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Stability of Spinning Spacecraft with Partially Liquid-Filled Tanks

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This paper presents general stability conditions for a spinning spacecraft with partially liquid-filled tanks by using Rumyanstev and McIntyre methods. These methods are compared for accuracy and limitation by applying them to a specific case of a spinning spacecraft with two partially liquid-filled tanks. The stability conditions require that, for a stable motion, the spin to transverse moment of inertia must be greater than $1+C$, where C is a positive definite function of the spacecraft parameters. Numerical spacecraft parameters are also used to determine minimum inertia ratios, $1+C$.

Nomenclature

E = total energy
H = angular momentum
h = distance from tank center to the surface of the fluid
I_{ij} = elements of inertia matrix of the spacecraft
I^p_{ij} = inertia elements of propellant about its own c.m.
I_s = moment of inertia about the spin axis of the spacecraft
L = distance of the tank center plane from system c.m.
M = mass of the spacecraft
m = mass of propellant per tank
q = generalized coordinates
R = tank radius
r_1 = distance from spin axis to tank center
r_2 = distance from spin axis to fluid surface
S = surface area of the fluid
U = potential energy
y_0 = distance from tank center to c.m. of propellant tank
ρ = density of the fluid
$\Psi_1 \Psi_2$ = spin axis tilt angles about the transverse (1 and 2) axes, respectively
ω = angular velocity about the spin axis

Introduction

THE stability and dynamics of spinning spacecraft have been the subject of numerous papers.[1-3] The motion of a spinning spacecraft with liquid propellant is described by very complex equations consisting of nonlinear ordinary differential equations for the rigid spacecraft body and partial differential equations for the liquid in the tanks supplemented by appropriate initial and boundary conditions. In order to solve these equations, several simplifying assumptions are made. A significant simplification is possible if the only question is stability. If a steady-state solution exists, it can only be a rotation of the complete spacecraft, rigid body, and propellant, like a rigid body. Otherwise, the relative motion between the liquids and the walls of their containers would lead to energy dissipation and thus to change in the motion.

For a perfectly rigid body, stable spin motion can occur only about the axis of maximum or minimum moment of inertia. For a body with flexible elements, the only stable spin axis is the axis of maximum moment of inertia. This axis provides a minimum energy state for a given angular momentum. The stability condition can be stated as

$$(I_s/I_t) > 1$$

where I_s is the moment of inertia about the spin axis and I_t is the moment of inertia about the transverse axis.

In the above stability condition, the impact of liquid motion on the inertia properties is neglected. This assumption, however, will not be valid for a spacecraft with liquid perigee and/or apogee motor where a significant portion of the spacecraft mass may be liquid. By taking into account the change in the moment of inertia of the spacecraft due to propellant relative motion, the stability condition becomes

$$(I_s/I_t) > (1+C)$$

where C is positive definite and is a function of spacecraft parameters. It is also found that the spacecraft dry imbalance is amplified by propellant motion. This effect results in amplification of wobble and degradation of pointing performance. Hence the propellant motion is important not only for stability considerations but also for wobble amplification.

In this paper, the techniques for determining stable conditions and wobble amplification factors for a spacecraft with liquid propellant are analyzed. As an example, a spacecraft with two propellant tanks is considered. Numerical examples are also discussed.

Stability Conditions

Formulation of Stability Conditions

The stability conditions for a flexible spinning spacecraft have been formulated by several investigators. The basic approach is the same. Total energy is used as Liapunov function and the spacecraft is assumed force-free, resulting in constant angular momentum. In this paper, the stability condition formulations by Rumyanstev[4,5] and McIntyre and Miyagi[6] are discussed.

Rumyanstev Stability Conditions

Rumyanstev has performed an extensive stability analysis for rigid bodies containing fluid. Rumyanstev's method is based on the system total energy E, defined in the steady-state motion as follows:

$$E = ½(H^2/I_s) + U \tag{1}$$

where the first term is the kinetic energy and the second term is the potential energy. The potential energy, U, is defined as

$$U = -U_1 - \rho\int_\tau U_2 \mathrm{d}\tau + U_2^* \tag{2}$$

where U_1 corresponds to the effective forces applied to the rigid body, U_2 to the body forces acting on the fluid, and U_2^* to the surface tension forces.

