the two players. A shorthand is used in naming strategies: 3002 is the strategy of using 3 on the large state and 2 on one of the small ones, with the lucky small state being randomly chosen among the three. On account of the symmetry among the three small states, the last three digits in any pure strategy can be shown in nondecreasing order, with the understanding that all six permutations are equally likely when the mixed strategy is played. If Blue plays row 1022 against Red's column 4001, then Blue will lose the big state, and can only hope to win all three small states. This can happen only if the 0 in 1022 lines up against one of the 0s in 4001, which happens 2/3 of the time. Blue must also win the resulting coin flip, which happens half of the time, so the resulting entry in the game matrix is 1/3. Blue's optimal mixed strategy is computed by solving a linear program, and shown in the last column of Table 2. The average payoff against each column is shown in the last row, and of course all of these numbers are at least 0.5, the value of the game.

	5000	**4001**	**3011**	**3002**	**2111**	**2012**	**2003**	**1112**	**1022**	**1013**	**1004**	**0122**	**0113**	**0014**	**0005**	**0023**	x()
5000	0.5	0.75	0.5	0.75	0	0.5	0.75	0	0.5	0.5	0.75	0	0	0.5	0.75	0.5	0
4001	0.25	0.5	0.83	0.92	0.5	0.75	0.92	0.33	0.67	0.75	0.92	0.17	0.33	0.75	0.92	0.67	0.39
3011	0.5	0.167	0.5	0.5	0.75	0.92	1	0.58	0.83	0.92	1	0.33	0.58	0.92	1	0.83	0
3002	0.25	0.083	0.5	0.5	1	0.92	0.92	0.83	0.83	0.83	0.92	0.67	0.67	0.83	0.92	0.75	0.16
2111	1	0.5	0.25	0	0.5	0.5	0.5	0.75	1	1	1	0.5	0.75	1	1	1	0.1
2012	0.5	0.25	0.08	0.08	0.5	0.5	0.5	0.92	0.92	0.96	1	0.75	0.83	0.96	1	0.88	0
2003	0.25	0.083	0	0.08	0.5	0.5	0.5	1	1	0.92	0.92	1	0.83	0.83	0.92	0.92	0
1112	1	0.667	0.42	0.17	0.25	0.08	0	0.5	0.5	0.5	0.5	0.83	0.92	1	1	1	0.31
1022	0.5	0.333	0.17	0.17	0	0.08	0	0.5	0.5	0.5	0.5	0.92	1	1	1	0.92	0
1013	0.5	0.25	0.08	0.17	0	0.04	0.08	0.5	0.5	0.5	0.5	1	0.92	0.96	1	0.96	0
1004	0.25	0.083	0	0.08	0	0	0.08	0.5	0.5	0.5	0.5	1	1	0.92	0.92	1	0
0122	1	0.833	0.67	0.33	0.5	0.25	0	0.17	0.08	0	0	0.5	0.5	0.5	0.5	0.5	0.04
0113	1	0.667	0.42	0.33	0.25	0.17	0.17	0.08	0	0.08	0	0.5	0.5	0.5	0.5	0.5	0
0014	0.5	0.25	0.08	0.17	0	0.04	0.17	0	0	0.04	0.08	0.5	0.5	0.5	0.5	0.5	0
0005	0.25	0.083	0	0.08	0	0	0.08	0	0	0	0.08	0.5	0.5	0.5	0.5	0.5	0
0023	0.5	0.333	0.17	0.25	0	0.13	0.08	0	0.08	0.04	0	0.5	0.5	0.5	0.5	0.5	0
	0.591	0.5	0.58	0.5	0.5	0.52	0.55	0.5	0.65	0.68	0.76	0.5	0.62	0.86	0.94	0.81	1

Table 2: The solution of $G_{5,5}$. The first column and first row are the pure strategies for Blue and Red, respectively. The last column is an optimal mixed strategy for Blue, and the last row is the resulting payoff against each of Red's strategies. The interior numbers are probabilities that Blue wins.

