SUPPLEMENTARY MATERIALS: PROOF OF THEOREM 2.1

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In this supplementary appendix, we reproduce the proof of Theorem 2.1 given in [1, Ch. 4] with a few minor modifications to make the text self-contained. For ease of reference, we repeat our basic notational setup from the main article here. If $K = 2N + 1$ is an odd integer, the zero-centered equispaced grid of length K in $[-\pi, \pi)$ consists of the points

$$x_k = kh, \qquad -N \le k \le N, \tag{1}$$

where $h = 2\pi/K$ is the grid spacing. We consider the perturbed grid

$$\widetilde{x}_k = x_k + s_k h, \qquad |s_k| \le \alpha, \tag{2}$$

where the parameter α is a fixed value in the range $0 \le \alpha < 1/2$. The kth trigonometric Lagrange basis function associated with the perturbed grid is denoted by $\widetilde{\ell}_k$, i.e.

$$\widetilde{\ell}_k(x) = \prod_{\substack{j=-N \\ j \ne k}}^{N} \frac{\sin\left(\frac{x - \widetilde{x}_j}{2}\right)}{\sin\left(\frac{\widetilde{x}_k - \widetilde{x}_j}{2}\right)}.$$

We have $\widetilde{\ell}_k(x_j) = 1$ if $j = k$ and 0 if $j \ne k$. From (5.2) and (5.3) in the main article, we have

$$\widetilde{\Lambda}_N = \max_{x \in [-\pi, \pi]} \sum_{k=-N}^{N} |\widetilde{\ell}_k(x)|. \tag{3}$$

Our argument can be loosely outlined as follows. The bulk of the work is devoted to bounding $|\widetilde{\ell}_0(x)|$, which takes several steps to accomplish. Taking x as fixed, we determine the choice of the points $\widetilde{x}_j$ that maximizes $|\widetilde{\ell}_0(x)|$ and then bound the maximum using integrals. Since the resulting bound is independent of $\widetilde{x}_j$, we can exploit symmetry to obtain bounds on $|\widetilde{\ell}_k(x)|$ for $k \ne 0$. We then sum these bounds over k to obtain a bound on $\widetilde{\Lambda}_N$.

We begin with the following result, which shows that to bound $|\widetilde{\ell}_0(x)|$ we need consider only grids in which all the points, possibly aside from $\widetilde{x}_0$, are perturbed by the maximum amount of αh.

LEMMA 1. *For all $x \in [-\pi, \pi]$ and $-N \le j \le N$, $j \ne 0$,*

$$\left| \frac{\sin\left(\frac{x - \widetilde{x}_j}{2}\right)}{\sin\left(\frac{\widetilde{x}_0 - \widetilde{x}_j}{2}\right)} \right| \le \max\left(\left| \frac{\sin\left(\frac{x - (j-\alpha)h}{2}\right)}{\sin\left(\frac{\widetilde{x}_0 - (j-\alpha)h}{2}\right)} \right|, \left| \frac{\sin\left(\frac{x - (j+\alpha)h}{2}\right)}{\sin\left(\frac{\widetilde{x}_0 - (j+\alpha)h}{2}\right)} \right| \right). \tag{4}$$

Proof. The statement is trivially true if $x = \widetilde{x}_0$. If $x \ne \widetilde{x}_0$, then from

$$\frac{d}{dt} \frac{\sin\left(\frac{x-t}{2}\right)}{\sin\left(\frac{\widetilde{x}_0 - t}{2}\right)} = \frac{1}{2} \frac{\sin\left(\frac{x - \widetilde{x}_0}{2}\right)}{\left[\sin\left(\frac{\widetilde{x}_0 - t}{2}\right)\right]^2},$$

we see that $t \mapsto \sin\bigl((x-t)/2\bigr)/\sin\bigl((\widetilde{x}_0-t)/2\bigr)$ has no critical points in $[-\pi,\pi]$ apart from $t=\widetilde{x}_0$, where it is singular. In particular, it has no critical points in any of the intervals $[(j-\alpha)h,(j+\alpha)h]$ for $-N\le j\le N$, $j\ne 0$, and therefore must assume its extreme values on these intervals at the endpoints. Since $\widetilde{x}_j \in [(j-\alpha)h,(j+\alpha)h]$ for each j, we are done. ∎

Which of the two arguments to the maximum function on the right-hand side of (4) is larger depends on both x and j. We need to understand the exact conditions under which each one takes over. Our first step in this direction is the following lemma, which tells us when the two are equal.

LEMMA 2. *For $0<\alpha<1/2$ and $-N\le j\le N$, $j\ne 0$, the equation*

$$\left|\frac{\sin\left(\frac{x-(j-\alpha)h}{2}\right)}{\sin\left(\frac{\widetilde{x}_0-(j-\alpha)h}{2}\right)}\right| = \left|\frac{\sin\left(\frac{x-(j+\alpha)h}{2}\right)}{\sin\left(\frac{\widetilde{x}_0-(j+\alpha)h}{2}\right)}\right| \tag{5}$$

has exactly two solutions in $[-\pi,\pi]$: $x=\widetilde{x}_0$ and $x=x_j^*$, where[1]

$$x_j^* = 2\arctan\left(\frac{\cos(jh)-\cos(\alpha h)+\tan\bigl(\widetilde{x}_0/2\bigr)\sin(jh)}{\tan\bigl(\widetilde{x}_0/2\bigr)\bigl(\cos(jh)+\cos(\alpha h)\bigr)-\sin(jh)}\right).$$