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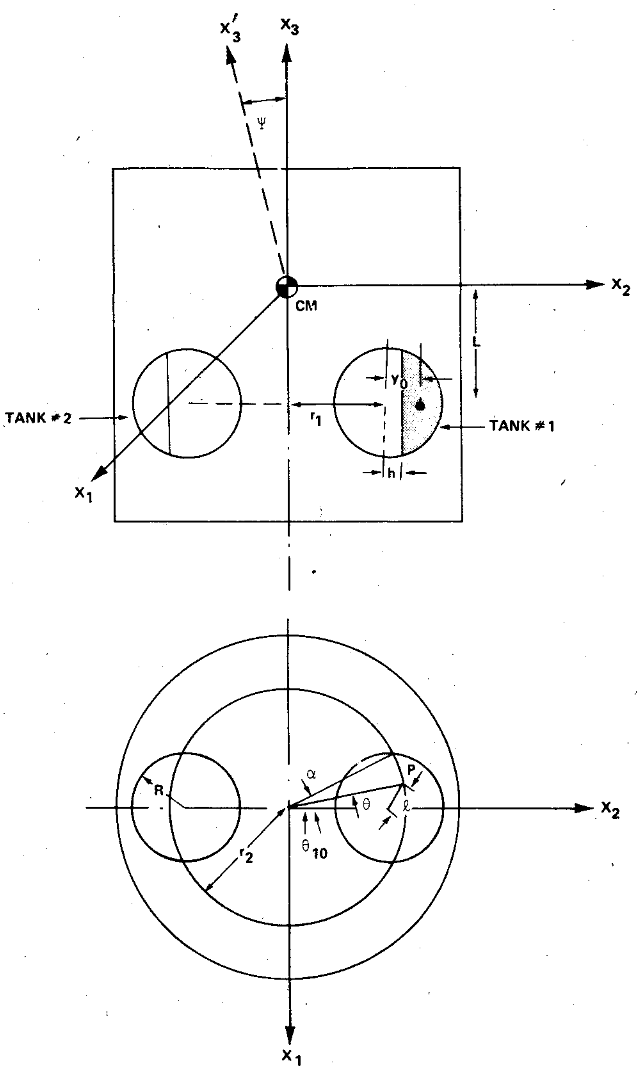

Fig. 1 Nominal state of two-tank propellant system.

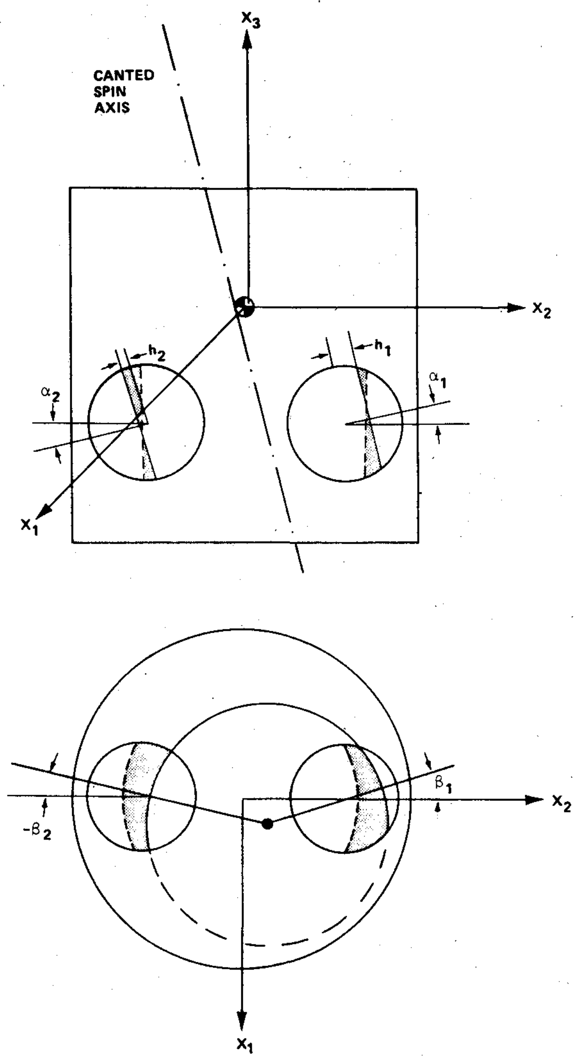

Fig. 2 Canted spin axis state of two-tank propellant system.

The Rumyanstev stability condition for the steady motion of a rigid body with a fluid filled cavity requires that E has an isolated minimum E_0. The Rumyanstev condition also implies that, in the absence of external forces, the system will have minimum energy when in a stable condition.

Consider a rigid body with fluid in the propellant tanks as shown in Fig. 1. The coordinate system (0, $X_1 X_2 X_3$) is fixed in the body with the origin at the center of mass (c.m.) of the whole body and the coordinate axes along the principal inertia axes of the body. During the steady-state motion, the body is spinning about X_3. The fluid surface under steady-state motion is shown in Fig. 1. Figure 2 shows a perturbed motion where the body is spinning about the X_3' axis. The perturbed motion from the steady-state motion is described in terms of the generalized coordinates q_j.

Let us consider the change in E due to the perturbation from the steady-state motion, $q_j = 0$. The perturbation can be considered in two parts: displacement into the perturbed position of the entire system as a single rigid body; and deformation of the fluid configuration with respect to the rigid body. In Fig. 2, the fluid deformation is shown by hatching and is denoted by τ_1.

The changes in E, I_s, and U can be written as follows:

$$\Delta E = \Delta_1 E + \Delta_2 E \tag{3a}$$

$$\Delta I_s = \Delta_1 I_s + \Delta_2 I_s \tag{3b}$$

$$\Delta U = \Delta_1 U + \Delta_2 U \tag{3c}$$

where Δ_1 is the change due to the perturbed motion of the entire system as a single rigid body, and Δ_2 is the change due to fluid deformation.

