Mode synthesis technique for dynamic analysis of structures*

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A mode synthesis technique is presented for determining the normal modes, natural frequencies, and responses of three-dimensional complex structure with flexible joints. Lagrange's equations are used to develop the equations of motion of the structures. Based on this technique a computer program called MODSYN has been developed for both free-free and cantilever systems. An example demonstrates the accuracy of this method.

Subject Classification: [43]40.20; [43]20.40.

LIST OF SYMBOLS

B_i^j	Distance of the center of gravity of the ith structure from its attachment point to the main structure in the jth direction
B_0^j	Distance from the center of gravity of the system to the center of gravity of the main structure in the jth direction for the undeformed configuration
f	Total number of generalized coordinates used to describe the motion of the system
f_i	Number of elastic modes used to describe the motion of the ith structure ($i=0,1,\ldots$)
$F_i^j(k)$	External force at the kth mass of the ith structure (both main and substructure) in the jth direction
$J_i^{j,j+1}$	Product moment of inertia of the ith substructure about its attachment point
$J_0^{j,j+1}$	Product moment of inertia of the main structure about the center of gravity of the system
J_i^{jj}	Moment of inertia of the ith substructure about its attachment point
J_0^{jj}	Mass moment of inertia of the main structure about the center of gravity of the system ($j=1,2,3$)
$[K]$	Stiffness matrix defining the coupling effect of various masses of the system
l_i	Mass point of the main structure where the ith substructure is attached
m_i	Mass of the ith structure
m_0	Mass of the main structure
$[M]$	Inertia matrix
M_{in}^j	Modal moment at the attachment point of the ith structure in the jth direction due to its nth mode
$M_i^j(k)$	External moment of the kth mass of the ith structure in the jth direction
n_i	Number of mass points in the ith structure
$\{q\}$	Column matrix of generalized coordinates
q_{in}	Generalized coordinates describing participation of the nth uncoupled mode of the ith beam in free vibration of the system ($i=0,1,\ldots;n=1,2,\ldots,f_i$)
q_{iR_j}	Generalized coordinates describing a relative rotation motion in the jth direction of the joint spring of the branch
q_{iT_j}	Generalized coordinates describing a relative translation motion in the jth direction of the joint spring of the ith branch
q_{0R_j}	Generalized coordinate describing rigid body rotation in the jth direction
q_{0T_j}	Generalized coordinate describing rigid body translation in the jth direction
Q_i	Generalized force
Q_{in}^j	Modal shear at the attachment point of the ith structure in the jth direction due to its nth mode
T	Kinetic energy
U	Potential energy
$y_i^j(k)$	Coordinate of the center of gravity of the kth mass of the ith structure in the jth direction measured in relationship to the center of gravity of the system ($i=0$ for the main structure; $i=1,\ldots$ for the substructure; $j=1,2,3$) in the undeformed configuration
$Y_i^j(k)$	Translational displacement of the kth mass of the ith structure in the jth direction
$\theta_i^j(k)$	Rotational displacement of the kth mass of the ith structure in the jth direction
μ_{in}	Modal mass in the nth normal mode of the ith structure
$\phi_{in}^j(k)$	Modal translational displacement of the kth mass of the ith structure in the jth direction, corresponding to the nth normal mode of the ith structure
$\phi_{iR_l}^j(k)$	Modal translation displacement of the kth mass of the ith structure in the jth direction, corresponding to the rigid body lth rotation mode ($i=0,1,\ldots;j=1,2,3;l=1,2,3$)
$\phi_{iT_l}^j(k)$	Modal translation displacement of the kth mass of the ith structure in the jth direction, corresponding to the rigid body lth translation mode
$\psi_{in}^j(k)$	Modal rotational displacement of the kth mass of the ith structure in the jth direction, corresponding to the nth normal mode of the ith structure
$\psi_{iR_l}^j(k)$	Modal rotational displacement of the kth mass of the ith structure in the jth direction, corresponding to the rigid body lth rotation mode

$\psi^j_{iT_l}(k)$ Modal rotational displacement of the kth mass of the ith structure in the jth direction, corresponding to the rigid body lth translation mode

ω_{in} Natural frequency of the nth normal mode of the ith structure

INTRODUCTION

Basically, in modal synthesis the structure is treated as an assembly of substructures, each of which is analyzed as a separate unit. The equations of motion of the complete structure are formulated by synthesizing the properties of the components, such as mode shapes and interface compatibility conditions. During the past decade, new methods of variances of the methods falling within the general scope of the modal synthesis technique have been developed by many investigators.[1-8] A brief review and comments on these methods have been given by Hurty.[9]

Recently the modal synthesis technique has been used by growing numbers of industries, such as General Dynamics for the coupled analysis of the INTELSAT IV-A satellite (built by Hughes for INTELSAT) and the Atlas-Centaur launch vehicle, McDonnell Douglas for the coupled analysis of the MARISAT satellite (built by Hughes for COMSAT General) and the Delta launch vehicle, and Hughes for the dynamic analysis of MARISAT. In addition to the savings in computer time and space, this technique has many other advantages. In the analysis of a large structure whose substructures are built by different contractors, this technique reduces to a minimum necessary technical communication across component interfaces. It is also desirable for analyzing a very large structure whose components are tested separately. It can be used to combine the mode shapes of the components obtained by tests to analyze the complete structure.

This paper presents a modal synthesis technique based on the energy approach. The displacement shape of the structure is expressed as the superimposition of the rigid modes and the finite number of normal modes of the main structure and the substructures. Lagrange's equations are then used to develop the equation of motion in matrix form. The technique discussed in this paper proceeds along the same lines as Ref. 6 except that translational and rotational springs are added at the interfaces of the main structure and the substructures, and the technique is generalized for three-dimensional analysis. The advantage of this technique is that, instead of using the mode shapes of the substructures, it uses the modal forces and moments to determine the natural frequencies of the system. This reduces the amount of data required across the component interfaces for the analysis.