The marginal distribution of the number of units allocated to the large state in the optimal strategy of $G_{5,5}$ is (4,31,10,16,39,0)/100 on the allocations (0,1 2,3 4,5), to be compared to the uniform Blotto strategy of (20,20,20,20,20,0)/100. Both strategies agree that one should never allocate all 5 units to the big state, but not about much else. One can summarize the rest of the optimal allocations in $G_{5,5}$ as "If the big state gets 3 or more, then randomly pick some single small state for the remainder. If the big state gets 2 or less, then divide the remainder as evenly as possible over all three small states". However, the idea that a player will either concentrate on one small state or treat them all as evenly as possible does not hold up in larger versions of the game. In $G_{9,9}$ when both budgets are 9, for example, the optimal mixed strategy includes 4113.The game $G_{9,9}$ is not shown here because it is too large, but the interested reader can find it on sheet "G99" of the workbook *ECollege.xlsm.* That workbook also includes sheets for $G_{5,5}$ through $G_{8,8}$. No exploitable pattern emerges, at least not to the author, as finer divisions of the budgets are permitted.

In $H_{b,r}$, Red is not restricted to spending his budget in integer amounts. Let $\mathbf{y} = (y_1, y_2, y_3, y_4)$ be a possible strategy for Red, and suppose that **y** is composed entirely of integers. If any component of **y** matches Blue's allocation, then the tie is resolved by flipping a coin to decide the winner in that state. Red will be tempted to increase that component by a very small amount, say ε, in order to convert potential ties into outright victories. He cannot do this without decreasing the allocation in some other state, thereby converting potential ties in that state into outright losses, but that same decrease can safely be large enough (3ε, to be precise) to permit increases in all states other than the one decreased. In effect, Red has the option of giving up one unit of budget in order to win all ties. This observation permits an analysis of $H_{b,r}$ by conventional means, since it is now possible to list all of Red's strategies: some strategies sum to r and have ties decided by coin flipping, while the rest sum to $r - 1$ and have all ties won by Red. The value of $H_{9,9}$ turns out to be 0.39. In that game Red avoids the strategies that sum to r, always finding it attractive to sacrifice one unit of budget in order to win all ties. The detailed game matrix is displayed and solved on sheet "H99" of *Ecollege.xlsm*.

Thus the best known strategy for Blue in the game EG((2,1,1,1),**0**,x,x), a game whose value is 0.5 by symmetry, can only guarantee to win 39% of the time. At least this number is larger than 25%, but it is disappointingly far from 50%. The prospects of solving EG((2,1,1,1),**0**,x,x) by this brute-force technique are not encouraging. The prospects of solving the larger EG game that corresponds to the USEC are even less so.

6. Summary

We have considered three cases where electoral politics is just a matter of money:

1. **Worst Case Blue**. Blue must spend before Red, and must consequently have about twice Red's budget to be competitive. The problem of finding the exact smallest Blue budget that will ensure victory is identified as the WCKP, and an algorithm for solving it is offered for the case where there are no biases. Randomization is not involved in optimal play.
2. **Continuous Public Spending**. This is probably the closest of the three cases to reality. Both parties watch each other closely, spending money in continuous time until election day. A complete solution is given, except for the possibility of equal

budgets. The impact of biases in the various states is equivalent to a budgetary offset. Again, randomization is not involved.

3. **Secret Spending**. Both parties spend money in secret, and then the election is decided by comparing spending in the various states. Randomization is essential to optimal play. A blotto game BG is solved, more or less, and used to approximate the probability that Blue wins the EG election. We have no evidence that the approximation is a good one (the opposite, in fact), and are able to find rigorous solutions of EG games only when the number of states is small. The EG game that corresponds to the USEC continues to resist solution.

For budgets near parity, the influence of money is extreme in case 2. This may partially explain why Republicans and Democrats often have budgets that are near parity in actual campaigns, since the consequences of falling behind are disastrous.

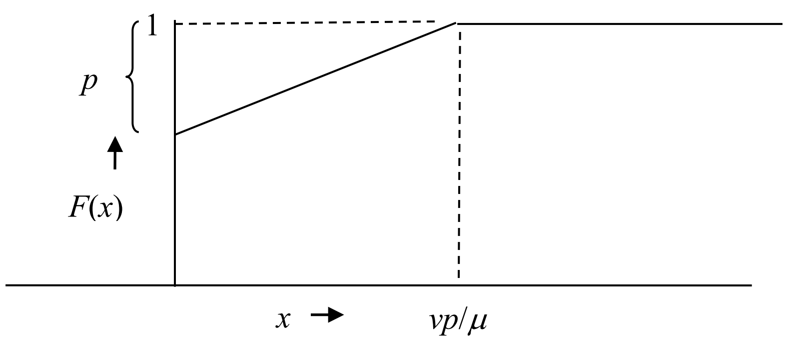
Appendix: Solution of a two-person zero-sum Blotto game