From Eq. (2), considering only U_2, we get

$$\Delta_2 U = -\rho \int_{\tau_1} U_2 \mathrm{d}\tau \tag{4}$$

From Eq. (1),

$$\begin{aligned} \Delta E &= \tfrac{1}{2} H^2 \{ [1/(I_{s0} + \Delta I_s)] - (1/I_s) \} + \Delta U \\ &\simeq \tfrac{1}{2}\omega^2 [-\Delta_1 I_s - \Delta_2 I_s + (\Delta_1 I_s^2 / I_{s0}) \\ &\quad + (\Delta_2 I_s^2 / I_{s0}) + (2\Delta_1 I_s \Delta_z / I_{s0})] + \Delta U \end{aligned} \tag{5}$$

From Eq. (5),

$$\begin{aligned} \Delta_2 E &= -\tfrac{1}{2}\omega^2 \Delta_2 I_s - \rho \int_{\tau_1} U_2 \mathrm{d}\tau + (\omega^2 / 2 I_{s0}) \\ &\quad \times [(\Delta_2 I_s)^2 + 2\Delta_1 I_s \cdot \Delta_2 I_s] \ldots \\ &= -\rho \int_{\tau_1} [\tfrac{1}{2}\omega^2 (X_1'^2 + X_2'^2) \\ &\quad + U_2 (X_1', X_2', X_3')] \mathrm{d}\tau + \ldots \end{aligned} \tag{6}$$

and

$$\Delta_1 E = \tfrac{1}{2} \sum_{i,j=1} \left(\frac{\partial^2 E}{\partial q_j \partial q_i} \right)_0 q_i q_j \tag{7}$$

The subscript 0 means that the quantity is calculated for the unperturbed position of the system.

The following section concerns the determination of the integral in Eq. (6), which is contributed by the fluid deformation with respect to the rigid body. Let the integrand of Eq. (6) be defined in terms of X_1, X_2, X_3 as

$$\phi(X_1, X_2, X_3, q) = \tfrac{1}{2}[\omega^2(X_1'^2 + X_2'^2)] + U_2(X_1', X_2', X_3') \tag{8}$$

where $X_i' \rightarrow X_i, q_j$.

For steady-state motion, the fluid surface has the form

$$\phi(X_1, X_2, X_3, 0) = \tfrac{1}{2}[\omega^2(X_1^2 + X_2^2)] + U_2(X_1, X_2, X_3) = C_0 \tag{9}$$

Under pertubed motion, the free surface is given by

$$f = (H^2/2I_s^2)(X_1'^2 + X_2'^2) + U_2(X_1', X_2', X_3') = C \tag{10}$$

The only difference between Eqs. (9) and (10) is that I_{s0} is used in Eq. (9) and I_s in Eq. (10). By substituting X_i' in terms of X_i and q_j into Eq. (10), one obtains

$$\phi_1(X_1, X_2, X_3, q_j) = C = C_0 + \Delta C \tag{11}$$

The difference between the functions ϕ_1 and ϕ is determined as follows:

$$\begin{aligned}\phi_1 &= \tfrac{1}{2}[(H^2/I_s^2)(X_1'^2 + X_2'^2)] + U_2(X_1', X_2', X_3') \\ &= \tfrac{1}{2}[(H^2/I_{s0}^2)(X_1'^2 + X_2'^2)][1 - 2(\Delta I_s/I_{s0}) + \ldots] \\ &\quad + U_2(X_1', X_2', X_3') \\ &= \tfrac{1}{2}\omega^2(X_1'^2 + X_2'^2) \\ &\quad + U_2(X_1', X_2', X_3') - (\omega^2/I_{s0})(X_1'^2 + X_2'^2)\Delta I_s \\ &= \phi - (\omega^2/I_{s0})(X_1'^2 + X_2'^2)\Delta I_s - \ldots\end{aligned} \tag{12}$$

Since the volume of the fluid bounded by the free surfaces, Eqs. (9) and (10), will have the same volume, the volume of the fluid undergoing deformation must be zero, i.e.,

$$\int_{\tau_1} d\tau = 0 \tag{13}$$

In first approximation,

$$\iint_Q dX_1 dX_2 \int_{X_{30}}^{X_{31}} dX_3 = 0 \tag{14}$$

where Q denotes the region of the plane (X_1, X_2) bounded by the projections on this plane of the closed curve s, and s is the locus of the points of intersection of the fluid-free surface under steady-state motion with the walls of the cavity. X_{30} and X_{31} denote, respectively, the values of the variable X_3 for the points on the surface Eqs. (9) and (11). For the integration of Eq. (14), it is convenient to replace X_3 with the following new variable:

$$\mu = \phi(X_1, X_2, X_3, q) - C_0 \tag{15}$$

For X_{30},

$$\begin{aligned}\mu_0 &= \phi(X_1, X_2, X_{30}, q) - C_0 \\ &= \phi(X_1, X_2, X_{30}, 0) + \sum_{j=1}^{n-1} \left(\frac{\partial \phi}{\partial q}\right)_0 q_j + \ldots - C_0 \\ &= \sum_{j=1} \left(\frac{\partial \phi}{\partial q_j}\right)_0 q_j\end{aligned} \tag{16}$$

