NOTES ON GAME THEORY

Alan R. Washburn

Naval Postgraduate School

Monterey, California

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1. Introduction

The Theory of Games was born suddenly in 1944 with the publication of *Theory of Games and Economic Behaviour* by John von Neumann and Oskar Morgenstern. Their choice of title was a little unfortunate, since it quickly got shortened to "Game Theory," with the implication being that the domain of applications consists merely of parlour games.

Nothing could be further from the truth. In fact, the authors hoped that their theory might form the basis of decision making in all situations where multiple decision makers can affect an outcome, a large class of situations that includes warfare and economics..

In the years since 1944, the only part of Game Theory where a notion of “solution” has been developed that is powerful enough to discourage further theoretical work is the part where there are exactly two players whose interests are in complete opposition. Game theorists refer to these games as two-person zero sum (TPZS) games. TPZS games include all parlour games and sports where there are two people involved, as well as several where more than two people are involved. Tic-tac-toe, chess, cribbage, backgammon, and tennis are examples of the former. Bridge is an example of the latter; there are four people involved, but only two players (sides). Team sports are also examples of the latter. Many of these games were originally conceived in imitation of or as surrogates for military conflict, so it should come as no surprise that many military problems can also be analyzed as TPZS games.

Parlour games that are not TPZS are those where the players cannot be clearly separated into two sides. Examples are poker and Monopoly (when played by more than two people). Most real economic “games” are not TPZS because there are too many players, and also because the interests of the players are not completely opposed. Such non-TPZS games are the object of continued interest in the literature, but will not be mentioned further in these notes. We will confine ourselves entirely to TPZS games. Washburn (1994) contains a more in-depth treatment of TPZS games. Luce and Raiffa (1957) is a dated but still worthwhile introduction to the general subject, or see Owen (1995) or Winston (1994).

2. Strategies and the Normal Form

In game theory, the word “strategy” has a very definite meaning; namely, *a complete rule for decision making*. For example, in a problem where one vehicle is pursuing another, one strategy for the pursuer is “turn hard left if the evader bears more than 10° left, or hard right if the evader bears more than 10° right, else go straight.” Given the strategy, the pursuer could be replaced by a computer program that would turn the pursuer’s vehicle in accordance with what is observed about the position of the evader. Note that a single strategy for the pursuer may result in many possible tracks of his vehicle, since the track depends on the evader’s strategy as well as the pursuer’s. However, if both the pursuer’s and evader’s strategies are known, then the outcome is determined. We assume that there is some numerical “payoff” associated with the outcome. For example, we might take the payoff to be the amount of time T required for the pursuer to catch the evader, with T being a function of the pursuer’s strategy s and the evader’s strategy r. Symbolically, the “payoff” is $T(r, s)$, with the evader wanting to choose r to make $T(r, s)$ large, and the pursuer wanting to choose s to make $T(r, s)$ small. Note that s and r are not necessarily numbers, but that $T(r, s)$ is.

The advantage of dealing with strategies like r and s is that every TPZS game can be represented, in principle, as a function of two variables. The main disadvantage is that many interesting games have so many strategies that it is impossible to enumerate them all. There are a great many computer programs, for example, that could be written for playing even such a simple game as tic-tac-toe. Each of these programs is a strategy. The number of strategies for playing checkers or chess is finite, but so large as to make the memory of even the largest computer pale by comparison. The number of strategies for playing most pursuit and evasion games is actually infinite, since the actions taken are continuous. So the conceptually simplifying idea of strategy will not be useful for actually solving large games such as these by enumeration. Nonetheless, there are some interesting games that can be solved in this manner, and there are some useful principles that can be determined from this approach in any case, so let’s imagine what would happen if we could enumerate all possible strategies, in which case we would have what is called the *normal form* of the game.

3. Saddle Points

Figure 3-1 shows the normal form of a small TPZS game as a matrix. By convention, we will always have the maximizing player (Player I) choose the row, and the minimizing player (Player II) choose the column.

Player II

	1	0	6	4
Player I	2	2	3	3
	1	9	9	9

Figure 3-1

Each choice is made in secret, and then the payoff is determined as the entry where the row and column intersect. We assume that the payoff matrix is known to both sides.

If Player I chooses row 2 in Figure 3-1, then the payoff will be at least 2 no matter what column II chooses. This is a larger minimum level than can be guaranteed by either rows 1 or 3, so Player I's security level (v_I) is 2. Choosing row 2 corresponds to utter pessimism on the part of Player I, since it guarantees 2 no matter what II does. Similarly, II's security level is $v_{II} = 2$, since the payoff will be at most 2 as long as II chooses the first column (remember that II wants the payoff to be small). Since $v_I = v_{II}$, we say that this game has a *saddle point*. We claim that Player I should choose row 2 because

a) he will get at least 2 by doing so,

and b) II can hold the payoff to 2 by choosing column 1, so Player I should not hope for more than 2.

Player I might consider this result disappointing, since row 3 has several 9's in it, and might consider playing row 3 in the hope that II will make a mistake. Note that Player I has little to lose and much to gain by choosing row 3 instead of row 2. Nonetheless, he should play row 2 if II is the rational, completely opposed player that we assume him to be, because such a player will choose column 1:

- column 1 ensures that the payoff will not exceed 2
- Player I can get at least 2 by playing row 2, so II should be content with guaranteeing that 2 will not be exceeded.