Blue and Red are the maximizing and minimizing players, respectively. Blue chooses x, a nonnegative number, and Red chooses y, another nonnegative number. The payoff to Blue is $A(x, y; v, z, \lambda, \mu) = vI(x + z - y) - \lambda x + \mu y$, where v, λ and μ are positive numbers and z is any real number. Blue (Red) chooses x (y) knowing everything but y (x). The function $I()$ is 1 for positive arguments, 0 for negative arguments, or 0.5 if the argument is 0.

We first observe that, since $I(-w) + I(w) = 1$ for all w, we have $A(x, y; v, z, \lambda, \mu) = v - A(y, x; v, -z, \mu, \lambda)$. It follows that it suffices to consider only nonnegative values for z, so we assume $z \geq 0$ in the following. From here on we refer to the objective function as $A(x,y)$, since the rest of the parameters are known to both sides.

Let $\delta = \frac{\mu z}{v}$. If $\delta \geq 1$, then $A(x, 0) \leq v$ and $A(0, y) \geq v$, so the value of the game is v and 0 is an optimal strategy for both sides. From here on we assume $\delta < 1$.

Let random variable X be Blue's choice, let $F(x)$ be the cumulative distribution function of X, and let $A(F, y)$ be Blue's expected payoff against Red's choice of y. We will find an optimal strategy for Blue among the class of distributions where X is 0 with probability $1 - p$, or, else uniform between 0 and some upper limit. Figure 1 shows a sketch of the typical CDF. For such distributions, the expected value of X is the area above the CDF: $E(X) = \frac{vp^2}{2\mu}$. The slope of the CDF over the interval $[0, vp/\mu]$ is required to be μ/v, so the only free parameter is p, which must be in the interval [0, 1].



Since $E(X)$ does not depend on y, we temporarily remove that term from the objective function by considering $f(y) \equiv A(F, y) + \lambda E(X)$. The worst choice of y from Blue's viewpoint will minimize this function. The probability that $X + z$ exceeds y is $1 - F(y - z)$. Therefore

$$f(y) = v(1 - F(y - z)) + \mu y = \begin{cases} v + \mu y \text{ if } y < z \\ g(p) \equiv vp + \mu z \text{ if } z < y \leq z + vp/\mu \end{cases}$$

The cancellation of the μy term in $g(p)$ is because of the aforementioned observation about slope, which makes $f(y)$ constant over the interval $(z, z + vp/\mu]$. If $y < z$, then the minimizing y is clearly 0. Red will not find it attractive to make $y = z$ (a slightly larger value would be better) or larger than vp/μ, since $f(y)$ reverts to increasing at slope μ in that region. Therefore the minimum of $f(y)$ over all nonnegative y is the smaller of v and $g(p)$. Blue wants to choose p to maximize the function $\min(v, g(p)) - \lambda E(X)$. Consider first the problem of maximizing $g(p) - \lambda E(X)$. This is a quadratic function of p for which the maximizing value of p can easily be shown to be μ/λ. If $g(\mu/\lambda)$ exceeds v, then p should instead make $g(p) = v$, which is done by making $p = 1 - \delta$. Thus the maximizing value of p is the smaller of $1 - \delta$ and μ/λ, depending on whether $\delta + \dfrac{\mu}{\lambda} \geq 1$. Substituting these values into the formula for $A(F,y)$, we find that, for all y,

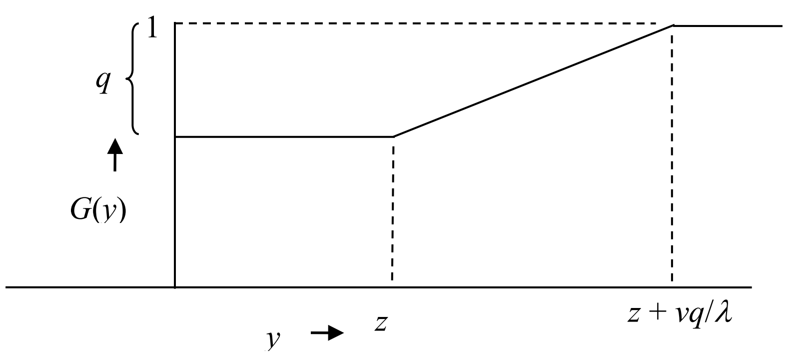
$$A(F, y) \geq \begin{cases} v - \dfrac{v\lambda}{2\mu}(1-\delta)^2 \text{ if } \delta + \dfrac{\mu}{\lambda} \geq 1 \\ v\dfrac{\mu}{2\lambda} + v\delta \text{ if } \delta + \dfrac{\mu}{\lambda} \leq 1 \end{cases}.$$