Proof. Multiplying through by the denominators of both sides and applying some trigonometric identities, we find that (5) can be reduced to

$$\begin{aligned}\left|\cos\left(\frac{\widetilde{x}_0-x}{2}+\alpha h\right)-\cos\left(\frac{\widetilde{x}_0+x}{2}-jh\right)\right| \\ = \left|\cos\left(\frac{\widetilde{x}_0-x}{2}-\alpha h\right)-\cos\left(\frac{\widetilde{x}_0+x}{2}-jh\right)\right|.\end{aligned} \tag{6}$$

If the expressions within the absolute value signs on either side of (6) are equal, then we have

$$\cos\left(\frac{\widetilde{x}_0-x}{2}+\alpha h\right)=\cos\left(\frac{\widetilde{x}_0-x}{2}-\alpha h\right).$$

In order to solve this equation, we consider two cases.

Case 1: $(\widetilde{x}_0-x)/2+\alpha h=(\widetilde{x}_0-x)/2-\alpha h+2n\pi$ for some integer n. Rearranging gives $\alpha h=n\pi$, and substituting for h, we arrive at $\alpha=Kn/2$. Since $\alpha<1/2$, this can hold only if $n=0$, in which case $\alpha=0$, but this is disallowed by our hypotheses.

Case 2: $(\widetilde{x}_0-x)/2+\alpha h=\alpha h-(\widetilde{x}_0-x)/2+2n\pi$ for some integer n. If this holds, then $\widetilde{x}_0-x=4n\pi$, but this can happen only if $n=0$, since $\widetilde{x}_0-x\in[-\pi-\alpha h,\pi+\alpha h]$, and this interval is contained in $[-2\pi,2\pi]$ because $\alpha h\le\pi$. Thus, $x=\widetilde{x}_0$.

We conclude that $x=\widetilde{x}_0$ is the only solution when the expressions within the absolute value signs on either side of (6) are equal. On the other hand, if they are equal but of opposite sign, we get

$$2\cos\left(\frac{\widetilde{x}_0+x}{2}-jh\right)=\cos\left(\frac{\widetilde{x}_0-x}{2}+\alpha h\right)+\cos\left(\frac{\widetilde{x}_0-x}{2}-\alpha h\right).$$

[1] Here, arctan denotes the principal branch of the inverse tangent function.

Simplifying the right-hand side to $2\cos\bigl((\widetilde{x}_0 - x)/2\bigr)\cos(\alpha h)$ and then expanding both sides out completely using trigonometric identities, we find that

$$\begin{aligned}\cos\left(\frac{\widetilde{x}_0}{2}\right)\cos\left(\frac{x}{2}\right)\cos(jh) - \sin\left(\frac{\widetilde{x}_0}{2}\right)\sin\left(\frac{x}{2}\right)\cos(jh)\\ + \sin\left(\frac{\widetilde{x}_0}{2}\right)\cos\left(\frac{x}{2}\right)\sin(jh) + \cos\left(\frac{\widetilde{x}_0}{2}\right)\sin\left(\frac{x}{2}\right)\sin(jh)\\ = \cos\left(\frac{\widetilde{x}_0}{2}\right)\cos\left(\frac{x}{2}\right)\cos(\alpha h) + \sin\left(\frac{\widetilde{x}_0}{2}\right)\sin\left(\frac{x}{2}\right)\cos(\alpha h).\end{aligned}$$

Dividing both sides of this through by $\cos(\widetilde{x}_0/2)\cos(x/2)$ and rearranging, we obtain

$$\tan\left(\frac{x}{2}\right) = \frac{\cos(jh) - \cos(\alpha h) + \tan\bigl(\widetilde{x}_0/2\bigr)\sin(jh)}{\tan\bigl(\widetilde{x}_0/2\bigr)\bigl(\cos(jh) + \cos(\alpha h)\bigr) - \sin(jh)}.$$

Taking the inverse tangent of both sides and multiplying by 2, we arrive at $x = x_j^*$. ∎

To move forward, we need a better understanding of the locations of the points x_j^*. The requisite inequalities are simple to state and are given in Lemma 4, but first we pause to establish a minor fact that we will need in their proof.

LEMMA 3. *For* $|t| \le \alpha$ *and* $-N \le j \le N$, $j \ne 0$,

$$\bigl|\sin\bigl((j+t)h\bigr)\bigr| > \sin\bigl((\alpha - |t|)h\bigr).$$

Proof. This is a consequence of the following chain of inequalities:

$$\begin{aligned}0 \le (\alpha - |t|)h < (1 - |t|)h \le (|j| - |t|)h\\ \le (|j| + |t|)h \le (N + |t|)h < \left(N + |t| + \frac{1}{2} - \alpha\right)h = \pi - (\alpha - |t|)h \le \pi.\end{aligned}$$

∎

LEMMA 4. *For* $-N \le j \le N$, $j \ne 0$, *and* $0 < \alpha < 1/2$,

$$(j - \alpha)h < x_j^* < (j + \alpha)h.$$

Proof. Let

$$f(t) = \frac{\cos(jh) - \cos(\alpha h) + t\sin(jh)}{t\bigl(\cos(jh) + \cos(\alpha h)\bigr) - \sin(jh)}.$$