I. BASIC FORMULATION

In the analysis the system is divided into main structure and substructures. The substructures are attached to the main structure through joint springs. The system can be free–free or cantilever. In the free–free system, the displacement is expressed as the superposition of the rigid body modes of the system and free–free uncoupled normal modes of the main structure and the cantilever modes of the substructures. In the cantilever system, the displacement is expressed as the superposition of the cantilever modes of the main structure and the substructures. In this paper, the free–free system will be analyzed first since the cantilever system is a special case in which rigid body modes are absent and the free–free modes of the main structure are replaced by cantilever modes.

For simplicity in the analysis, a lumped mass structural model, shown in Fig. 1, is assumed. The results of this analysis are also valid for the system in which the modal characteristics of the main structure and substructures are determined by exact analysis, finite element methods, or dynamic test data. For the analysis, the origin of the coordinate systems is assumed to be the center of gravity of the main structure.

A. Main structure

The displacement of the main structure is expressed as follows:

$$Y^j_0(k,t)=\sum_{n=T_1,T_2,T_3,R_1,R_2,R_3,1}^{f_0}\phi^j_{0n}(k)q_{0n}(t), \tag{1}$$

where $j=1,2,3$; $k=1,\ldots,n_0$; q_{0T_1}, q_{0T_2}, and q_{0T_3} represent the translational rigid body motion of the system; q_{0R_1}, q_{0R_2}, and q_{0R_3} represent the rotational rigid body

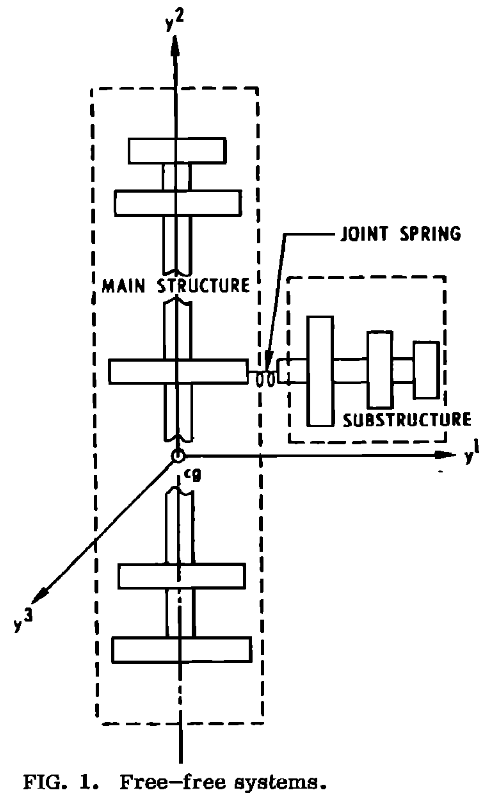

FIG. 1. Free–free systems.

motion of the system; and $q_{01}, \ldots, q_{0f_0}$ are the generalized coordinates of the main structure. Also,

$$\phi^j_{0T_i} = \delta_{ij}, \quad \text{Kronecker delta,} \tag{2a}$$

$$\phi^j_{0R_j} = 0 \tag{2b}$$

$$\left.\begin{aligned} \phi^j_{0R_{j+1}}(k) &= y_0^{j+2}(k) \\ \phi^j_{0R_{j+2}}(k) &= -y_0^{j+1}(k) \end{aligned}\right\} \text{for small rotational motion,} \tag{2c, 2d}$$

where the j's are in cyclic order; i.e., if $j=3$, then $j+1=1$ and $j+2=2$.

Similarly, the bending slope, θ_0^j, is expressed as

$$\theta_0^j(k,t) = \sum_{n=T_1,T_2,T_3,R_1,R_2,R_3,1}^{f_0} \psi^j_{0n}(k)\, q_{0n}(t), \tag{3}$$

where

$$\psi^j_{0T_1} = \psi^j_{0T_2} = \psi^j_{0T_3} = 0 \tag{4a}$$

$$\psi^j_{0R_i} = \delta_{ij}, \quad \text{Kronecker delta} \tag{4b}$$

B. Substructures

The displacements of the ith substructure are expressed as follows by using Eqs. (1) and (2):

$$\begin{aligned} Y_i^j(k,t) = \sum_{n=T_1,\ldots,R_3,1}^{f_0} &\{\phi^j_{0n}(l_i) + [y_i^{j+2}(k) - y_0^{j+2}(l_i)]\psi^{j+1}_{0n}(l_i) \\ &- [y_i^{j+1}(k) - y_0^{j+1}(l_i)]\psi^{j+2}_{0n}(l_i)\} q_{0n}(t) \\ &+ \sum_{n=T_1,\ldots,R_3,1}^{f_i} \phi^j_{in}(k) q_{in}(t), \end{aligned} \tag{5}$$

where $j=1,2,3$; $k=1,2,\ldots,n_i$; and l_i is the mass point of the main structure where the ith substructure is connected. Similarly, the bending slope θ_i^j of the substructure is expressed as follows by using Eq. (3):

$$\theta_i^j(k,t) = \sum_{n=T_1,\ldots,R_3,1}^{f_0} \psi^j_{0n}(l_i) q_{0n}(t) + \sum_{n=T_1,\ldots,R_3,1}^{f_i} \psi^j_{in}(k) q_{in}(t), \tag{6}$$