Even if the 9's in Figure 1 were changed to 900's, our theoretical advice to the two players would still be the same: Player I should play row 2, and II should play column 1. We will always give similar advice in any TPZS game that has a saddle point. To formalize, let a_{ij} be the payoff for row *i* and column *j*, and let $v_I = \max_i \left(\min_j a_{ij} \right)$, and $v_{II} = \min_j \left(\max_i a_{ij} \right)$. If $v_I = v_{II} \equiv v$, then the game has a saddle point, *v* is the *value of the game,* and the *optimal strategies* for Players I and II are the row and column that guarantee *v*.

Note that even in games that have saddle points, it is not true that a player's optimal strategy is the best reaction to every possible strategy choice of the opponent. For example, the best reaction of Player I to the choice of column 3 by II is to choose row 3, rather than what we have called I's optimal row. Nonetheless, the best *a priori* choice for Player I is row 2, since there is no reason to expect II to choose column 3.

It can be shown that all games with perfect information have saddle points. By "perfect information" we mean roughly that nothing of importance is concealed from a player when he makes his choice(s). Chess and backgammon are games of this class, but most card games are not because the opponent's cards are concealed. There is an optimal way to play chess, and a computer programmed with the optimal strategy would presumably win every time it got to make the first move. This observation has been of almost no help in writing chess playing programs because of the impossibility of writing all possible programs to test them against each other; nonetheless, an optimal strategy must exist. Games of pursuit and evasion are also often supposed to have perfect information, and therefore saddle points. In spite of the fact that they typically have infinitely many strategies, these games often have a simple enough structure to permit discovery of the saddle point through the techniques of differential games (Isaacs (1965)).

Exercises

1. Find v_I and v_{II} for each of the games in Figure 3-2. Note that $v_I \le v_{II}$ in all cases.
2. Prove that $v_I \le v_{II}$ for every game.

a)

3	6	1	4
5	2	4	2
5	5	3	5

b)

4	1
2	3

c)

8	0	4
–2	2	4

d)

10	2	0
8	2	3
0	1	7

e)

0	1	4
0	–3	0
1	2	1
–2	0	6

f)

2	1	1
1	8	8
–2	0	6
0	0	0
9	0	0

g)

8	3	0	7
0	3	4	2

h)

2	1	0	6
5	3	4	9
1	2	0	3
5	1	2	8
9	3	1	1
4	4	6	2

i)

2	4	9	0	5
8	3	2	5	0
1	3	8	0	4

Figure 3-2

4. Games Without Saddle Points

When a game has a saddle point, there is only one logical choice for the opponent, so there is no problem trying to guess what he will do. Either player could reveal his strategy choice and still ensure *v*. This is not so when there is no saddle point. In fact, the paramount characteristic of games without saddle points is the necessity of guessing the strategy of an opponent who will try to keep his strategy choice secret.

In World War II, for example, an anti-submarine aircraft frequently had to guess the depth to which a submarine had submerged in setting its depth charges. Ideally, the aircraft would choose the same depth as the submarine; otherwise, the probability of sinking the submarine would be low. Clearly, neither side should get into the habit of always choosing the same depth, since the essence of the problem is to be unpredictable. One way of ensuring unpredictability is to actually choose a strategy randomly. That this idea of using a mixed strategy is useful can be seen in Figure 4-1, where we have simplified the game so that each side has only three depth settings. The entries in the matrix are “sink probabilities.” Note that $v_I = .2$ and $v_{II} = .3$, which means that the game does not have a saddle point.