Note that $f\bigl(\tan(\widetilde{x}_0/2)\bigr) = \tan(x_j^*/2)$. A straightforward computation using the quotient rule and some trigonometric identities shows that

$$f'(t) = -\left(\frac{\sin(\alpha h)}{t\bigl(\cos(jh) + \cos(\alpha h)\bigr) - \sin(jh)}\right)^2,$$

which is always negative wherever it is defined. By Lemma 3, we have $|\sin(jh)| > \sin(\alpha h)$ for each j. Furthermore, note that $j \ne 0$ implies $N \ge 1$, so that $\alpha h < \pi/3$, and so $\cos(\alpha h) > 0$. Therefore, $|\cos(jh)+\cos(\alpha h)| \le 1+\cos(\alpha h)$ for each j. It follows that

$$\left|\frac{\sin(jh)}{\cos(jh)+\cos(\alpha h)}\right| > \frac{\sin(\alpha h)}{1+\cos(\alpha h)} = \tan\left(\frac{\alpha h}{2}\right).$$

Hence, the singularity in f is outside the interval $[\tan(-\alpha h/2), \tan(\alpha h/2)]$, and we conclude that

$$f\big(\tan(\alpha h/2)\big) \le f\big(\tan(\widetilde{x}_0/2)\big) \le f\big(\tan(-\alpha h/2)\big).$$

Next, consider the function g_+ and the number M_+ defined by

$$\begin{aligned} g_+(t) &= \frac{\cos(jh) - t + \tan(-\alpha h/2)\sin(jh)}{\tan(-\alpha h/2)\big(\cos(jh)+t\big) - \sin(jh)}, \\ M_+ &= \frac{\cos(jh) - 1 + \tan(-\alpha h/2)\sin(jh)}{\tan(-\alpha h/2)\big(\cos(jh)-1\big) - \sin(jh)}. \end{aligned}$$

Note that $g_+\big(\cos(\alpha h)\big) = f\big(\tan(-\alpha h/2)\big)$ and that

$$M_+ = \frac{\frac{\cos(jh)-1}{\sin(jh)} + \tan(-\alpha h/2)}{\tan(-\alpha h/2)\frac{\cos(jh)-1}{\sin(jh)} - 1} = \frac{\tan(jh/2) - \tan(-\alpha h/2)}{1+\tan(jh/2)\tan(-\alpha h/2)} = \tan\left(\frac{(j+\alpha)h}{2}\right).$$

Therefore, if we can show that $g_+\big(\cos(\alpha h)\big) < M_+$, we will have that $\tan(x_j^*/2) < \tan\big((j+\alpha)h/2\big)$, which implies that $x_j^* < (j+\alpha)h$, as desired. The remainder of the proof will be devoted to establishing this fact. The lower bound on x_j^* can be derived by considering the function g_- and the number M_- defined by

$$\begin{aligned} g_-(t) &= \frac{\cos(jh) - t + \tan(\alpha h/2)\sin(jh)}{\tan(\alpha h/2)\big(\cos(jh)+t\big) - \sin(jh)}, \\ M_- &= \frac{\cos(jh) - 1 + \tan(\alpha h/2)\sin(jh)}{\tan(\alpha h/2)\big(\cos(jh)-1\big) - \sin(jh)} \end{aligned}$$

and arguing similarly. We omit the details.

To show that $g_+\big(\cos(\alpha h)\big) < M_+$, we begin by noting that by multiplying the numerator and denominator of both $g_+(t)$ and M_+ by $\cos(-\alpha h/2)$ and applying some trigonometric identities, they can be rewritten as

$$g_+(t) = \frac{\cos(\alpha h/2)t - \cos\big((j+\alpha/2)h\big)}{\sin(\alpha h/2)t + \sin\big((j+\alpha/2)h\big)}, \qquad M_+ = -\frac{\cos(\alpha h/2) - \cos\big((j+\alpha/2)h\big)}{\sin(\alpha h/2) - \sin\big((j+\alpha/2)h\big)}.$$

Consider the affine function φ obtained by multiplying together the denominators in these new expressions for g_+ and M_+, where that of the latter is taken to include the leading minus sign:

$$\begin{aligned} \varphi(t) = -\sin(\alpha h/2)\Big(\sin(\alpha h/2) - \sin\big((j+\alpha/2)h\big)\Big)t& \\ - \sin\big((j+\alpha/2)h\big)\Big(\sin(\alpha h/2) - \sin\big((j+\alpha/2)h\big)\Big)&. \end{aligned}$$

We will show that $\varphi\big(\cos(\alpha h)\big) > 0$. First, note that $\varphi(t) = 0$ at $t = t_0 = -\sin\big((j+\alpha/2)h\big)/\sin(\alpha h/2)$ and that by Lemma 3, this point lies outside of the interval $[-1, 1]$.

Next, observe that $\sin(\alpha h/2) > 0$, that $\sin\bigl((j+\alpha/2)h\bigr)$ has the same sign as j, and that

$$\varphi'(t) = -\sin(\alpha h/2)\Bigl(\sin(\alpha h/2) - \sin\bigl((j+\alpha/2)h\bigr)\Bigr).$$

If $j < 0$, then $\sin(\alpha h/2) - \sin\bigl((j+\alpha/2)h\bigr) > 0$ trivially, so $\varphi'(t) < 0$. Thus, $\varphi(t) > 0$ for $t < t_0$. Inspecting the formula for t_0, we find that $t_0 > 0$ in this case. Since t_0 cannot lie in the interval $[-1, 1]$, it must further be true that $t_0 > 1$. As $\cos(\alpha h) \leq 1$, we have that $\cos(\alpha h) < t_0$, as desired. On the other hand, if $j > 0$, then $\sin(\alpha h/2) - \sin\bigl((j+\alpha/2)h\bigr) < 0$ by Lemma 3, and we have that $\varphi'(t) > 0$, so that $\varphi(t) > 0$ for $t > t_0$. But $t_0 < 0$ in this case, and since $\cos(\alpha h) > 0$, we have $\cos(\alpha h) > t_0$, and we are done.