For X_{31},

$$\mu_1 = \phi(X_1, X_2, X_{31}, q) - C_0$$

Using Eq. (12) for substituting ϕ in terms of ϕ_1,

$$\begin{aligned}\mu_1 &= \phi_1 + (\omega^2/I_{s0})(X_1'^2 + X_2'^2)\Delta I_s - C_0 - \ldots \\ &= \Delta C + (\omega^2/I_{s0})(X_1^2 + X_2^2)\Delta I_s - \ldots\end{aligned} \tag{17}$$

Substituting Eq. (15) into Eq. (14) yields

$$\iint_Q dX_1 dX_2 \int_{\mu_0 + C_0}^{\mu_1 + C_0} \left(\frac{\partial X_3}{\partial \phi}\right)_0 d\phi = 0$$

or

$$\iint_Q \left(\frac{\partial X_3}{\partial \phi}\right)_0 (\mu_1 - \mu_0) dX_1 dX_2 = 0 \tag{18}$$

Similarly, in a first approximation,

$$\Delta_2 I = \rho \iint_Q \left(\frac{\partial X_3}{\partial \phi}\right)_0 (X_1^2 + X_2^2)(\mu_1 - \mu_0) dX_1 dX_2 \tag{19}$$

By substituting μ_0 and μ_1 from Eqs. (16) and (17), respectively, into Eqs. (18) and (19), $\Delta_2 I$ and ΔC can be uniquely determined as linear functions of q_j. It can be shown that if

$$\iint_Q \left(\frac{\partial X_3}{\partial \phi}\right)_0 \mu_0 dX_1 dX_2 = \iint_Q \left(\frac{\partial X_3}{\partial \phi}\right)_0 (X_1^2 + X_2^2)\mu_0 dX_1 dX_2 = 0 \tag{20}$$

and

$$\frac{\partial I_s}{\partial q_j} = 0 \qquad (j = 1, \ldots, n-1)$$

then $\Delta_1 I_s = 0$, $\Delta_2 I_s = 0$, and $\Delta C = 0$ in a first approximation.

The integrand of Eq. (6) can be written as

$$\begin{aligned}&-\rho \int_{\tau_1} [\tfrac{1}{2}\omega^2(X_1'^2 + X_2'^2) + U_2(X_1', X_2', X_3')] d\tau \\ &= -\rho \iint_Q dX_1 dX_2 \int_{\mu + C_0}^{\mu_1 + C_0} \left(\frac{\partial X_3}{\partial \phi}\right)_0 \phi d\phi \\ &= -\tfrac{1}{2}\rho \iint_Q dX_1 dX_2 \left(\frac{\partial X_3}{\partial \phi}\right)_0 (\mu_1^2 - \mu_0^2) \\ &\quad + \rho C_0 \iint_Q dX_1 dX_2 \left(\frac{\partial X_3}{\partial \phi}\right)_0 (\mu_1 - \mu_0)\end{aligned} \tag{21}$$

From Eq. (18), the last term in Eq. (21) is zero. Combining Eqs. (3b), (6), (7), and (21) yields

$$\Delta E = \frac{1}{2}\sum\left[\frac{\partial^2 E}{\partial q_i \partial q_j}\right] q_i q_j - \frac{1}{2}\rho \iint_Q \frac{\partial X_3}{\partial \phi}$$
$$\times (\mu_1^2 - \mu_0^2)\,\mathrm{d}X_1\mathrm{d}X_2 + \frac{\omega^2}{2I_{s0}}$$
$$\times [(\Delta_2 I_s)^2 + 2\Delta_1 I_s \Delta_2 I_s] - \ldots \qquad (22)$$

The system will be stable if ΔE is positive definite.

McIntyre and Miyagi Stability Conditions

For the derivation of stability criteria, McIntyre and Miyagi have used the concept of change in spacecraft balance due to the deformation of flexible elements. The general stability principal for a spinning body is stated as follows: the spinning motion of a flexible body is stable if all small displacements of the flexible elements tilt the spin axis so that the combined elastic loads and the tilted centrifugal loads tend to decrease the displacement.

The stability conditions are derived from the total energy E, as defined in Eq. (1). In the perturbed position, the angular momentum H is constant. The inertia matrix I and the potential energy U are expanded about their steady-state conditions and terms up to the second order are retained:

$$I(q) = I_0 + \Delta_1 I(q) + \Delta_2 I(q) + \ldots \qquad (23)$$

$$U(q) = U_0 + \Delta_1 U(q) + \Delta_2 U(q) + \ldots \qquad (24)$$

In the steady state,

$$I = I_0 = \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix} \qquad (25)$$

The perturbed state is defined by the generalized coordinates q_j.