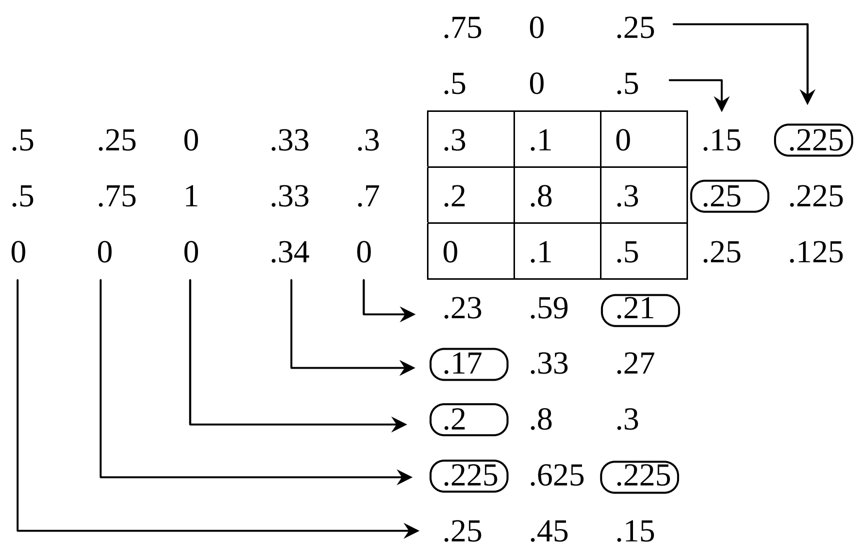

Figure 4-1

Now, actually, Player I (the aircraft) can guarantee a sink probability greater than v_I by using the mixed strategy $\underline{x} = (x_1, x_2, x_3) = (.3, .7, 0)$. (In general x_i will be the probability of using row *i*, y_j of column *j*.) If II uses his first strategy, then, according to the law of total probability, the payoff will be .3(.3) + .7(.2) = .23. If II uses his second or third strategy, the payoff will be .3(.1) + .7(.8) = .59 or .3(0) + .7(.3) = .21. The smallest of these three numbers is .21 (circled in Figure 4-1), which is greater than v_I. So the mixed strategy (.3, .7, 0) is better than any of the three “pure” strategies. Examination of several other mixed strategies for Player I is shown in Figure 4-1. The best of the five mixed strategies examined is (.25, .75, 0), which ensures a sink probability of .225. There are infinitely many more mixed strategies $\underline{x}$; but it is not necessary to examine them because it turns out that II can guarantee that the sink probability will not exceed .225, as can be seen in Figure 4-1 where II has circled the *largest* of three payoffs for each of the two mixed strategies $\underline{y}$ that he has tested. The value of the game is therefore $v = .225$, and the optimal mixed strategies for the two players are $\underline{x}^* = (.25, .75, 0)$ and $\underline{y}^* = (.75, 0, .25)$. In general, if there is a number v and a pair of mixed strategies $\underline{x}^*$ and $\underline{y}^*$ such that

- $\sum_i a_{ij} x_i^* \geq v$ for all j,

and

- $\sum_j a_{ij} y_j^* \leq v$ for all i,

then we will say that the value of the game is v and that $\underline{x}^*$ and $\underline{y}^*$ are *optimal strategies* (we will skip the word “mixed” except for emphasis). This definition applies to all TPZS games — if the matrix happens to have a saddle point, then $\underline{x}^*$ and $\underline{y}^*$ will consist of 0’s except in one row or column.

We have seen that the idea of randomization is useful in a particular game. It permits us to solve the game in the same sense that we earlier solved saddle point games; i.e., each player can ensure the same number v by using his optimal mixed strategy. Three questions now arise:

1) Does every game have a solution when mixed strategies are permitted?

2) If so, how can we find it?

3) If we can find it, is the solution meaningful if the game is to be played only once?

The answer to the first question is yes. John von Neumann proved that every TPZS game with finitely[*] many strategies has a value *v* and optimal mixed strategies $\underline{x}^*$ and $\underline{y}^*$. We will deal with methods for answering the second question in succeeding sections.

The third question is not really about games, but about expected values; it is simply the old question of whether an expected value computation is relevant to a gamble that is to be made only once. In the depth-charging example, the answer to the third question is clearly yes: .225 is a probability in the same sense that the entries in the matrix are probabilities in the first place. Even if a gamble is only to be made once, surely the best gamble is the one with the greatest probability of success. More generally, the answer is "yes, provided the payoff function is carefully chosen." For example, suppose that an attack on a redundant command and control system results in *N* of the redundant paths surviving, where *N* is random because of mixed strategies used by the two sides and possibly for other reasons. Then using the expected value of *N* for a payoff would be wrong, because it makes no difference how many paths survive, as long as the number is not zero. A better payoff would be the probability that no paths survive.

Many people encounter some intuitive resistance to the idea of randomization as an essential part of decision making in games. This is perhaps due to a prior acquaintance with single person decision making. Unpredictability is of no value in single person decision problems, even though it is vital in games:

Q: What stock did you decide to buy – ABC or XYZ?

A: I thought about it a long time and decided to flip a coin.

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[*] The finite restriction is necessary. For example, the TPZS game where the winner is the one who thinks of the largest number has no solution.

Upon overhearing this conversation, our conclusion would be that the investor is simply choosing an oblique way of saying that the two investment opportunities appear equal to him. Furthermore, if we overhear the comment often enough, we conclude that the investor does not really know much about the stock market; certainly we would feel cheated if we discovered that a stockbroker gave us advice on this basis. In single person decision problems, the idea of randomization is not logically necessary, so we tend to regard its employment as an admission of defeat or indifference.

There is a danger of carrying this attitude over to the consideration of competitive situations. For example, there is a tendency in the military to codify tactics; in situation X, the best response is Y, etc. If situation X is a guessing game, then this codification may be a mistake, since there is no such thing as a once-and-for-call "best tactic" in a guessing game. It is essential to be unpredictable, and one way of achieving this is to use the very same mixed strategies that are absurd in single person decision problems. Most people who have been involved in protracted conflict can tell horror stories about disasters that have ensued because of repetition of a tactic — bombers arriving at the same time every day, the same communication channel being selected time after time, etc. Some of these unfortunate incidents are no doubt due to carryover of the idea of the existence of a best tactic to situations where no such tactic exists.

There is nothing really new about our observation that unpredictability is (or ought to be) an essential part of conflict — military strategists have always stressed the need for surprise. What is new is explicit identification of the mixed strategy as a mathematical representation of unpredictability, with all the associated possibilities for computation and optimization.

Critic: "But it is quite possible to include the idea of unpredictability in conflict without having people carry a coin around in case they have to make a decision. Suppose, for example, that I were the row player in one of your matrix games.

> My approach would be to use my experience and my intelligence network (a vital part of conflict that you have completely ignored) to estimate what column will be chosen by my opponent. I will concede that the opponent's column choice might not be perfectly predictable, so that I would have to introduce a probability distribution $\underline{y}$ over columns, but the point is that $\underline{y}$ is estimated, rather than calculated. Then I would make whatever decision gives me the largest expected payoff. There is no coin required. My assessment of $\underline{y}$ will change from time to time, so I may appear to you to be acting at random in successive plays, but that's not the way I look at it."