It follows that $g_+\bigl(\cos(\alpha h)\bigr) < M_+$ is equivalent to the inequality

$$\begin{aligned}
-\Bigl(\cos(\alpha h/2)\cos(\alpha h) - \cos\bigl((j+\alpha/2)h\bigr)\Bigr)\Bigl(\sin(\alpha h/2) - \sin\bigl((j+\alpha/2)h\bigr)\Bigr)& \\
< \Bigl(\sin(\alpha h/2)\cos(\alpha h) + \sin\bigl((j+\alpha/2)h\bigr)\Bigr)\Bigl(\cos(\alpha h/2) - \cos\bigl((j+\alpha/2)h\bigr)\Bigr)&.
\end{aligned}$$

Expanding out the products, moving all terms involving $\cos(\alpha h)$ to the left and those not involving it to the right, and using some trigonometric identities to simplify the result, we find that this in turn is equivalent to

$$\Bigl(\sin\bigl((j+\alpha)h\bigr) - \sin(\alpha h)\Bigr)\cos(\alpha h) < \sin(jh).$$

Next, we expand $\sin\bigl((j+\alpha)h\bigr)$ and move all terms involving $\sin(jh)$ to the right, leaving us with

$$\bigl(\cos(jh) - 1\bigr)\sin(\alpha h)\cos(\alpha h) < \sin(jh)\bigl(1 - \bigl(\cos(\alpha h)\bigr)^2\bigr).$$

Using the identities $1 - \cos(jh) = \sin(jh)\tan(jh/2)$ and $1 - \bigl(\cos(\alpha h)\bigr)^2 = \sin\bigl(\alpha h\bigr)^2$, we can rearrange this one more time to find that our original inequality is equivalent to

$$\sin(jh)\bigl(\tan(\alpha h) + \tan(jh/2)\bigr) > 0.$$

If $j > 0$, then since $\sin(jh) > 0$, this is equivalent to $-\tan(jh/2) < \tan(\alpha h)$, which holds trivially, since the left-hand side is negative, while the right-hand side is positive. If $j < 0$, then $\sin(jh) < 0$, and the inequality is equivalent to $-\tan(jh/2) > \tan(\alpha h)$. Taking inverse tangents, we see that this is equivalent to $\alpha < -j/2$, and this inequality holds, since $-j \geq 1$ and $\alpha < 1/2$. This completes the proof. □

Assembling these results, we can prove the following statement about the right-hand side of (4).

Lemma 5. *We have*

$$\max\left(\left|\frac{\sin\left(\frac{x-(j-\alpha)h}{2}\right)}{\sin\left(\frac{\widetilde{x}_0-(j-\alpha)h}{2}\right)}\right|, \left|\frac{\sin\left(\frac{x-(j+\alpha)h}{2}\right)}{\sin\left(\frac{\widetilde{x}_0-(j+\alpha)h}{2}\right)}\right|\right) = \left|\frac{\sin\left(\frac{x-(j-\alpha)h}{2}\right)}{\sin\left(\frac{\widetilde{x}_0-(j-\alpha)h}{2}\right)}\right|$$

when $1 \le j \le N$ *and* $x \in [-\pi, \widetilde{x}_0] \cup [x_j^*, \pi]$ *or when* $-N \le j \le -1$ *and* $x \in [x_j^*, \widetilde{x}_0]$, *and*

$$\max\left(\left|\frac{\sin\left(\frac{x-(j-\alpha)h}{2}\right)}{\sin\left(\frac{\widetilde{x}_0-(j-\alpha)h}{2}\right)}\right|, \left|\frac{\sin\left(\frac{x-(j+\alpha)h}{2}\right)}{\sin\left(\frac{\widetilde{x}_0-(j+\alpha)h}{2}\right)}\right|\right) = \left|\frac{\sin\left(\frac{x-(j+\alpha)h}{2}\right)}{\sin\left(\frac{\widetilde{x}_0-(j+\alpha)h}{2}\right)}\right|$$

when $1 \le j \le N$ *and* $x \in [\widetilde{x}_0, x_j^*]$ *or when* $-N \le j \le -1$ *and* $x \in [-\pi, x_j^*] \cup [\widetilde{x}_0, \pi]$.

Proof. We will give the proof assuming $1 \le j \le N$; the proof for $-N \le j \le -1$ is similar. When $\alpha = 0$, there is nothing to prove, so we may assume $\alpha > 0$. By Lemma 2, the two arguments of the maximum function are equal only at $x = \widetilde{x}_0$ and $x = x_j^*$, and by Lemma 4, we have $-\pi < \widetilde{x}_0 < (j-\alpha)h < x_j^* < (j+\alpha)h < \pi$. Evaluating both arguments of the maximum function at $x = (j-\alpha)h$, we see that the first is zero, while the second is nonzero. Thus, the second must be the larger on $[\widetilde{x}_0, x_j^*]$. Evaluating at $x = (j+\alpha)h$, the situation is reversed, and by periodicity we find that the first must be the larger on $[-\pi, \widetilde{x}_0] \cup [x_j^*, \pi]$. ∎

This lemma is all we need for maximizing the factors in $|\widetilde{\ell}_0(x)|$ with respect to the $\widetilde{x}_j$ for $j \ne 0$. We would like to do something similar for $\widetilde{x}_0$. Unfortunately, the dependence on $\widetilde{x}_0$ of the various cases in this result tells us that we cannot go further and maximize any one factor over $\widetilde{x}_0$ independently of x. The next result shows that we can get around this by pairing up the factors at $\pm j$ for $1 \le j \le N$ instead of considering them in isolation.