The stability condition is that a 2×2 symmetric matrix, K, be positive definite, where

$$K = \begin{bmatrix} I_{33} - I_{22} - b\cdot\Gamma^{-1}b & b\cdot\Gamma^{-1}a \\ a\cdot\Gamma^{-1}b & I_{33} - I_{11} - a\cdot\Gamma^{-1}a \end{bmatrix} > 0 \qquad (26)$$

The elements of the matrix K are defined as follows:

$$\Delta_1 I_{33} = a\cdot q \qquad \Delta_1 I_{23} = b\cdot q$$
$$(2\Delta_2 U/\omega^2) - \Delta_2 I_{33} + (\Delta_1 I_{33}^2 / I_{33}) = q\cdot\Gamma q \qquad (27)$$

where q is an n-dimensional vector of generalized coordinates, a and b are n-dimensional vectors, and Γ a nonsymmetric matrix.

In the above discussion, it is assumed that the X_3 axis is a principal axis. Assume an imperfectly balanced rigid body such that the steady-state spin axis tilt satisfies

$$\Psi_{10} = -I_{23}/(I_{33} - I_{22}) \qquad \Psi_{20} = I_{13}/(I_{33} - I_{11}) \qquad (28)$$

It is shown that, for the flexible body, the tilt is given by

$$\begin{Bmatrix} \Psi_1 \\ \Psi_2 \end{Bmatrix} = K^{-1}\begin{bmatrix} (I_{33} - I_{22})\Psi_{10} \\ (I_{33} - I_{11})\Psi_{20} \end{bmatrix} \qquad (29)$$

where K is given by Eq. (26). In the rigid body case, K reduces to K_0 where

$$K_0 = \begin{bmatrix} I_{33} - I_{22} & 0 \\ 0 & I_{33} - I_{11} \end{bmatrix} > 0$$

For the flexible body case, it is shown that

$$K_0 > K \rightarrow K^{-1} > K_0^{-1}$$

Hence the flexibility amplifies the spin axis tilt over that which would exist if the body were rigid. Furthermore, the amplification increases without limit as the stability boundary, defined by Eq. (26), is approached.

Example

Consider a spacecraft with two propellant tanks as shown in Fig. 1. During the steady motion, the spacecraft spins about its maximum principal axis X_3. In the perturbed state, the spin axis is perturbed. It is assumed that there are no external forces on the spacecraft. The body forces on the fluid due to the force function U_2, such as gravity or thrust, are assumed to be absent.

Rumyanstev Method

Rumyanstev method is used in Ref. 4 to determine stability conditions for a spinning spacecraft with a partially filled circular ring. To apply the Rumyanstev method to this example, some modifications and approximations are made. It is assumed that the tanks are interconnected to allow liquid to migrate from one tank to another.

Let $\vec{i}, \vec{j}, \vec{k}$ be the unit vectors along the axes X_1, X_2, X_3, respectively, and $\vec{i}'$, $\vec{j}'$, $\vec{k}'$ be the unit vectors along the perturbed axis X_1', X_2', X_3', respectively. Then

$$\vec{k}' = \lambda_1\vec{i} + \lambda_2\vec{j} + \lambda_3\vec{k} \qquad (30)$$

and

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1 \qquad (31)$$

where λ_1, λ_2, and λ_3 are the cosines of the angles between $\vec{k}'$ and $\vec{i}, \vec{j}$, and $\vec{k}$, respectively. In the steady-state motion, the fluid surface is

$$r^2 = X_1^2 + X_2^2 = r_2^2 \qquad (32)$$

The fluid-free surface equation of the perturbed motion in the cylindrical coordinates is

$$\phi(r,\theta,X_3,\lambda_1,\lambda_2) = \tfrac{1}{2}\omega^2[r^2 - r^2(\lambda_1^2\cos^2\theta + \lambda_2^2\sin^2\theta)$$
$$+ X_3^2(\lambda_1^2 + \lambda_2^2) - 2r^2\sin\theta\cos\theta\lambda_1\lambda_2 - 2rX_3$$
$$\times(\lambda_1\cos\theta + \lambda_2\sin\theta)(1 - \lambda_1^2 - \lambda_2^2)^{1/2}] \qquad (33)$$

where λ_1 and λ_2 can be considered as the generalized coordinates of the perturbation.

Equation (14) in the cylindrical coordinates can be written as

$$\int \mathrm{d}\theta \int \mathrm{d}X_3 \int_{\mu_0}^{\mu_1}\left(r\frac{\partial r}{\partial\phi}\right)_0 \mathrm{d}\mu = 0 \qquad (34)$$

From Eqs. (32) and (33),

$$\left(\frac{\partial\phi}{\partial r}\right)_0 = \omega^2 r_2 \qquad (35)$$

and

$$\mu_0=\left(\frac{\partial\phi}{\partial\lambda_1}\right)_0\lambda_1+\left(\frac{\partial\phi}{\partial\lambda_2}\right)_0\lambda_2$$

$$=-\omega^2 r_2 X_3(\lambda_1\cos\theta+\lambda_2\sin\theta) \tag{36}$$

For the present symmetric configuration, it can be assumed that Eq. (20) holds: i.e., $\Delta_1 I_s=\Delta_2 I_s=\Delta C=\mu_1=0$ in the first approximation.