Our response to this criticism is twofold. First, suppose that v is the value of the game, but that the critic is able to achieve more than v on the average. Then the opposing commander is subject to criticism, since he could hold the average payoff to v by using his optimal strategy $\underline{y}^*$. In fact, $\underline{y}^*$ is simply the worst possible (from the critics standpoint) value for $\underline{y}$, and therefore the best from the standpoint of the opponent. It is possible, of course, that the opponent will make a mistake, in which case it should certainly be taken advantage of. But using an assumed distribution $\underline{y}$ instead of a worst cast distribution $\underline{y}^*$ is responding to the opponent's perceived intentions, rather than to his capabilities. Second, the critic's procedure is of little use to an analyst who must assess the average payoff in a game long before the conflict occurs. The analyst may be interested, for example, in whether it is wise to change some of the entries in a payoff matrix. The value of the game is the only defensible way of turning the payoff matrix into a number so that comparisons between games can be made.

Exercise

Obtain upper and lower bounds on the value of the following modification of Figure 4-1 by guessing mixed strategies for the two sides:

.3	.1	0
.2	.8	.1
0	.4	.5

Are you able to find an optimal pair $\underline{x}^*$ and $\underline{y}^*$? If not, what bounds on v are implied by your best candidates?

5. Solving 2-by-2 Games

Let the payoff matrix be as in Figure 5-1, with $(1 - x, x)$ and $(1 - y, y)$ being the mixed strategies for the two sides. If x is chosen in such a manner that each of II's columns have the same average payoff v, then we must have $(x)a + (1 - x)c = (x)b + (1 - x)d = v$.

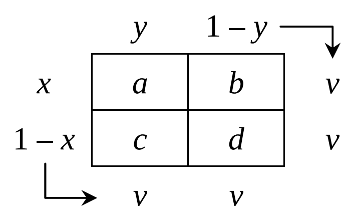

Figure 5-1

These equations can be solved for x and v. If we define $D = a + d - b - c$, the solution is $v = (ad - bc)/D$ and $x = (d - c)/D$. Similarly, the solution of $(y)a + (1 - y)b = (y)c + (1 - y)d = v$ is the same value for v and $y = (d - b)/D$. The game does not have a saddle point if and only if x and y are both between 0 and 1. If x and y are both probabilities, then they must be the optimal probabilities x^* and y^*. To summarize, if the game does not have a saddle point, then

$$D = a + d - b - c$$

$$v = (ad - bc)/D$$

$$x^* = (d - c)/D$$

$$y^* = (d - b)/D$$

Example.

If $(a,b,c,d) = (0,2,3,0)$, then $v = 1.2$, $x^* = .6$, and $y^* = .4$.

Example.

If $(a,b,c,d) = (3,2,1,1)$, then substitution gives $v = 1$, $x^* = 0$, and $y^* = -1$. This is not the solution of the game because y^* is not a probability. This game has a saddle point, and the value of the game is 2.

6. Solving 2-by-*n* or *m*-by-2 Games

Consider the game in Figure 6-1.

x	9	3	0	4
$1 - x$	1	3	6	0

Figure 6-1

In solving this game, we will take advantage of the fact, which is true in all TPZS games, that the optimal mixed strategy can be safely announced to the opponent. Even though the specific strategy choices must be kept secret, the distribution from which they are drawn can be public knowledge. Figure 6-2 shows the consequences of using the mixed strategy $(x, 1 - x)$ when Player II always chooses the worst possible (from Player I's standpoint) column. For example, when II uses the first column against $(x, 1 - x)$, then the payoff is $(x)9 + (1 - x)1$, which is a linear function of x — line 1 in Figure 6-2. Note that the line can be drawn by simply connecting 1 on the left-hand vertical axis to 9 on the right-hand vertical axis. Similar lines can be drawn for the other three columns. The four lines can be used to predict which strategy II will choose against any x; for example, II will choose column 4 if $x = .1$ because line 4 lies below all of the others at that point. Evidently, any point on the lower envelope of the four lines (shown shaded) is achievable by Player I. Furthermore, the highest point of the lower envelope is at (x^*, v). But (x^*, v) is exactly the point we would have computed in a 2×2 game where II had only the strategies corresponding

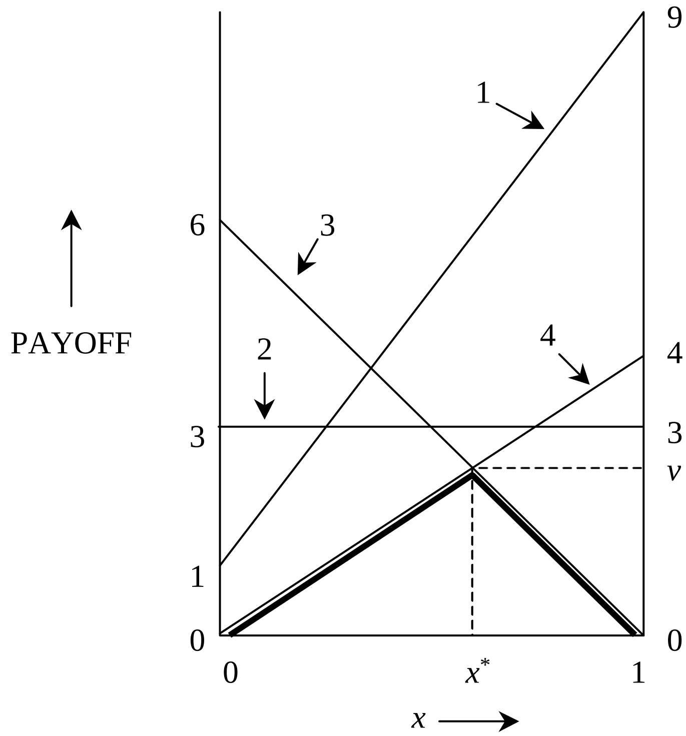

Figure 6-2

to lines 3 and 4. Consequently, we can solve this game by solving the game $(a,b,c,d) = (0,4,6,0)$: the solution is $v = 2.4$, $x^* = .6$, and $y^* = .4$. To verify that the original game has in fact been solved, we construct Figure 6-3.