Note that we state the result only for $x \in [-\pi, 0]$. The reason is that, by symmetry, any bound we obtain on $|\widetilde{\ell}_0(x)|$ for $x \in [-\pi, 0]$ that is independent of x must also hold for $x \in [0, \pi]$. We will therefore ignore the case of $x \in [0, \pi]$ until we reach the end of our argument, at which point we will see that it has been taken care of for free. Alternatively, one could write out an analogous argument that assumes $x \in [0, \pi]$ instead.

LEMMA 6. *For* $x \in [-\pi, 0]$ *and* $1 \le j \le N$,

$$\left|\frac{\sin\left(\frac{x-\widetilde{x}_{-j}}{2}\right)\sin\left(\frac{x-\widetilde{x}_j}{2}\right)}{\sin\left(\frac{\widetilde{x}_0-\widetilde{x}_{-j}}{2}\right)\sin\left(\frac{\widetilde{x}_0-\widetilde{x}_j}{2}\right)}\right| \le \begin{cases} \left|\dfrac{\sin\left(\frac{x+(j-\alpha)h}{2}\right)\sin\left(\frac{x-(j-\alpha)h}{2}\right)}{\sin\left(\frac{jh}{2}\right)\sin\left(\frac{(2\alpha-j)h}{2}\right)}\right| & -\pi \le x \le x_{-j}^* \\ \left|\dfrac{\sin\left(\frac{x+(j+\alpha)h}{2}\right)\sin\left(\frac{x-(j-\alpha)h}{2}\right)}{\sin\left(\frac{(2\alpha+j)h}{2}\right)\sin\left(\frac{(2\alpha-j)h}{2}\right)}\right| & x_{-j}^* \le x \le 0. \end{cases}$$

Proof. Fix x, and define the functions f_1, f_2, and f_3 by

$$f_1(t) = \frac{\sin\left(\frac{x+(j-\alpha)h}{2}\right)\sin\left(\frac{x-(j-\alpha)h}{2}\right)}{\sin\left(\frac{t+(j-\alpha)h}{2}\right)\sin\left(\frac{t-(j-\alpha)h}{2}\right)} = \frac{\cos\big((j-\alpha)h\big)-\cos(x)}{\cos\big((j-\alpha)h\big)-\cos(t)}$$

$$f_2(t) = \frac{\sin\left(\frac{x+(j+\alpha)h}{2}\right)\sin\left(\frac{x-(j-\alpha)h}{2}\right)}{\sin\left(\frac{t+(j+\alpha)h}{2}\right)\sin\left(\frac{t-(j-\alpha)h}{2}\right)} = \frac{\cos(jh)-\cos(x+\alpha h)}{\cos(jh)-\cos(t+\alpha h)}$$

$$f_3(t) = \frac{\sin\left(\frac{x+(j-\alpha)h}{2}\right)\sin\left(\frac{x-(j+\alpha)h}{2}\right)}{\sin\left(\frac{t+(j-\alpha)h}{2}\right)\sin\left(\frac{t-(j+\alpha)h}{2}\right)} = \frac{\cos(jh)-\cos(x-\alpha h)}{\cos(jh)-\cos(t-\alpha h)}.$$

Note that only the denominators of these functions vary with t; the numerators are constant. By Lemma 5, we have

$$\left|\frac{\sin\left(\frac{x-\widetilde{x}_{-j}}{2}\right)\sin\left(\frac{x-\widetilde{x}_j}{2}\right)}{\sin\left(\frac{\widetilde{x}_0-\widetilde{x}_{-j}}{2}\right)\sin\left(\frac{\widetilde{x}_0-\widetilde{x}_j}{2}\right)}\right| \leq \begin{cases} |f_1(\widetilde{x}_0)| & -\pi \leq x \leq x^*_{-j} \\ |f_2(\widetilde{x}_0)| & x^*_{-j} \leq x \leq \widetilde{x}_0 \\ |f_3(\widetilde{x}_0)| & \widetilde{x}_0 \leq x \leq x^*_j. \end{cases} \tag{7}$$

Recalling that $\widetilde{x}_0 \in [-\alpha h, \alpha h]$, by maximizing $|f_1(t)|$, $|f_2(t)|$, and $|f_3(t)|$ over $t \in [-\alpha h, \alpha h]$ under the appropriate conditions on x, we will show that this inequality may be replaced by

$$\left|\frac{\sin\left(\frac{x-\widetilde{x}_{-j}}{2}\right)\sin\left(\frac{x-\widetilde{x}_j}{2}\right)}{\sin\left(\frac{\widetilde{x}_0-\widetilde{x}_{-j}}{2}\right)\sin\left(\frac{\widetilde{x}_0-\widetilde{x}_j}{2}\right)}\right| \leq \begin{cases} |f_1(\alpha h)| & -\pi \leq x \leq x^*_{-j} \\ |f_2(\alpha h)| & x^*_{-j} \leq x \leq 0, \end{cases}$$

and this is the inequality we are trying to establish. We consider three cases.