Let $\rho = \mu/\lambda$, and define the function

$$Val(\rho, \delta) = \begin{cases} 1 - 0.5(1-\delta)^2/\rho \text{ if } \delta + \rho \geq 1 \\ 0.5\rho + \delta \text{ if } \delta + \rho \leq 1 \end{cases}$$

Then the above observations show that Blue can guarantee a payoff of at least $vVal(\rho, \delta)$, regardless of y.

We next consider mixed strategies for Red. Let random variable Y represent Red's choice, and let $G(y)$ be the CDF of Y. There is clearly no good argument for Red's choosing y in the open interval $(0, z)$, so we consider only CDFs of the form shown in the figure below.



Random variable Y is 0 with probability $1 - q$, or otherwise uniform over the interval $[z, z + vq/\lambda]$. The expected value of Y is $E(Y) = qz + \dfrac{vq^2}{2\lambda}$. Let $A(x,G)$ be the expected payoff if Blue uses x against G. The probability that x exceeds $Y - z$ is $G(x + z)$. For $0 < x \leq vq/\lambda$, $G(x+z) = 1 - q + \lambda x / v$, so, in that interval, $A(x, G) = v(1-q) + \mu E(Y)$, which does not depend on x. Since Blue has no motive to choose x outside of that interval, Red should choose q to minimize $v(1-q)+\mu(qz+\frac{vq^2}{2\lambda})$, another quadratic equation. The minimizing value is $q=\frac{\lambda}{\mu}(1-\delta)$, or else 1 if that product exceeds 1. The product exceeds 1 when $\delta+\frac{\mu}{\lambda}<1$. Substituting the minimizing q into the formula for $A(x,G)$, we find

$$A(x,G)\le\begin{cases} v-\frac{v\lambda}{2\mu}(1-\delta)^2 \text{ if } \delta+\frac{\mu}{\lambda}\ge 1 \\ v\frac{\mu}{2\lambda}+v\delta \text{ if } \delta+\frac{\mu}{\lambda}\le 1 \end{cases}$$

In other words, Red can guarantee that the expected payoff will not exceed $vVal(\rho,\delta)$, regardless of x. This is the same value that Blue can guarantee, so $vVal(\rho,\delta)$ is the value of the game, and the optimized mixed strategies F and G are optimal for Blue and Red, respectively.

Let $E(I)$ be the expected value of the random variable $I(X+z-Y)$, so that

$$vVal(\rho,\delta)=vE(I)-\lambda E(X)+\mu E(Y).$$

It is sometimes useful to have expressions for each of the three components of the game value. We record below the correct expressions when X and Y are determined by the optimal mixed strategies derived above, including formulas for $z\le 0$.

For $z\ge 0$, define $\delta=\frac{\mu z}{v}$ and $\rho=\frac{\mu}{\lambda}$.

For $\delta+\rho\le 1$, $E(I)=0.5\rho$, $\lambda E(X)=0.5v\rho$, and $\mu E(Y)=v(\delta+0.5\rho)$

For $\delta+\rho\ge 1$, $E(I)=1-\frac{1-\delta^2}{2\rho}$, $\lambda E(X)=v\frac{(1-\delta)^2}{2\rho}$, and $\mu E(Y)=v\frac{1-\delta^2}{2\rho}$

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For $\delta+\rho\ge 1$, $E(I)=\frac{1-\delta^2}{2\rho}$, $\mu E(Y)=v\frac{(1-\delta)^2}{2\rho}$, and $\lambda E(X)=v\frac{1-\delta^2}{2\rho}$

The redundant definitions on the boundaries where $z=0$ and $\delta+\rho=1$ are to emphasize that all three expected values are continuous functions across those boundaries.

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