where

$$\phi^j_{iT_k} = \delta_{jk}, \quad \text{Kronecker delta,} \tag{7a}$$

$$\phi^j_{iR_j} = 0 \tag{7b}$$

$$\phi^j_{iR_{j+1}}(k) = [y_i^{j+2}(k) - y_0^{j+2}(l_i)] \tag{7c}$$

$$\phi^j_{iR_{j+2}}(k) = -[y_i^{j+1}(k) - y_i^{j+1}(l_i)] \tag{7d}$$

$$\psi^j_{iT_1} = \psi^j_{iT_2} = \psi^j_{iT_3} = 0 \tag{7e}$$

$$\psi^j_{iR_k} = \delta_{jk}, \quad \text{Kronecker delta} \tag{7f}$$

II. KINETIC ENERGY

In terms of the generalized coordinates, the kinetic energy of the system can be written as follows:

$$T = \sum_{i=0} \sum_{k=1}^{n_i} \sum_{j=1}^{3} \{\tfrac{1}{2} m_i(k)[\dot{Y}_i^j(k)]^2 + \tfrac{1}{2} I_i^{jj}[\dot{\theta}_i^j(k)]^2 - I_i^{j,j+1}(k)\, \dot{\theta}^j(k)\, \dot{\theta}^{j+1}(k)\}. \tag{8}$$

The kinetic energy is expressed in terms of generalized coordinates by substituting Eqs. (1)–(7) into Eq. (8). The expression is complex and can be simplified by using the following conditions:

(a) *Conservation of linear and angular momentum for the normal free-free modes of the main structure*: That is, preservation of translational and rotational equilibrium, as shown in the following equations:

$$\sum_{k=1}^{n_0} m_0(k)\phi^j_{0n}(k) = 0, \quad n = 1, 2, \ldots, f_0; j = 1, 2, 3, \tag{9}$$

$$\sum_{k=1}^{n_0} \{I_0^{jj}(k)\psi^j_{0n}(k) - I_0^{j,j+1}\psi^{j+1}_{0n}(k) - I_0^{j,j+2}\psi^{j+2}_{0n}(k) + m_0(k)[y_0^{j+1}(k)\phi^{j+2}_{0n}(k) - y_0^{j+2}(k)\,\phi^{j+1}_{0n}(k)]\} = 0\,. \tag{10}$$

(b) *Orthogonality condition of the normal modes*. To further simplify the remaining terms, the following notation is introduced:

m_0 = mass of the main structure,

$$= \sum_{k=1}^{f_0} m_0(k)\,,$$

B_0^j = distance from the center of gravity of the system to the center of gravity of the main structure in the jth direction,

$$M_0 B_0^j = \sum_{k=1}^{n_0} m_0(k)\, y^j(k)\,,$$

J_0^{jj} = mass moment of inertia of the main structure about the center of gravity of the system,

$$= \sum_{k=1}^{n_0} [I_0^{jj}(k) + m_0(k)\{[y_0^{j+1}(k)]^2 + [y_0^{j+2}(k)]^2\}]$$

$$J_0^{j,j+1} = \sum_{k=1}^{n_0} [I_0^{j,j+1}(k) + m_0(k) y_0^j(k)\, y_0^{j+1}(k)]\,,$$

m_i = mass of the ith structure,

$$= \sum_{k=1}^{n_i} m_i(k)\,,$$

B_i^j = distance of the center of gravity of the ith structure from its attachment point to the main structure in the jth direction,

$$m_i B_i^j = \sum_{k=1}^{n_i} m_i(k)[y_i^j(k) - y_0^j(l_i)]\,,$$

$$J_i^{jj} = \sum_{k=1}^{n_i} [I_i^{jj}(k) + m_i(k)\{[y_i^{j+1}(k) - y_0^{j+1}(l_i)]^2 + [y_i^{j+2}(k) - y_0^{j+2}(l_i)]^2\}]\,,$$

$$J_i^{j,j+1} = \sum_{k=1}^{n_i} \{I_i^{j,j+1}(k) + m_i(k)[y_i^j(k) - y_i^j(l_i)] \times [y_i^{j+1}(k) - y_0^{j+1}(l_i)]\}\,,$$

Q^j_{in} = modal shear at the attachment point of the ith structure in the jth direction due to its nth mode,

$$-(Q^j_{in}/\omega^2_{in}) = \sum_{k=1}^{n_i} m_i(k)\phi^j_{in}(k)\,,$$

M^j_{in} = modal moment at the attachment point of the ith structure in the jth direction due to its nth mode

$$(M_{in}^j/\omega_{in}^2)=\sum_{k=1}^{n_i}[I_i^{jj}(k)\psi_{in}^j(k)-I_i^{j,j+1}\psi_{in}^{j+1}(k)-I_i^{j,j+2}\psi_{in}^{j+2}(k)$$

$$+m_i(k)\{[y_i^{j+1}(k)-y_0^{j+1}(l_i)]\phi_{in}^{j+2}(k)-[y_i^{j+2}(k)-y_0^{j+2}(l_i)]\phi_{in}^{j+1}(k)\}]$$

The simplified kinetic energy can now be expressed in terms of total masses, mass moments of inertia, natural frequencies, modal masses, and modal forces (modal shear and moment at the base) of the main structure and the substructures; i.e.,

$$T=\sum_{j=1}^{3}\tfrac{1}{2}[m_0\dot{q}_{0T_j}^2+2m_0B_0^{j+2}\dot{q}_{0T_j}\dot{q}_{0R_{j+1}}-2m_0B_0^{j+1}\dot{q}_{0T_j}\dot{q}_{0R_{j+2}}+J_0^{jj}(\dot{q}_{0R_j})^2-2J_0^{j,j+1}\dot{q}_{0R_j}\dot{q}_{0R_{j+1}}]$$