Case 1: $-\pi \leq x \leq x^*_{-j}$. In this case, the right-hand side of (7) is governed by f_1. The denominator of f_1 has a critical point in $[-\alpha h, \alpha h]$ at $t = 0$, and it takes on identical values at the endpoints $\pm\alpha h$. Since

$$0 < \alpha h < (1-\alpha)h \leq (j-\alpha)h \leq (N-\alpha)h < \left(N + \frac{1}{2}\right)h = \pi,$$

we have $\cos\big((j-\alpha)h\big) \leq \cos(\alpha h) \leq 1$, and so $\big|\cos\big((j-\alpha)h\big) - \cos(\alpha h)\big| \leq \big|\cos\big((j-\alpha)h\big) - 1\big|$. Thus, the denominator is smallest in magnitude at $t = \pm\alpha h$. Since the numerator of f_1 does not vary with t, we are done.

Case 2: $x^*_{-j} \leq x \leq -\alpha h$. Here, the behavior of (7) is determined by f_2. The only critical point of the denominator f_2 in $[-\alpha h, \alpha h]$ is at the left endpoint, where it takes the value $\cos(jh) - 1$. At the right endpoint, the denominator is $\cos(jh) - \cos(2\alpha h)$. From

$$0 < 2\alpha h < h \leq jh \leq Nh < \left(N + \frac{1}{2}\right)h = \pi,$$

we see that $\cos(jh) \leq \cos(2\alpha h) \leq 1$, and so we have $|\cos(jh) - \cos(\alpha h)| \leq |\cos(jh) - 1|$. Thus, the denominator is smallest in magnitude at $t = \alpha h$, and we are done, as in the previous case.

Case 3: $-\alpha h \leq x \leq 0$. In this case, for $-\alpha h \leq \widetilde{x}_0 \leq x$, the right-hand side of (7) is governed by f_3, while for $x \leq \widetilde{x}_0 \leq \alpha h$, it is governed by f_2. From the previous case, we know that the maximum absolute value of $f_2(t)$ for $t \in [-\alpha h, \alpha h]$ occurs at $t = \alpha h$, and a virtually identical argument shows that the maximum absolute value of $f_3(t)$ over the same range occurs at $t = -\alpha h$. We are thus left to compare $|f_3(-\alpha h)|$ and $|f_2(\alpha h)|$. Since these two quantities have the same denominator, we need only compare their numerators. The conditions on x imply that

$$0 \leq \alpha h + x \leq \alpha h - x \leq 2\alpha h \leq jh < \pi,$$

the later inequalities following as in the developments of the previous case. Therefore, $\cos(jh) \leq \cos(x - \alpha h) \leq \cos(x + \alpha h)$, which implies that $|\cos(jh) - \cos(x - \alpha h)| \leq |\cos(jh) - \cos(x + \alpha h)|$. It follows that $|f_2(\alpha h)| \geq |f_3(-\alpha h)|$, as desired. □

We can now prove the following result, which gives a bound on $|\widetilde{\ell}_0(x)|$ for $x \in [-\pi, 0]$ that is independent of the points $\widetilde{x}_j$. First, we introduce some additional notation that we will need for the remainder of our argument. Define $x_0^* = 0$ and $x_{-N-1}^* = -\pi$. For $0 \le k \le N$, let $R_k^* = [x_{-k-1}^*, x_{-k}^*]$ and $R_k = [(-k-1-\alpha)h, (-k+\alpha)h]$. Observe that $\bigcup_{k=0}^N R_k^* = [-\pi, 0]$. Again for $0 \le k \le N$, let

$$P_k(x) = \prod_{j=1}^{N} \left|\sin\left(\frac{x-(j-\alpha)h}{2}\right)\right| \times \prod_{j=1}^{k} \left|\sin\left(\frac{x+(j-\alpha)h}{2}\right)\right| \times \prod_{j=k+1}^{N} \left|\sin\left(\frac{x+(j+\alpha)h}{2}\right)\right|,$$

and let

$$Q_k = \prod_{j=1}^{N} \left|\sin\left(\frac{(2\alpha-j)h}{2}\right)\right| \times \prod_{j=1}^{k} \left|\sin\left(\frac{jh}{2}\right)\right| \times \prod_{j=k+1}^{N} \left|\sin\left(\frac{(2\alpha+j)h}{2}\right)\right|.$$

Define

$$M_k = \max_{x\in[-\pi,0]\cap R_k} \frac{P_k(x)}{Q_k},$$

and note that M_k does not depend on the points $\widetilde{x}_j$.

Lemma 7. *For $0 \le k \le N$ and $x \in R_k^*$, we have $|\widetilde{\ell}_0(x)| \le M_k$.*

Proof. Multiply together the inequalities derived in Lemma 6 for $1 \le j \le N$, and note that $R_k^* \subset R_k$ by Lemma 4. ∎

Next, we turn to bounding M_k. Our strategy will be to reduce the products $P_k(x)$ and Q_k to sums by taking logarithms and then bounding the sums using integrals. We begin with $P_k(x)$, which requires more work than Q_k because of its dependence on x. The bound that we need is given by Lemma 14, but before presenting it, we first establish several minor technical results that we will need in its proof.