Hence Eq. (22) can be rewritten as

$$\Delta E=\Delta_1 E+\Delta_2 E$$

$$=\frac{1}{2}\sum\left(\frac{\partial^2 E}{\partial q_i q_j}\right)_0 q_i q_j$$

$$+\frac{1}{2}\rho\iint\left(\frac{\partial X_3}{\partial\phi}\right)_0\mu_0^2\mathrm{d}X_1\mathrm{d}X_2 \tag{37}$$

Since the potential energy U is zero,

$$E=\tfrac{1}{2}(H^2/I_{33}^1) \tag{38}$$

where

$$I_{33}^1=I_{11}\lambda_1^2+I_{22}\lambda_2^2+I_{33}(1-\lambda_1^2-\lambda_2^2) \tag{39}$$

Using Eqs. (39-41) results in

$$\Delta_1 E=\frac{1}{2}\left(\frac{\partial^2 E}{\partial q_i q_j}\right)_0 q_i q_j$$

$$=\tfrac{1}{2}\omega^2[(I_{33}-I_{11})\lambda_1^2+(I_{33}-I_{22})\lambda_2^2] \tag{40}$$

To determine the integral in Eq. (37), contributed by the fluid deformation, consider the jth tank and a point P on the fluid surface in the center plane. The distance between P and the center of the tank, l, is given by

$$l^2=r_1^2+r_2^2-2r_1r_2\cos(\theta-\theta_{j0}) \tag{41}$$

Let the extreme values of X_3 on the fluid surface at an angle θ be $X_{3_{\min}}$ and $X_{3_{\max}}$.

$$X_{3_{\max}}=-L+(R^2-l^2)^{1/2}$$

$$=-L+[2r_1r_2\cos(\theta-\theta_{j0})-\cos\alpha]^{1/2} \tag{42}$$

$$X_{3_{\min}}=-L-\{2r_1r_2\cos(\theta-\theta_{j0})-\cos\alpha]\}^{1/2} \tag{43}$$

The integral in Eq. (37) is

$$\frac{1}{2}\rho\iint\left(\frac{\partial X}{\partial\phi}\right)_0\mu_0^2\mathrm{d}X_1\mathrm{d}X_2$$

$$=\sum_{j=1}^{2}\frac{1}{2}\rho\int_\theta\mathrm{d}\theta\int\left(r\frac{\partial r}{\partial\phi}\right)_0\mu_0^2\mathrm{d}X_3 \tag{44}$$

Substituting Eqs. (35), (36), (42), and (43) in the above equation yields

$$=-\sum_{j=1}^{2}\frac{1}{6}\rho\omega^2 r_2\int_{\theta_{j0}-\alpha}^{\theta_{j0}+\alpha}[(X_{3_{\max}}^j)^3-(X_{3_{\min}}^j)^3]$$

$$\times[\lambda_1^2\cos^2\theta+\lambda_2^2\sin^2\theta+\lambda_1\lambda_2 2\sin\theta\cos\theta)\mathrm{d}\theta \tag{45}$$

Substitution of $X_{3_{\max}}$ and $X_{3_{\min}}$ from Eqs. (42) and (43), respectively, into Eq. (45) and integration over θ determines $\Delta_2 E$. The integration is simplified considerably if it is assumed that the height of the fluid surface is constant and equal to the average height. Assuming it to be $2h_s$ results in

$$X_{3_{\max}}^j=-L+h_s \qquad X_{3_{\min}}^j=-L-h_s \tag{46}$$

Substituting Eq. (46) into Eq. (45) and noting that $\theta_{10}=\pi/2$ and $\theta_{20}=3\pi/2$ yields

$$-\sum_{j=1}^{2}\frac{1}{3}\rho\omega^2 r_2^2 h_s(3L^2+h_s^2)\int_{\theta_{j0}-\alpha}^{\theta_{j0}+\alpha}[(\lambda_1^2/2)(1+\cos2\theta)$$

$$+(\lambda_2^2/2)(1-\cos2\theta)+\sin2\theta\lambda_1\lambda_2]\mathrm{d}\theta$$

$$=-\tfrac{2}{3}\rho\omega^2 r_2^2 h_s(3L^2+h_s^2)\{\lambda_1^2[\alpha-(\sin2\alpha/2)]$$

$$+\lambda_2^2[\alpha+(\sin2\alpha/2)]\} \tag{47}$$

Combining Eqs. (40) and (47) results in

$$\Delta E=\tfrac{1}{2}\omega^2\{[I_{33}-\{I_{11}+(4/3)\rho r_2^2 h_s[\alpha-(\sin2\alpha/2)]$$

$$\times(3L^2+h_s^2)\}]\lambda_1^2$$

$$+[I_{33}-\{I_{22}+(4/3)\rho r_2^2 h_s[\alpha+(\sin2\alpha/2)]$$

$$\times(3L^2+h_s^2)]\}\lambda_2^2\} \tag{48}$$

For stability, ΔE should be positive definite. Thus the stability conditions are

$$I_{33}>[I_{11}+A_1] \tag{49}$$

$$I_{33}>[I_{22}+A_2] \tag{50}$$

where

$$A_1=(4/3)\rho r_2^2 h_s[\alpha-(\sin2\alpha/2)](3L^2+h_s^2)$$

$$A_2=(4/3)\rho r_2^2 h_s[\alpha+(\sin2\alpha/2)](3L^2+h_s^2) \tag{51}$$

In the above derivation, it is assumed that the fluid surface height is constant. Assuming a circular fluid surface s of radius R_s, the equivalent height is given by the following equation:

$$\int_{-\alpha}^{\alpha}\mathrm{d}\theta\int_{-R_s\cos(\pi/2\cdot\theta/\alpha)}^{R_s\cos(\pi/2\cdot\theta/\alpha)}X_3^2\mathrm{d}X_3=\int_{-\alpha}^{\alpha}\mathrm{d}\theta\int_{-h_s}^{h_s}X_3^2\mathrm{d}X_3 \tag{52}$$

From Eq. (52),

$$h_s=R_s(4\pi/3)^{1/3} \tag{53}$$

or

$$h_s=(s/\pi)^{1/2}(4/3\pi)^{1/3} \tag{54}$$

where $s=\pi R_s^2$.

McIntyre and Miyagi Method

The McIntyre method is based on the study of the change in spacecraft balance due to the deformation of flexible elements. This method requires a closer look at the deformation of the fluid in the tanks.

The situation for a slightly canted spin axis is shown in Fig. 2. The fluid rotation about the tank center is described by the angles α_1, α_2, β_1, and β_2. The fluid level in tank 1 is lower, as this tank's lower distance from the canted spin axis forces some propellant through the manifold into tank 2. To describe this effect, the fifth variable is taken to be the change in the distance from the tank center to the fluid surface in tank 1. The generalized coordinates are defined as follows:

$$q_1 = \tfrac{1}{2}(\alpha_1+\alpha_2) \qquad q_2 = \tfrac{1}{2}(\alpha_1-\alpha_2) \qquad q_3 = \tfrac{1}{2}(\beta_1+\beta_2)$$

$$q_4 = \tfrac{1}{2}(\beta_1-\beta_2) \qquad q_5 = h_1 - h = \mathrm{d}h_1 \tag{55}$$

To determine the matrix K in Eq. (26), Γ, a, and b must be determined.

The changes in the inertia matrix due to propellant motion are

$$\begin{aligned}
&\Delta_1 I_{13} = -2mLy_0q_4 \qquad \Delta_1 I_{33} = 0\\
&\Delta_1 I_{23} = -2K_1q_1 - 2\rho SL(r_1+a_0)q_5\\
&\Delta_2 I_{33} = -4[(my_0)^2/M_s]q_4^2 - 2K_1(q_1^2+q_2^2)\\
&\qquad -2mr_1y_0(q_3^2+q_4^2) - 2\rho S(r_1+h)q_5^2
\end{aligned} \tag{56}$$

where

$$K = I^p_{33} + my_0(r_1+y_0) - I^p_{22}$$

and a_0 is the distance along the y axis from the tank center to the c.m. of the small element of fluid which has migrated between tanks.

The other parameters in Eq. (56) are defined in the nomenclature. From Eq. (56),

$$a = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -2my_0L \\ 0 \end{Bmatrix} \qquad b = \begin{Bmatrix} -2K_1 \\ 0 \\ 0 \\ \\ -2\rho SL(r_1+a_0) \end{Bmatrix} \qquad \Gamma = \begin{bmatrix} 2K_1 & 0 & 0 & 0 & 0 \\ 0 & 2K_1 & 0 & 0 & 0 \\ 0 & 0 & 2mr_1y_0 & 0 & 0 \\ 0 & 0 & 0 & 2mr_1y_0+4(my_0)^2/M & 0 \\ 0 & 0 & 0 & 0 & -2\rho S(r_1+h) \end{bmatrix} \tag{57}$$

Substituting Eq. (57) into Eq. (26), the stability matrix K becomes

$$K = \begin{bmatrix} I_{33}-I_{22}-2K_1-\dfrac{2\rho SL(r+a_0)^2}{(r_1+h)+[2\rho S(r_1+a_0)^2]/M} & 0 \\ 0 & I_{33}-I_{11}-\dfrac{2mL^2y_0}{r_1+(2my_0/M)} \end{bmatrix} \tag{58}$$