$$+\frac{1}{2}\sum_{n=1}^{f_0}\mu_{0n}\dot{q}_{0n}^2+\sum_{i=1}\sum_{j=1}^{3}\Big[\frac{1}{2}\sum_{n=T_1,\ldots,R_3,1}^{f_0}\ \sum_{p=T_1,\ldots,R_3,1}^{f_0}\{m_i\phi_{0n}^j(l_i)\phi_{0p}^j(l_i)+2m_iB_i^{j+2}\phi_{0n}^j(l_i)\,\psi_{0p}^{j+1}(l_i)$$

$$-2m_iB_i^{j+1}\psi_{0n}^{j+2}(l_i)\phi_{0p}^j(l_i)+J_i^{jj}\psi_{0n}^j(l_i)\psi_{0p}^j(l_i)-2J_i^{j,j+1}\psi_{0n}^j(l_i)\psi_{0p}^{j+1}(l_i)\}\dot{q}_{0n}\dot{q}_{0p}$$

$$+\sum_{n=1}^{f_i}\ \sum_{p=T_1,\ldots,R_3,1}^{f_0}\Big\{-\Big(\frac{Q_{in}^j}{\omega_{in}^2}\Big)\phi_{0p}^j(l_i)+(M_{in}^j/\omega_{in}^2)\psi_{0p}^j(l_i)\Big\}\dot{q}_{in}\dot{q}_{0p}+\tfrac{1}{2}\{m_i(\dot{q}_{iT_j})^2+2m_iB_i^{j+2}\dot{q}_{iT_j}\dot{q}_{iR_{j+1}}-2m_iB_i^{j+1}\dot{q}_{iT_j}\dot{q}_{iR_{j+2}}$$

$$+J_i^{jj}(\dot{q}_{iR_j})^2-2J_i^{j,j+1}\dot{q}_{iR_j}\dot{q}_{iR_{j+1}}\Big\}+\dot{q}_{iT_j}\Big\{\sum_{n=T_1,\ldots,R_3,1}^{f_0}[m_i\phi_{0n}^j(l_i)+m_iB_i^{j+2}\psi_{0n}^{j+1}(l_i)-m_iB_i^{j+1}\psi_{0n}^{j+2}(l_i)]\,\dot{q}_{0n}(t)\sum_{n=1}^{f_i}-\Big(\frac{Q_{in}^j}{\omega_{in}^2}\Big)\dot{q}_{in}(t)\Big\}$$

$$+\dot{q}_{iR_j}\Big\{\sum_{n=T_1,\ldots,R_3,1}^{f_0}[J_i^{jj}\psi_{0n}^j(l_i)+m_iB_i^{j+1}\phi_{0n}^{j+2}(l_i)-m_iB_i^{j+2}\phi_{0n}^{j+1}(l_i)-J_i^{j,j+1}\psi_{0n}^{j+1}(l_i)-J_i^{j+2,j}\psi_{0n}^{j+2}(l_i)]\dot{q}_{0n}+\sum_{n=1}^{f_i}\Big(\frac{M_{in}^j}{\omega_{in}^2}\Big)\dot{q}_{in}(t)\Big\}\Big]$$

$$+\frac{1}{2}\sum_{i=1}\sum_{n=1}^{f_i}\mu_{in}\dot{q}_{in}^2\ . \tag{11}$$

III. POTENTIAL ENERGY

If the orthogonality conditions of the normal mode are used, the potential energy can be written as follows:

$$U=\sum_{i=0}\sum_{n=1}^{f_i}\tfrac{1}{2}\mu_{in}\omega_{in}^2q_{in}^2+\sum_{i=1}\ \sum_{n=T_1,\ldots,R_3}\tfrac{1}{2}K_{in}q_{in}^2\ , \tag{12}$$

where $K_{iT_1}, \ldots, K_{iR_3}$ are the translational and rotational spring constants of the joint spring in the ith substructure.

IV. EQUATIONS OF MOTION

The kinetic energy and potential energy equations [Eqs. (11) and (12)] have been evaluated in terms of the generalized coordinates. It is now possible to introduce Lagrange's equation,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i}+\frac{\partial U}{\partial q_i}=Q_i\ . \tag{13}$$

In matrix form, the equations of motion appear as follows:

$$[M]\{\ddot{q}\}+[K]\{q\}=\{Q\}\ , \tag{14}$$

where

$[M]$ = inertia matrix describing the coupling of various masses of the system,

$\{q\}$ = column matrix of the generalized coodinates,

$[K]$ = stiffness matrix defining the coupling effect of various stiffnesses of the system,

$\{Q\}$ = column matrix of the generalized forces due to external forces at the mass point.

To make a clear presentation, each matrix is decomposed into several elementary matrices, and the stiffness and inertia matrices are presented as the sums of these matrices:

$$[M]=[M_0]=\sum_{i=1}([M_i]+[M_{is}]) \tag{15}$$

and

$$[K]=[K_0]+\sum_{i=1}([K_i]+[K_{is}])\ , \tag{16}$$

where

$[M_0]$ = inertia matrix of the main structure,

$[M_i]$ = inertia matrix of the ith substructure,

$[M_{is}]$ = inertia matrix for the joint spring of the ith substructure,

$[K_0]$ = stiffness matrix of the main structure,

$[K_i]$ = stiffness matrix of the ith substructure,

$[K_{is}]$ = stiffness matrix for the joint spring of the ith substructure.

These square inertia and stiffness matrices are expanded as shown in the following subsections. Their elements correspond to the generalized coordinates at their right-hand side.