Lemma 8. *For $0 \le k \le N$ and $x \in R_k$,*

$$\left|\sin\left(\frac{x+(k-\alpha)h}{2}\right)\sin\left(\frac{x+(k+1+\alpha)h}{2}\right)\right| \le \left|\sin\left(\frac{(\alpha+1/2)h}{2}\right)\right|^2.$$

Proof. The derivative of the expression inside the absolute value signs on the left-hand side of this inequality is $(1/2)\sin\bigl(x+(k+1/2)h\bigr)$, which vanishes inside R_k only at $x = -(k+1/2)h$. The maximum absolute value of the expression must occur at this point, since it is zero at the endpoints of R_k. Substituting this value in for x in the left-hand side, we arrive at the right-hand side. ∎

Lemma 9. *For $1 \le k \le N$ and $x \in R_k$,*

$$\left|\sin\left(\frac{x-(1-\alpha)h}{2}\right)\sin\left(\frac{x+(1-\alpha)h}{2}\right)\right| \ge \left|\sin\left(\frac{(k+1-2\alpha)h}{2}\right)\sin\left(\frac{(k-1)h}{2}\right)\right|.$$

Proof. Let $f(x)$ be the expression inside the absolute value signs on the left-hand side of this inequality. Applying some trigonometric identities, we find that $f(x) = \cos\bigl((1-\alpha)h\bigr)/2 - \cos(x)/2$. If $1 \le k \le N-1$, then since

$$0 \le (1-\alpha)h \le (k-\alpha)h \le -x \le (k+1+\alpha)h \le (N+\alpha)h < \pi,$$

we have $\cos(x) \le \cos\big((1-\alpha)h\big)$, and so $f(x) \ge 0$ for $x \in R_k$. The same string of inequalities shows that $f'(x) = \sin(x)/2$ is negative on R_k, so f is decreasing on R_k. Therefore, the smallest absolute value of f is obtained by evaluating at the right endpoint $x = (-k+\alpha)h$, and this produces the expression on the right-hand side of the inequality to be established.

For the $k = N$ case, we note that f has a critical point in R_N at the midpoint $x = -\pi$. Since $f''(x) = \cos(x)/2$, we have $f''(-\pi) = -1/2$, and so this point is a local maximum. Thus, the minimum must occur at one of the two endpoints. Noting that f is even about π, the value of f must be the same at both endpoints, so we may as well pick the right endpoint $x = (-N+\alpha)h$. Since $0 \le (1-\alpha)h \le (N-\alpha)h \le \pi$, the value of f at this endpoint is nonnegative, completing the proof. ∎

Lemma 10. *For $K \ge 3$ and $x \in R_0$,*

$$\left|\sin\left(\frac{x-(1-\alpha)h}{2}\right)\sin\left(\frac{x+(1+\alpha)h}{2}\right)\right| \le \left|\sin\left(\frac{h}{2}\right)\right|^2.$$

Proof. As in the previous argument, let $f(x)$ be the expression inside the absolute value signs on the left-hand side of the inequality, and note that $f(x) = \cos(h)/2 - \cos(x+\alpha h)/2$. Since

$$-\pi < -h \le x + \alpha h \le 2\alpha h \le h < \pi,$$

for $x \in R_0$, we have $\cos(h) \le \cos(x+\alpha h)$ for $x \in R_0$, and it follows that f is negative on R_0. Since $\cos(x+\alpha h) \le 1$, we have $0 \ge f(x) \ge \cos(h)/2 - 1/2$. This lower bound is attained for $x \in R_0$ at $x = -\alpha h$. Thus, f attains its maximum absolute value on R_0 at $x = -\alpha h$, and substituting this value into the original expression for f yields the claimed inequality. ∎

Lemma 11. *For $K \ge 3$ and $x \in R_1$, the following inequalities hold:*

$$\begin{aligned}
\left|\sin\left(\frac{x-(1-\alpha)h}{2}\right)\right| &\ge \left|\sin\big((1-\alpha)h\big)\right|,\\
\left|\sin\left(\frac{x+(1-\alpha)h}{2}\right)\right| &\le \left|\sin\left(\frac{(1+2\alpha)h}{2}\right)\right|,\\
\left|\sin\left(\frac{x+(2+\alpha)h}{2}\right)\right| &\le \left|\sin\left(\frac{(1+2\alpha)h}{2}\right)\right|.
\end{aligned}$$

Proof. The first inequality follows from

$$-\pi \le -\frac{3}{2}h \le \frac{x-(1-\alpha)h}{2} \le (\alpha-1)h \le 0,$$

the second from

$$-\pi \le -\frac{(1+2\alpha)h}{2} \le \frac{x+(1-\alpha)h}{2} \le 0,$$

and the third from

$$0 \le \frac{x+(2+\alpha)h}{2} \le \frac{(1+2\alpha)h}{2} \le \pi.$$

□

LEMMA 12. *For* $0 \le k \le N$ *and* $x \in R_k$,

$$[x+(k-\alpha)h]\log\left(-\frac{x+(k-\alpha)h}{2}\right)-[x+(k+1+\alpha)h]\log\left(\frac{x+(k+1+\alpha)h}{2}\right) \le -(1+2\alpha)h\log\left(\frac{(\alpha+1/2)h}{2}\right).$$

Proof. Let $f(x)$ be the expression on the left-hand side of this inequality. The derivative of f is

$$f'(x)=\log\left(-\frac{x+(k-\alpha)h}{x+(k+1+\alpha)h}\right),$$

and this vanishes in R_k only at the point $x=-(k+1/2)h$. Since $f(x)$ tends to $-\infty$ as x approaches the endpoints of R_k, f must assume its maximum value on R_k at this point. Evaluating f at this point yields the right-hand side of the claimed inequality.
□

LEMMA 13. *For* $x \in R_0$,

$$\big(x-(1-\alpha)h\big)\log\left(-\frac{x-(1-\alpha)h}{2}\right) - \big(x+(1+\alpha)h\big)\log\left(\frac{x+(1+\alpha)h}{2}\right) \le -2h\log\left(\frac{h}{2}\right).$$