	0	0	.4	.6	
.6	9	3	0	4	2.4
.4	1	3	6	0	2.4
	5.8	3	2.4	2.4	

Figure 6-3

Note that:

- The only use of Figure 6-2 is to reveal which 2×2 game to solve.

- As long as Player I uses (.6, .4), the payoff will be 2.4 if II uses (0, 0, y, $1 - y$) for any y. Nonetheless, the unique optimal strategy for II is (0, 0, .4, .6).

To solve an m×2 game, we follow a similar procedure except that (y^*, v) is the lowest point on the upper envelope of the m lines. There will always be some lowest point that is at the intersection of a non-decreasing line and a non-increasing line. Player I can safely use only these two lines, and the value of the game is the same as the value of the resulting 2×2 game. For example,

The game:

4	0
3	1
0	4

The graph:

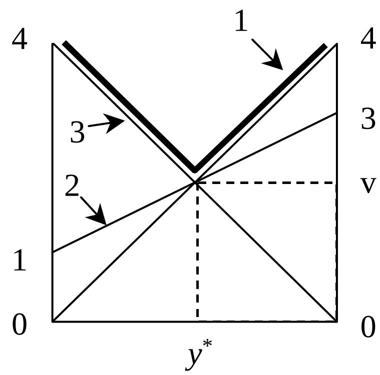

Figure 6-4

Since all 3 lines go through the minimum point on the upper envelope, we can "cross out" either row 1 or row 2 (but not row 3!) from the original game. If row 1 is crossed out, the solution is $\underline{x}^*$ = (0, 2/3, 1/3), $\underline{y}^*$ = (.5, .5), and v = 2. If row 2 is crossed out the solution is $\underline{x}^*$ = (.5, 0, .5), $\underline{y}^*$ = (.5, .5), and v = 2. The optimal strategy for Player I is not unique (any average of (0, 2/3, 1/3) and (.5, 0, .5) is also optimal), but v and $\underline{y}^*$ are. Of course, similar things can happen in 2×n games, except that the non-uniqueness is in $\underline{y}^*$.

7. Dominance

Compare columns 1 and 4 in Figure 6-3. Each element of column 4 is smaller than the corresponding element of column 1, so II prefers column 4 to column 1 no matter what row is chosen by Player I. Column 4 *dominates* column 1, so column 1 can be "crossed out" — II will use it with probability 0 in his optimal strategy. More generally, column *j* dominates column *k* if $a_{ij} \le a_{ik}$ for all *i*. Similarly, Player I will not use row *k* if it is uniformly smaller than row *i*; row *i* dominates row *k* if $a_{ij} \ge a_{kj}$ for all *j*. Roughly speaking, large columns and small rows can be safely removed from the payoff matrix. There is no instance of row dominance in Figure 6-3, but consider Figure 7-1. In diagram a, column 4 dominates column 2, and row 2 dominates row 3. If column 2 and row 3 are crossed out, diagram b results. Notice that column 3 dominates columns 1 and 4 in diagram b, even though it did not in diagram a. Crossing out these two columns results in diagram c. There is no further dominance in diagram c, but the game can now be solved using the procedures of Section 6. The value of the game is 3.8; the optimal solution and a verification of that solution are shown in diagram d.

The principal value of the dominance idea is to reduce the effective size of the game. Actually, if any linear combination of rows dominates row *k*, where the weights in the linear combination form a probability distribution, then row *k* can be eliminated. Similarly for columns. For example, column 2 in Figure 6-3 is not dominated by any other column, but it is dominated by .5 of the third column plus .5 of the fourth; with this observation, that game can be solved directly as a 2×2 game.

Exercises

1. Solve all of the games in Figure 3-2.
2. There are two contested areas, and the attacker (I) and defender (II) must each secretly divide their units between the two areas. The attacker captures an area if and only if he assigns (strictly) more units than the defender. The payoff is the number of areas captured by the attacker. The attacker has 3 units and the defender has 4. Construct the payoff matrix (it should have 4 rows and 5 columns), and solve the game. This is an example of a "Blotto" game.

a)

1	4	0	2	9
3	6	2	2	8
1	1	2	0	3
7	6	5	6	1

b)

1		0	2	9
3		2	2	8
7		5	6	1

c)

		0		9
		2		8
		5		1

d)

	0	0	.7	0	.3	
0	1	4	0	2	9	2.7
.4	3	6	2	2	8	(3.8)
0	1	1	2	0	3	2.3
.6	7	6	5	6	1	3.8
	5.4	6.0	(3.8)	4.4	3.8	

Figure 7-1

3. Modify Figure 3-2d to reflect the idea that Player II has an intelligence network that will be successful in discovering I's strategy choice with probability .5. A typical strategy for

II is now “choose column *j* if intelligence fails, else choose the best reaction to I’s row choice.” The new game should still be a 3×3. Solve it.