A. Main structure matrices

The main structure matrices are expanded as follows:

$$[K_0]\{q_0\}=\begin{bmatrix}\mu_{01}\omega_{01}^2 & & & & & & \\ & \mu_{02}\omega_{02}^2 & & & & & \\ & & \mu_{03}\omega_{03}^2 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \mu_{0f_0}\omega_{0f_0}^2\end{bmatrix}\begin{Bmatrix}q_{01}\\ q_{02}\\ q_{03}\\ \cdot\\ \cdot\\ \cdot\\ q_{0f_0}\end{Bmatrix}, \tag{17}$$

$$[M_0]\{q_0\}=\begin{bmatrix}m_0 & 0 & 0 & 0 & m_0B_0^3 & -m_0B_0^2 & & & & \\ & m_0 & 0 & -m_0B_0^3 & 0 & m_0B_0^1 & & & & \\ & & m_0 & m_0B_0^2 & -m_0B_0^1 & 0 & & & & \\ & & & J_0^{11} & -J_0^{12} & -J_0^{13} & & & & \\ & & & & J_0^{22} & -J_0^{23} & & & & \\ & & & & & J_0^{33} & & & & \\ & & & & & & \mu_{01} & & & \\ & & & \text{symmetric} & & & & \cdot & & \\ & & & & & & & & \cdot & \\ & & & & & & & & & \cdot \\ & & & & & & & & & & \mu_{0f_0}\end{bmatrix}\begin{Bmatrix}q_{0T_1}\\ q_{0T_2}\\ q_{0T_3}\\ q_{0R_1}\\ q_{0R_2}\\ q_{0R_3}\\ q_{01}\\ \cdot\\ \cdot\\ \cdot\\ q_{0f_0}\end{Bmatrix}, \tag{18}$$

B. Substructure matrices

The substructure matrices are expanded to yield

$$[K_i]\{q_i\}=\begin{bmatrix}\mu_{i1}\omega_{i1}^2 & & & & \\ & \mu_{i2}\omega_{i2}^2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot & \\ & & & & & \mu_{if_i}\omega_{if_i}^2\end{bmatrix}\begin{Bmatrix}q_{i1}\\ q_{i2}\\ \cdot\\ \cdot\\ \cdot\\ q_{if_i}\end{Bmatrix}, \tag{19}$$

$$[M_i]\{q_i\}=\begin{bmatrix}\mu_{i1} & & & & \\ & \mu_{i2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot & \\ & & & & & \mu_{if_i}\end{bmatrix}\begin{Bmatrix}q_{i1}\\ q_{i2}\\ \cdot\\ \cdot\\ \cdot\\ q_{if_i}\end{Bmatrix}$$

$$
+\sum_{j=1}^{3}\left[+m_i\begin{Bmatrix}\delta_{1j}\\ \delta_{2j}\\ \delta_{3j}\\ \phi^{j}_{0R_1}(l_i)\\ \phi^{j}_{0R_2}(l_i)\\ \phi^{j}_{0R_3}(l_i)\\ \phi^{j}_{01}(l_i)\\ \vdots\\ \phi^{j}_{0f_0}(l_i)\end{Bmatrix}\begin{Bmatrix}\delta_{1j}\\ \delta_{2j}\\ \delta_{3j}\\ \phi^{j}_{0R_1}(l_i)\\ \phi^{j}_{0R_2}(l_i)\\ \phi^{j}_{0R_3}(l_i)\\ \phi^{j}_{01}(l_i)\\ \vdots\\ \phi^{j}_{0f_0}(l_i)\end{Bmatrix}^T+m_iB_i^{j+2}\begin{Bmatrix}0\\ 0\\ 0\\ \delta_{1,j+1}\\ \delta_{2,j+1}\\ \delta_{3,j+1}\\ \psi^{j+1}_{01}(l_i)\\ \vdots\\ \psi^{j+1}_{0f_0}(l_i)\end{Bmatrix}\begin{Bmatrix}\delta_{1j}\\ \delta_{2j}\\ \delta_{3j}\\ \phi^{j}_{0R_1}(l_i)\\ \phi^{j}_{0R_2}(l_i)\\ \phi^{j}_{0R_3}(l_i)\\ \phi^{j}_{01}(l_i)\\ \vdots\\ \phi^{j}_{0f_0}(l_i)\end{Bmatrix}^T+m_iB_i^{j+2}\begin{Bmatrix}\delta_{1j}\\ \delta_{2j}\\ \delta_{3j}\\ \phi^{j}_{0R_1}(l_i)\\ \phi^{j}_{0R_2}(l_i)\\ \phi^{j}_{0R_3}(l_i)\\ \phi^{j}_{01}(l_i)\\ \vdots\\ \phi^{j}_{0f_0}(l_i)\end{Bmatrix}\begin{Bmatrix}0\\ 0\\ 0\\ \delta_{1,j+1}\\ \delta_{2,j+1}\\ \delta_{3,j+1}\\ \psi^{j+1}_{01}(l_i)\\ \vdots\\ \psi^{j+1}_{0f_0}(l_i)\end{Bmatrix}^T\right.
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$$
-m_iB_i^{j+1}\begin{Bmatrix}0\\ 0\\ 0\\ \delta_{1,j+2}\\ \delta_{2,j+2}\\ \delta_{3,j+2}\\ \psi^{j+2}_{01}(l_i)\\ \vdots\\ \psi^{j+2}_{0f_0}(l_i)\end{Bmatrix}\begin{Bmatrix}\delta_{1j}\\ \delta_{2j}\\ \delta_{3j}\\ \phi^{j}_{0R_1}(l_i)\\ \phi^{j}_{0R_2}(l_i)\\ \phi^{j}_{0R_3}(l_i)\\ \phi^{j}_{01}(l_i)\\ \vdots\\ \phi^{j}_{0f_0}(l_i)\end{Bmatrix}^T-m_iB_i^{j+1}\begin{Bmatrix}\delta_{1j}\\ \delta_{2j}\\ \delta_{3j}\\ \phi^{j}_{0R_1}(l_i)\\ \phi^{j}_{0R_2}(l_i)\\ \phi^{j}_{0R_3}(l_i)\\ \phi^{j}_{01}(l_i)\\ \vdots\\ \phi^{j}_{0f_0}(l_i)\end{Bmatrix}\begin{Bmatrix}0\\ 0\\ 0\\ \delta_{1,j+2}\\ \delta_{2,j+2}\\ \delta_{3,j+2}\\ \psi^{j+2}_{01}(l_i)\\ \vdots\\ \psi^{j+2}_{0f_0}(l_i)\end{Bmatrix}^T+J_i^{jj}\begin{Bmatrix}0\\ 0\\ 0\\ \delta_{1j}\\ \delta_{2j}\\ \delta_{3j}\\ \psi^{j}_{01}(l_i)\\ \vdots\\ \psi^{j}_{0f_0}(l_i)\end{Bmatrix}\begin{Bmatrix}0\\ 0\\ 0\\ \delta_{1j}\\ \delta_{2j}\\ \delta_{3j}\\ \psi^{j}_{01}(l_i)\\ \vdots\\ \psi^{j}_{0f_0}(l_i)\end{Bmatrix}^T
$$