Assuming $h \simeq a_0$ and substituting $[K]$ from Eq. (58) into Eq. (29), the wobble angles become

$$\Psi_1 = \frac{I_{33}-I_{22}}{I_{33}-\left[I_{22}+2K_1+\dfrac{2\rho SL^2r_2}{1+(2\rho Sr_2/M)}\right]}\Psi_{10} \tag{59}$$

and

$$\Psi_2^2 = \frac{I_{33}-I_{11}}{I_{33}-\left[I_{11}+\dfrac{2mL^2y_0}{r_1+(2my_0/M)}\right]}\Psi_{20} \tag{60}$$

and the stability conditions are

$$I_{33} > I_{22}+2K_1+\frac{2\rho SL^2r_2}{1+(2\rho Sr_2/M)} \tag{61}$$

and

$$I_{33} > I_{11}+\frac{2mL^2y_0}{r_1+(2my_0/M)} \tag{62}$$

In summary, the stability conditions for a spinning spacecraft with two propellant tanks, as shown in Fig. 1, are

$$I_{33} > (I_{11}+A_1) \quad \text{or} \quad (I_{33}/I_{11}) > (1+C_1) \tag{63}$$

and

$$I_{33} > (I_{22}+A_2) \quad \text{or} \quad (I_{33}/I_{22}) > (1+C_2) \tag{64}$$

where, from Rumyanstev,

$$A_1 = (4/3)\rho r_2^2 h_s[\alpha - (\sin 2\alpha/2)](3L^2+h_s^2) \tag{65}$$

$$A_2 = (4/3)\rho r_2^2 h_s[\alpha + (\sin 2\alpha/2)](3L^2+h_s^2) \tag{66}$$

and from McIntyre and Miyagi

$$A_1 = \frac{2mL^2y_0}{r_1+(2my_0/M)} \tag{67}$$

$$A_2 = 2K_1 + \frac{2\rho SL^2r_2}{1+(2\rho Sr_2/M)} \tag{68}$$

Discussion

In these stability conditions, the terms containing L^2 in A_2 correspond to the contribution of propellant migration between tanks. The effect of propellant migration is basically dependent on two spacecraft parameters: distance of the tank center plane from spacecraft c.m., L, and surface area of the fluid, s.

During the derivation of the stability conditions, several approximations are made to simplify the analysis. In the Rumyanstev method, the height of the fluid is assumed to be constant. In the McIntyre and Miyagi approach, the fluid is assumed to be a rigid body rotating like a pendulum about the tank center. The fluid surface is also assumed to be flat instead of a curved surface. This will introduce greater errors for a spacecraft with smaller distance between spin axis and tank center. These approximations contribute to the difference in the above stability conditions. One important difference is that the parameter y_0, the distance from tank center to c.m. of propellant in the tank, is an important parameter in the McIntyre/Miyagi method because the fluid is assumed to be rigid and rotating like a pendulum about the center, as discussed earlier. However, this parameter does not influence the stability condition in the Rumyanstev method because only the fluid surface tilt is considered.

Numerical Examples

In these examples, transfer orbit configurations are analyzed. The first example refers to a spacecraft with a solid

Table 1 Stability conditions

Example	Tank fill fraction, %	Minimum inertia ratio, $I_{33}/I_{22}=1+C$ Rumyanstev method	McIntyre/Miyagi method
First	30	1.0306	1.0312
	50	1.0294	1.03145
	70	1.0226	1.0255
Second	25	1.1217	1.1053
	50	1.1166	1.1099
	75	1.0801	1.0678

apogee motor and the second to a spacecraft with a liquid apogee motor.

First Example

The following spacecraft parameters are assumed:

$$\begin{aligned} \rho &= 10^3\ \text{kg/m}^3 \\ r_l &= 0.8\ \text{m} \\ L &= 0.3\ \text{m} \\ R &= 0.35\ \text{m} \\ M &= 2500\ \text{kg} \\ I_{22} &= 2000\ \text{kg-m}^2 \end{aligned}$$

The stability conditions, determined for three different tank fill fractions (30, 50, and 70%) by both the Rumyanstev and McIntyre/Miyagi methods, are given in Table 1.

Second Example

For this example, the following spacecraft parameters are assumed:

$$\begin{aligned} \rho &= 10^3\ \text{kg/m}^2 \\ r_l &= 1.2\ \text{m} \\ L &= 0.3\ \text{m} \\ R &= 0.6\ \text{m} \\ M &= 2500\ \text{kg} \\ I_{22} &= 2000\ \text{kg/m}^2 \end{aligned}$$

The stability conditions, determined for three different tank fill fractions (25, 50, and 75%) by both the Rumyanstev and McIntyre/Miyagi methods, are given in Table 1.

A comparison of the stability conditions from these two methods in Table 1 indicates that, for the given sets of spacecraft parameters, the difference is generally small (less than 10%). However, the difference could be significant for certain spacecraft parameters, such as a large y_0, which influences one method more than the other.

Conclusions

The Liapunov method is used in the derivation of the stability conditions for a flexible spinning spacecraft. Different approaches and approximations used by investigators to calculate the change in moment of inertia due to liquid propellant motion have resulted in different stability conditions. For the spacecraft parameters given as examples, the differences in the stability conditions by the Rumyanstev and the McIntyre/Miyagi methods are generally small. However, for certain spacecraft parameters, such as a large distance from tank center plane to c.m. of the propellant in the tank, the difference could be significant.

The stability conditions are derived for a single spinner. This method can be extended to a stable dual-spin configuration, where the rotor spin moment of inertia exceeds the transverse inertia. In this case, the spin inertia of the rotor will represent the total spin inertia. Frequently, the system c.m. motion is neglected in analyzing the stability of a spinning body. In such cases, the stability boundaries are somewhat narrower, since the c.m. movement has a relieving effect.

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