4. The same as Exercise 3 except use 3-2c.

8. Completely Mixed Square Games

It sometimes happens that it is clear from the description of a game that every strategy will be used with positive probability. In this case, we say that the game is *completely mixed*. Obtaining the solution of such a game amounts to solving some simultaneous linear equations. For example, consider the game in Figure 8-1.

1/2	0	0
0	2/3	0
0	0	3/4

Figure 8-1

One interpretation of this game is a hide-and-seek situation where Player I must guess which of 3 cells II is hidden in, and where the probability of detection is not 1 even if Player I guesses correctly (before reading further, make a guess at which row will be used most often by Player I). There is no dominance, so the game cannot be reduced. Suppose Player I were to use any mixed strategy that avoids some row entirely, and announce it (as he can safely do if the strategy is optimal) to II. Then II would play that column always, and the resulting probability of detection would be 0, which is clearly less than the value of the game. Therefore, Player I will not avoid any row entirely; that is, $x_1^* > 0$, $x_2^* > 0$, and $x_3^* > 0$. Let $v_i = \sum_j a_{ij} y_j^*$ for i = 1,2,3, and let v be the value of the game. v_i is the probability of detection when Player I uses row i against II’s optimal mixed strategy. We must have $v_i \le v$ and also $\sum_i x_i^* v_i = v$. Since $x_i^* > 0$, the only way this can be true is if $v_i = v$ for all i (the only way a positively weighted average of several numbers can be v when none of them is larger than v is if they all equal v — this is a special case of the complementary slackness result of the next section). Similar arguments show that $y_j^* > 0$ and $\sum_i a_{ij} x_i^* = v$ for all j. We now have two sets of four equations in four unknowns. The equations involving $\underline{x}^*$ are

$$\frac{1}{2}x_1^* = \frac{2}{3}x_2^* = \frac{3}{4}x_3^* = v \qquad \text{and} \qquad x_1^* + x_2^* + x_3^* = 1.$$

The solution is

$$v = \frac{1}{\left(\frac{2}{1}+\frac{3}{2}+\frac{4}{3}\right)} = \frac{6}{29} \qquad \text{and} \qquad \underline{x}^* = \left(\frac{12}{29}, \frac{9}{29}, \frac{8}{29}\right).$$

Similar calculations with the other set of equations show that Player II uses exactly the same mixed strategy; the final verification is Figure 8-2. Player I uses row 1 most often — is that what you guessed earlier?

	12/29	9/29	8/29	
12/29	1/2	0	0	6/29
9/29	0	2/3	0	6/29
8/29	0	0	3/4	6/29
	6/29	6/29	6/29	

Figure 8-2

It will always be true that if one can somehow guess which strategies will be "active" (used with probability greater than 0), then solution of the game amounts to solving a set of linear equations (the equations may be complicated, but at least solving equations is a familiar problem). This approach is the basis of the GUTS (GUess The Solution) technique of solving large games that are conceptually simple enough to limit the number of guesses that have to be made and checked out. To further illustrate the GUTS method, consider the following modification of the hide-and-seek game: Player I installs a fixed device in the first box that has probability p of detecting II regardless of which box Player I looks in. p is a constant, but we wish to leave it unspecified for the moment. Figure 8-3 shows the revised payoff matrix.

y_1	y_2	y_3	
$(1 + p)/2$	0	0	w
p	2/3	0	w
p	0	3/4	w

Figure 8-3

The first entry is $(1 + p)/2$ because we assume that the new device acts independently of Player I's look. The common row value is called w rather than v because w has not yet been shown to be the value of the game.

Intuitively, if p is large enough, then II will avoid the first column entirely. The effect of this should be that the game will be played as if the first column and therefore row were missing, in which case the probability of detection will be $p_0 \equiv 6/17$, the value of the remaining 2×2 game. In fact, this is clearly the case if $p \geq 2/3$, by dominance. However, suppose for the moment that p is small enough to keep the game completely mixed. We first determine $\underline{y}$ by solving four simultaneous equations:

$$y_1(1+p)/2 = w$$

$$y_1 p + 2/3 y_2 = w$$

$$y_1 p + 3/4 y_3 = w$$

$$y_1 + y_2 + y_3 = 1$$

The solution of these equations is (recall $p_0 = 6/17$)

$$y_1 = 2p_0/(2p_0 + 1 - p)$$

$$y_2 = (3/2)p_0(1-p)/(2p_0 + 1 - p)$$

$$y_3 = (4/3)p_0(1-p)/(2p_0 + 1 - p)$$

$$w = p_0(1+p)/(2p_0 + 1 - p)$$

Note that (y_1, y_2, y_3) is a probability distribution for $0 \le p \le 1$. Therefore the value of the game does not exceed w. To see if the value of the game is equal to w, we must solve for $\underline{x}$. The equations are

$$\begin{aligned} x_1(1+p)/2 \quad + \quad x_2 p \quad + \quad x_3 p &= u \\ 2/3 x_2 \qquad\qquad &= u \\ 3/4 x_3 &= u \\ x_1 \quad + \quad x_2 \quad + \quad x_3 &= 1 \end{aligned}$$

The solution of these equations is

$$\begin{aligned} x_1 &= 2(p_0 - p)/(2p_0 + 1 - p) \\ x_2 &= (3/2)p_0(1+p)/(2p_0 + 1 - p) \\ x_3 &= (4/3)p_0(1+p)/(2p_0 + 1 - p) \\ u &= p_0(1+p)/(2p_0 + 1 - p) \end{aligned}$$

$\underline{x}$ is a probability distribution only for $p \le p_0$, since $x_1 < 0$ when $p > p_0$. The quantities u and w are equal for all p, but u (or w) is the value of the game only for $p \le p_0$. When $p > p_0$, Player I does not use the first row, II does not use the first column, and the value of the game is $v = p_0$; this is as we earlier guessed, except that the critical value of p is p_0, rather than 2/3. In summary,

for $p \le p_0$, $\underline{x}^* = \underline{x}$, $y^* = \underline{y}$, and $u = v = w$

for $p \ge p_0$, $\underline{x}^* = (0, 9/17, 8/17) = y^*$, $v = p_0$ and $\mu = w \ne v$.