$$
\left.-J_i^{j,j+1}\begin{Bmatrix}0\\ 0\\ 0\\ \delta_{1j}\\ \delta_{2j}\\ \delta_{3j}\\ \psi^{j}_{01}(l_i)\\ \vdots\\ \psi^{j}_{0f_0}(l_i)\end{Bmatrix}\begin{Bmatrix}0\\ 0\\ 0\\ \delta_{1,j+1}\\ \delta_{2,j+1}\\ \delta_{3,j+1}\\ \psi^{j+1}_{01}(l_i)\\ \vdots\\ \psi^{j+1}_{0f_0}(l_i)\end{Bmatrix}^T-J_i^{j,j+1}\begin{Bmatrix}0\\ 0\\ 0\\ \delta_{1,j+1}\\ \delta_{2,j+1}\\ \delta_{3,j+1}\\ \psi^{j+1}_{01}(l_i)\\ \vdots\\ \psi^{j+1}_{0f_0}(l_i)\end{Bmatrix}\begin{Bmatrix}0\\ 0\\ 0\\ \delta_{1j}\\ \delta_{2j}\\ \delta_{3j}\\ \psi^{j}_{01}(l_i)\\ \vdots\\ \psi^{j}_{0f_0}(l_i)\end{Bmatrix}^T\right]\begin{Bmatrix}q_{0T_1}\\ q_{0T_2}\\ q_{0T_3}\\ q_{0R_1}\\ q_{0R_2}\\ q_{0R_3}\\ q_{01}\\ \vdots\\ q_{0f_0}\end{Bmatrix}
$$

$$
+\left[\left\{\begin{matrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ -(Q^j_{i1}/\omega^2_{i1}) \\ \vdots \\ -(Q^j_{if_i}/\omega^2_{if_i}) \end{matrix}\right\}\left\{\begin{matrix} \delta_{1j} \\ \vdots \\ \phi^j_{0R_3} \\ \phi^j_{01} \\ \vdots \\ \phi^j_{0f_0} \\ \vdots \\ 0 \\ \vdots \\ 0 \end{matrix}\right\}^T+\left\{\begin{matrix} \delta_{1j} \\ \vdots \\ \phi^j_{0R_3} \\ \phi^j_{01} \\ \vdots \\ \phi^j_{0f_0} \\ \vdots \\ 0 \\ \vdots \\ 0 \end{matrix}\right\}\left\{\begin{matrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ -(Q^j_{i1}/\omega^2_{i1}) \\ \vdots \\ -(Q^j_{if_i}/\omega^2_{if_i}) \end{matrix}\right\}^T\right.
$$

$$
\left.+\left\{\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ (M^j_{i1}/\omega^2_{i1}) \\ \vdots \\ (M^j_{if_i}/\omega^2_{if_i}) \end{matrix}\right\}\left\{\begin{matrix} 0 \\ 0 \\ 0 \\ \delta_{1j} \\ \delta_{2j} \\ \delta_{3j} \\ \psi^j_{01}(l_i) \\ \vdots \\ \psi^j_{0f_0}(l_i) \\ 0 \\ \vdots \\ 0 \end{matrix}\right\}^T+\left\{\begin{matrix} 0 \\ 0 \\ 0 \\ \delta_{1j} \\ \delta_{2j} \\ \delta_{3j} \\ \psi^j_{01}(l_i) \\ \vdots \\ \psi^j_{0f_0}(l_i) \\ 0 \\ \vdots \\ 0 \end{matrix}\right\}\left\{\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ (M^j_{i1}/\omega^2_{i1}) \\ \vdots \\ (M^j_{if_i}/\omega^2_{if_i}) \end{matrix}\right\}^T\right]\left\{\begin{matrix} q_{0T_1} \\ q_{0T_2} \\ q_{0T_3} \\ q_{0R_1} \\ q_{0R_2} \\ q_{0R_3} \\ q_{01} \\ \vdots \\ q_{0f_0} \\ q_{i1} \\ \vdots \\ q_{if_i} \end{matrix}\right\} . \tag{20}
$$

C. Joint spring matrices

Expansion of the joint spring matrices is performed as follows:

$$
[K_{is}]\{q_i\}=\begin{bmatrix} K_{T_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & K_{T_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{T_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{R_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{R_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{R_3} \end{bmatrix}\left\{\begin{matrix} q_{iT_1} \\ q_{iT_2} \\ q_{iT_3} \\ q_{iR_1} \\ q_{iR_2} \\ q_{iR_3} \end{matrix}\right\} . \tag{21}
$$