Player I's behavior is reasonable — as the device in the first box becomes more reliable, he is more inclined to depend on it and search one of the other two boxes. II's behavior is surprising. As p increases, II is *more* inclined to hide in the first box (as long as $p \le p_0$), whereas one would suspect intuitively that II would be *less* inclined to hide there. It is only when $p > p_0$ that II avoids the first box. It is intuition that is wrong here, rather than the solution; the reader should convince himself of this. Another interesting feature is that there is no advantage to Player I for making $p > p_0$; once a system is so good that the opponent avoids it, there is no point in making it better.

There are two important points in this section. First, in circumstances where it is clear that the solution of a square game will be completely mixed, solving the game amounts to solving simultaneous linear equations. Second, the GUTS procedure is fail-safe as long as one verifies that the resulting pair of strategies is in equilibrium.

Exercise

1. Verify that the "completely mixed" assumption for the game in Figure 4-1 results in $\underline{x} = (29/52,\ 7/52,\ 16/52)$ and $u = 10.1/52 = .194$. Is it possible that the solution for $\underline{y}$ would be a probability vector?

9. The General Finite Case – Linear Programming

The mathematical problem of maximizing a function of several variables subject to constraints on those variables is called a mathematical programming problem. If the function to be maximized (the objective function) and all the constraints are linear functions of the variables, then we have the special case of linear programming (LP). LP is a powerful optimization tool because of the availability of computer codes capable of solving problems with hundreds of thousands of variables and constraints. It is therefore handy that large TPZS games can be solved using Linear Programming.

We can formulate the solution of a TPZS game as an LP problem as follows: consider the set of n "value constraints" where $x_1, \ldots, x_m$ has the usual meaning and x_0 is a new variable.

$$\sum_{i=1}^{m} a_{ij} x_i - x_0 \geq 0, \qquad j = 1, \ldots, n.$$

The j^{th} of these inequalities states that the average payoff exceeds x_0 when II uses strategy j and I uses mixed strategy $\underline{x} = (x_1, \ldots, x_m)$. If all n of the inequalities hold, then x_0 is a security level for Player I in the sense that the average payoff will be at least x_0 no matter what II does. Player I's LP, then, is to make x_0 as large as possible subject to the n value constraints and additional constraints to the effect that (x_1, …, x_m) must be a probability distribution. The objective function is a linear function of the m+1 variables (the last m coefficients being 0), and so is each of the constraints. An LP code will provide the solution $\left(x_0^*, x_1^*, \ldots, x_m^*\right)$ that maximizes the objective function subject to the constraints. The value of the game is $v = x_0^*$, and the optimal mixed strategy for Player I is $\underline{x}^* = \left(x_1^*, \ldots, x_m^*\right)$.

Similarly, since LP codes can also be used for minimization, $\underline{y}^*$ can be found by solving Player II's problem:

$$\begin{aligned} &\text{minimize} && y_0 \\ &\text{subject to} && \sum_{j=1}^{n} a_{ij} y_j - y_0 \le 0; \quad i = 1, \ldots, m \\ & && \sum_{j=1}^{n} y_j = 1, \\ &\text{and} && y_1, \ldots, y_n \ge 0 \end{aligned}$$

If the solution is $\left(y_0^*, y_1^*, \ldots, y_n^*\right)$, then the value of the game is $v = y_0^*$, and the optimal mixed strategy for II is $\left(y_1^*, \ldots, y_n^*\right)$.

The fact that x_0^* from I's program is equal to y_0^* from II's program is not obvious, except for the fact that we already know that all games have a solution, so both numbers must be the value of the game. Actually, one can prove $x_0^* = y_0^*$ directly by observing that I's and II's programs are duals (Winston (1994)) of each other, thereby providing a direct and constructive proof that every finite TPZS game has a solution. The complementary slackness result of linear programming also has a game theoretic interpretation. It predicts that for every i either $x_i = 0$ or else the i^{th} constraint in II's program is active (= rather than < holds), and similarly either $y_j = 0$ or else the j^{th} constraint in I's program is active (= rather than > holds). This is the basis of the equation solving technique for completely mixed games.

It may seem that a standard method for solving finite TPZS games can now be identified:

1) Convert the game to normal form by identifying first the strategies for each side and then the payoff function.
2) Use LP to solve the resulting matrix game.

This “brute force” procedure is sometimes the best way to proceed, but the process of carrying out step 1 has some disadvantages. First, it is likely to destroy whatever structure the game had when first formulated. For example, given only the normal form of tic-tac-toe (a very large matrix full of 1’s and 0’s, and –1’s), it would be very difficult to recover the simple nature of the game. Tic-tac-toe is not difficult to solve if the structure is understood (most schoolboys do it — it has a saddle point at 0 for “tie”), but it would appear to be formidable if presented only if normal form. Second, there is a large class of games where the number of strategies is too large for LP; such games may still have solutions, but the brute force technique is not the way to find them. It is not possible in these brief notes to discuss methods that have a better chance of success, but see Washburn (1994).