Define

$$L_n^j(l_i)=m_i\phi_{0n}^j(l_i)+m_iB_i^{j+2}\psi_{0n}^{j+1}(l_i)-m_iB_i^{j+1}\psi_{0n}^{j+2}(l_i) \tag{22a}$$

$$M_n^j(l_i)=J_i^{jj}\psi_{0n}^j(l_i)+m_iB_i^{j+1}\phi_{0n}^{j+2}(l_i)-m_iB_i^{j+2}\phi_{0n}^{j+1}(l_i)-J_i^{j,j+1}\psi_{0n}^{j+1}(l_i)-J_i^{j+2,j}\psi_{0n}^{j+2}(l_i)\ . \tag{22b}$$

Then

$$[M_{is}]\{q_i\}=\begin{bmatrix} & & & L_{T_1}^1 & \cdots & M_{T_1}^3 & & & \\ & & & \vdots & & \vdots & & & \\ & & & L_{f_0}^1 & \cdots & M_{f_0}^3 & & & \\ L_{T_1}^1 & \cdots & L_{f_0}^1 & m_i & \cdots & -m_iB_i^2 & -(Q_{i1}^1/\omega_{i1}^2) & \cdots & -(Q_{if_i}^1/\omega_{if_i}^2) \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ M_{T_1}^3 & \cdots & M_{f_0}^3 & -m_iB_i^2 & \cdots & J_i^{33} & (M_{i1}^3/\omega_{i1}^2) & \cdots & (M_{if_i}^3/\omega_{if_i}^2) \\ & & & -(Q_{i1}^1/\omega_{i1}^2) & \cdots & (M_{i1}^3/\omega_{i1}^2) & & & \\ & & & \vdots & & \vdots & & & \\ & & & -(Q_{if_i}^1/\omega_{if_i}^2) & \cdots & (M_{if_i}^3/\omega_{if_i}^2) & & & \end{bmatrix}\begin{Bmatrix} q_{0T_1} \\ \vdots \\ q_{0f_0} \\ q_{iT_1} \\ \vdots \\ q_{iR_3} \\ q_{i1} \\ \vdots \\ q_{if_i} \end{Bmatrix}. \tag{23}$$

If

$$\overline{\phi}_{in}^j(k)=\phi_{0n}^j(l_i)+\{y_i^{j+2}(k)-y_0^{j+2}(l_i)\}\psi_{0n}^{j+1}(l_i)-\{y_i^{j+1}(k)-y_0^{j+1}(l_i)\}\psi_{0n}^{j+2}(l_i) \tag{24}$$

then Eqs. (1)–(7), which express the displacements of the structure in terms of generalized coordinates, can be expressed in matrix form as follows:

$$\begin{Bmatrix} Y_0^1(1) \\ \vdots \\ \theta_0^3(n_0) \\ \vdots \\ Y_i^1(1) \\ \vdots \\ \theta_i^3(n_i) \end{Bmatrix}=\begin{bmatrix} \phi_{0T_1}^1(1) & \cdots & \phi_{0f_0}^1(1) & & & & \\ \vdots & & \vdots & & & & \\ \psi_{0T_1}^3(n_0) & \cdots & \psi_{0f_0}^3(n_0) & & & & \\ \vdots & & \vdots & & & & \\ \overline{\phi}_{iT_1}^1(1) & \cdots & \overline{\phi}_{1f_0}^1(1) & \cdots & \phi_{iT_1}^1(1) & \cdots & \phi_{if_i}(1) \\ \vdots & & \vdots & & \vdots & & \vdots \\ \psi_{0T_1}^3(n_i) & \cdots & \psi_{0f_0}^3(n_i) & \cdots & \psi_{iT_1}^3(n_i) & \cdots & \psi_{if_i}^3(n_i) \end{bmatrix}\begin{Bmatrix} q_{0T_1} \\ \vdots \\ q_{0f_0} \\ \vdots \\ q_{iT_1} \\ \vdots \\ q_{if_i} \end{Bmatrix} \tag{25}$$

or

$$\{Y\}=[\phi]\{q\}\ .$$

The modal force $\{Q\}$ is obtained from the external force vector $\{F\}$ by the following transformation:

$$\{Q\}=[\phi]^T\{F\}\ , \tag{26}$$

where

$$\{F\}=\begin{Bmatrix} F_0^1(1) \\ \vdots \\ M_0^3(n_0) \\ \vdots \\ F_i^1(1) \\ \vdots \\ M_i^3(n_i) \end{Bmatrix} . \tag{27}$$

V. SOLUTIONS OF EQUATIONS OF MOTION

In matrix form, the differential equations of motion appear as

$$[M]\{\ddot{q}\}+[K]\{q\}=\{Q\} , \tag{28}$$

where the inertia matrix $[M]$ and the stiffness matrix $[K]$ are square, symmetric, and of the order $f\times f$. For the cantilever system, the stiffness matrix $[K]$ will be nonsingular; hence the modal characteristics and response can be easily obtained. For the free–free system, the matrix $[K]$ will be singular. The solution for this system requires separating the generalized coordinates into rigid body coordinates, $q_{0T_1}, \ldots, q_{0R_3}$, and elastic coordinates.

Partitioning Eq. (28) gives

$$\begin{bmatrix} M_{11} & M_{12} \\ 6\times 6 & \\ M_{21} & M_{22} \end{bmatrix} \begin{Bmatrix} \ddot{q}_R \\ \ddot{q}_E \end{Bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} q_R \\ q_E \end{Bmatrix} = \begin{Bmatrix} Q_R \\ Q_E \end{Bmatrix} , \tag{29}$$

where

$$\{q_R\}=\begin{Bmatrix} q_{0T_1} \\ q_{0T_2} \\ q_{0T_3} \\ q_{0R_1} \\ q_{0R_2} \\ q_{0R_3} \end{Bmatrix} \quad \{q_E\}=\begin{Bmatrix} q_{01} \\ \vdots \\ q_{0f_0} \\ \vdots \\ q_{iT_1} \\ \vdots \\ q_{if_i} \\ \vdots \end{Bmatrix} . \tag{30}$$

For the free–free system,

$$[K_{11}]=[K_{12}]=[K_{21}]=0 . \tag{31}$$

If Eq. (29) is rewritten as

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{Bmatrix} \ddot{q}_R \\ \ddot{q}_E \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & K_{22} \end{bmatrix} \begin{Bmatrix} q_R \\ q_E \end{Bmatrix} = \begin{Bmatrix} Q_R \\ Q_E \end{Bmatrix} , \tag{32}$$

the partitioned matrix is expanded as

$$[M_{11}]\{\ddot{q}_R\}+[M_{12}]\{\ddot{q}_E\}=\{Q_R\} , \tag{33a}$$

$$[M_{21}]\{\ddot{q}_R\}+[M_{22}]\{\ddot{q}_E\}+[K_{22}]\{q_E\}=\{Q_E\} , \tag{33b}$$

the first equation is multiplied by $-[M_{21}][M_{11}]^{-1}$, and the two equations are added,

$$([M_{22}]-[M_{21}][M_{11}]^{-1}[M_{12}])\{\ddot{q}_E\}+[K_{22}]\{q_E\} =\{Q_E\}-[M_{21}][M_{11}]^{-1}\{Q_R\} . \tag{34}$$

If it is assumed that

$$[\bar{M}]=[M_{22}]-[M_{21}][M_{11}]^{-1}[M_{12}] , \tag{35a}$$

$$[\bar{K}]=[K_{22}] , \tag{35b}$$

$$\{\bar{Q}_E\}=\{Q_E\}-[M_{21}][M_{11}]^{-1}\{Q_R\} , \tag{35c}$$

rewriting Eq. (34) results in

$$[\bar{M}]\{\ddot{q}_E\}+[\bar{K}]\{q_E\}=\{\bar{Q}_E\} . \tag{36}$$

Now the natural modes, natural frequencies, and response can be obtained by using the standard approach. The rigid motion can be obtained by solving Eq. (33) for $\{q_R\}$. The motion of the structure can be obtained by using Eq. (25).

VI. COMPUTER PROGRAM

Based on the method discussed earlier, a computer program called MODSYN has been developed for both free–free and cantilever systems. The following example demonstrates the accuracy of this method. The structure is divided into two components as shown in Fig. 2.

For simplicity it is assumed that

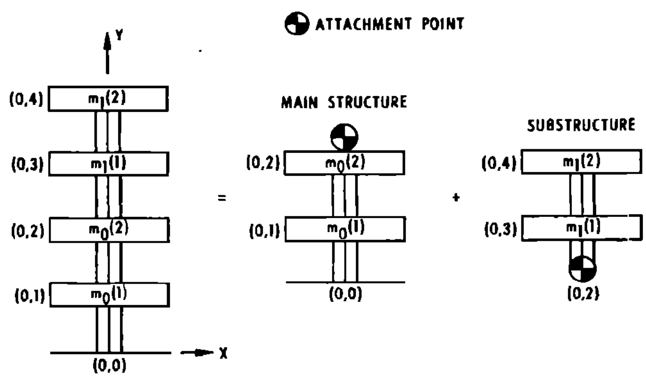

FIG. 2. Components of the structure.

TABLE I. Natural frequencies of the complete structure and the main structure obtained from the NASTRAN program.

Mode number	Frequency (Hz) Structure	Main structure and substructure
1	79.45	217.18
2	341.77	701.56
3	683.12	1715.53
4	919.86	2945.52
5	1627.96	
6	2238.60	
7	2868.70	
8	3323.54	

$$m_0(1) = m_0(2) = m_1(1) = m_1(2) = 1 \text{ slug } (14.6 \text{ kg}) \, ,$$

$$I_0^{\epsilon\epsilon}(1) = I_0^{\epsilon\epsilon}(2) = I_1^{\epsilon\epsilon}(1) = I_1^{\epsilon\epsilon}(2) = 1 \text{ slug ft}^2 \; (1.356 \text{ kg m}^2) \; .$$

The structure consists of concentrated masses which are connected by beams. The areas of the cross sections and bending stiffnesses (EI) of the beams are assumed to be the same and equal to 1 ft^2 (0.09 m^2) and 10^7 lb ft^2 (4.12 N m^2), respectively. The main structure and substructure are assumed to be the same. The cantilever modal characteristics of the complete structure and the main structure (or substructure) are obtained by using NASTRAN (see Table I). The natural frequencies of the complete structure have been obtained with the MODSYN program by using the four modes (total), the first two modes, and the fundamental mode for the main structure and for the substructure (see Table II). The percentage errors are obtained by comparing the exact natural frequencies of the structure from NASTRAN and the natural frequencies from MODSYN. For the case in which all four modes of the substructure are considered, the results are exact, as expected.

VII. CONCLUSIONS

The mode synthesis technique presented in this paper reduces significantly the technical communication across component interfaces of a complex structure for determination of natural frequencies. The accuracy of the results obtained with this method is good. This technique is especially preferred for in-orbit flexible dynamic analysis of a spacecraft.

TABLE II. Natural frequencies of the complete structure obtained from the MODSYN program.

	4 Modes	2 Modes		1 Mode	
Mode number	Frequency (Hz)	Frequency (Hz)	Percentage error	Frequency (Hz)	Percentage error
1	79.45	79.46	0.0125	80.8	1.69
2	341.77	343.80	0.5900	389.5	13.96
3	683.13	684.53	0.2000		
4	919.85	937.80	1.9500		
5	1627.90				
6	2238.60				
7	2868.17				
8	3323.50				

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