In summary, we can say that in theory LP can be used to solve any game with finitely many strategies, given the availability of a payoff matrix. Frequently, however, exploitation of some kind of special structure will produce a solution with less effort.

Exercise

Check that the complementary slackness result holds for all games solved so far.

10. Formulation of Military TPZS Games

Selection of the payoff function is simplest in games where there are only two possible outcomes (I wins/II wins, II detected/II not detected, II shot down/I shot down, II sunk/II not sunk, etc.), in which case the payoff is the probability that the first outcome happens. In some games there are resource losses L_I and L_{II} on both sides, in which case it is tempting to use $L_{II} - L_I$ as a payoff. The question of whether the losses are commensurate is likely to arise, since the units may be different or may be valued differently by the two sides. This issue does not arise if all the variable losses are II's, in which case the payoff is L_{II}. For example, if Player I conducts an attack with units that can only be used once, then the only issue is how much damage they do to II.

The payoff function could also be the average amount of time until some critical event happens, desired large by Player I and small by II.

We give below a brief discussion of several games that have been formulated and solved in applications. Our intent is not so much to communicate the details of the solutions as to indicate the type of problem that can be successfully dealt with. In all cases, the payoff is one of the possibilities discussed above.

Hide-and-seek Games

The payoff is the probability that I detects II, or sometimes a capture distance, and the strategies are choices of position for each side. "Position" might mean physical location, or it might be depth if a submarine is being sought, or frequency if an emitter is being sought. There is an example in Section 8 where detection is impossible unless the position choices coincide. This requirement is not necessary; the payoff can be an arbitrary function of the two position choices. This is one case where LP is the natural solution method. For example, MEDOPS, an ASW planning tool, uses LP to reduce a sensor-depth versus submarine-depth game to an equivalent sweep width (the game value) for use in subsequent calculations.

Duels (Games of Timing)

Each participant makes his shot(s) at a time of his own choice, with the probability of hitting the opponent being a prespecified increasing function of time. Either the rules of the game are rigged so that both players are concerned only with the fate of Player II, or else it is assumed that both players being hit is equivalent to neither player being hit (otherwise the game is not zero-sum, and the duelists might be well advised to call the whole thing off). There are infinitely many strategies, but such games have been solved using the GUTS procedure. The mixed strategies are probability distributions of when to act. Analysis of these games was prompted by military questions about when to open fire.

Blotto Games

This class of games is named after the legendary Colonel Blotto, who has to divide his attacking force among several forts without knowing how the defenders are distributed. In the generalization, each side has a certain total force that must be divided among N "areas." The payoff is a sum of payoffs in each area, and the payoff in each area depends only on the forces assigned to that area. Exercise 2 at the end of Chapter 7 is a Blotto game. There have been diverse applications of this idea:

a) Areas are segments of the ocean, I's resource is ballistic missile submarines, and II's resource is ASW forces. The payoff is the average number of I's submarines surviving an attack by II. The engagement rules in this application are typically such that a saddle point exists.

b) Areas are ICBM silos, I's resource is attacking ICBM's, and II's resource is defending ABM's. The payoff is the average number of silos killed. Mixed strategies are involved.

c) Areas are routes, I's resource is tons of supplies, and II's resource is interdiction forces. The payoff is the amount of supplies getting through. The interdiction forces could be troops, submarines, etc.

Communications

A jammer (II) attempts to interfere with information transmission by adding noise to a transmission. Both transmitter and jammer are power limited, but otherwise free to emit arbitrary signals. The payoff is the rate of transmission of information. It can be shown (Welch (1958)) that the jammer's optimal strategy is to transmit Gaussian noise. This is in agreement with the conventional wisdom in Electronic Warfare to the effect that non-Gaussian jamming may be particularly effective against a specific communication scheme, but that simple Gaussian noise jamming is "robust" in the sense of being effective against all possible communication schemes. It is interesting that many natural noises are Gaussian; this is evidence for the widely held point of view that Mother Nature is actually diabolical.

Tactical Air War

Many modern military aircraft can be used to perform multiple functions. Berkovitz and Dresher (1959) take these functions to be "counter-air," "air defense," and "ground support," and solve a multiple-day campaign in which only the survivors of day n could be assigned to one of the three roles on day $n + 1$. The payoff depends only on the ground support assignments for the several days, but consistently assigning all aircraft to ground support is not optimal because large aircraft losses due to the enemy's counter-air operations would result. An interesting feature of the solution is that the weak side randomizes more than the strong side; this is also often true in Blotto games. Continuous-time versions of this problem have also been formulated as differential games (Taylor (1974)).

Barrier Operations

In patrolling a barrier, a searcher has the choice of going slow so that he can hear well, or fast so that he can cover as much ground as possible. A penetrator has the choice of going slow to be quiet, or fast to minimize the length of the encounter. The probability of detection depends continuously on the speed choice of both parties, and a game results. Langford (1973) solves a version of this game.

Pursuit and Evasion Games

Define the "launch envelope" of a missile to be those initial positions for which an optimally programmed missile can catch its target regardless of how the target maneuvers. Determination of this envelope is properly in the province of differential games, a topic invented by Isaacs (1965). The payoff is 1 or 0 depending on whether the target is caught or not, and strategies are programs for control surfaces as a function of sensor inputs. Solutions have been obtained only for highly simplified physical models, but have nonetheless proved enlightening for actual design.

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