Proof. As in the previous argument, let $f(x)$ be the expression on the left-hand side of the inequality. We have

$$f'(x)=\log\left(-\frac{x+(\alpha-1)h}{x+(\alpha+1)h}\right),$$

and this vanishes in R_0 only at the point $x=-\alpha h$. Moreover,

$$f''(x)=\frac{2h}{(x+\alpha h)^2-h^2}.$$

The denominator of this function is a quadratic polynomial with positive leading coefficient and zeroes at $(-1-\alpha)h$ and $(1-\alpha)h$. Since $x\in R_0$, we have $(-1-\alpha)h \le x \le \alpha h < (1-\alpha)h$, and it follows that f'' is negative everywhere on R_0. This implies that f has a global maximum on R_0 at the critical point at $-\alpha h$ that we just found. Evaluating $f(-\alpha h)$ produces the right-hand side of the inequality to be established.
□

LEMMA 14. *For sufficiently large* K *and* $x \in R_k$, $0 \le k \le N$, *we have*

$$P_k(x) \le 5\cdot 2^{-K}K$$

for $k=0,1$ *and*

$$P_k(x) \le 3\cdot 2^{-K}K^{2\alpha}\left|\sin\left(\frac{(k+1-2\alpha)h}{2}\right)\sin\left(\frac{(k-1)h}{2}\right)\right|^{\alpha-1/2}$$

for $2 \le k \le N$.

Proof. Let $S_k(x) = \log P_k(x)$. For $1 \le j \le N$, define $a_j(x)$, $b_j(x)$, and $c_j(x)$ by

$$\begin{aligned}
a_j(x) &= \log \left| \sin\left(\frac{x-(j-\alpha)h}{2} \right) \right|, \\
b_j(x) &= \log \left| \sin\left(\frac{x+(j-\alpha)h}{2} \right) \right|, \\
c_j(x) &= \log \left| \sin\left(\frac{x+(j+\alpha)h}{2} \right) \right|.
\end{aligned}$$

For brevity, we will typically suppress the argument when referring to these quantities, writing a_j in place of $a_j(x)$, and so forth. Let

$$\begin{aligned}
A_k(x) &= \sum_{j=1}^{N-1} \frac{1}{2} h(a_j + a_{j+1}), \\
B_k(x) &= \sum_{j=1}^{k-1} \frac{1}{2} h(b_j + b_{j+1}), \\
C_k(x) &= \sum_{j=k+1}^{N-1} \frac{1}{2} h(c_j + c_{j+1}),
\end{aligned}$$

and note that

$$hS_k(x) = A_k(x) + B_k(x) + C_k(x) + \frac{1}{2} h(a_1 + a_N + b_1 + b_k + c_{k+1} + c_N).$$

The sums $A_k(x)$, $B_k(x)$, and $C_k(x)$ are composite trapezoidal rule approximations to the integral of $\log\bigl|\sin\bigl((x+t)/2\bigr)\bigr|$ (with respect to t) over certain subintervals of $[-\pi, \pi]$. Since this function is concave-down everywhere on $[-\pi, \pi]$, these approximations will yield lower bounds on the corresponding integrals [2, p. 54]. More precisely, we have

$$\begin{aligned}
A_k(x) &\le \int_{-(N-\alpha)h}^{-(1-\alpha)h} \log \left| \sin\left(\frac{x+t}{2} \right) \right| \, dt, \\
B_k(x) &\le \int_{(1-\alpha)h}^{(k-\alpha)h} \log \left| \sin\left(\frac{x+t}{2} \right) \right| \, dt, \\
C_k(x) &\le \int_{(k+1+\alpha)h}^{(N+\alpha)h} \log \left| \sin\left(\frac{x+t}{2} \right) \right| \, dt,
\end{aligned}$$

where the inequality for $B_k(x)$ holds for $1 \le k \le N$ and the inequality for $C_k(x)$ holds for $0 \le k \le N-1$. We consider four cases.

Case 1: $2 \le k \le N-1$. In this case, the preceding developments yield

$$\begin{aligned}
hS_k(x) \le \int_{-\pi}^{\pi} - \int_{-\pi}^{-(N-\alpha)h} - \int_{-(1-\alpha)h}^{(1-\alpha)h} - \int_{(k-\alpha)h}^{(k+1+\alpha)h} - \int_{(N+\alpha)h}^{\pi} \log \left| \sin\left(\frac{x+t}{2} \right) \right| \, dt \\
+ \frac{1}{2} h(a_1 + a_N + b_1 + b_k + c_{k+1} + c_N).
\end{aligned}$$

Now we just need to bound the integrals and loose terms on the right-hand side of this inequality. It turns out that the first integral can be evaluated explicitly [3, 4.384-7]:

$$\int_{-\pi}^{\pi} \log \left| \sin\left(\frac{x+t}{2}\right) \right| \, dt = -\pi \log(4). \tag{8}$$

For the second and fifth integrals, we have the following bound, which can be derived by applying the trapezoidal rule to the integral from $(N+\alpha)h$ to $2\pi - (N-\alpha)h$ and using the periodicity of the integrand:

$$-\int_{-\pi}^{-(N-\alpha)h} - \int_{(N+\alpha)h}^{\pi} \log \left| \sin\left(\frac{x+t}{2}\right) \right| \, dt \le -\frac{1}{2}h(a_N + c_N). \tag{9}$$

The fourth integral requires some care, since it has a singularity in the interval of integration at the point $t = -x$. (Recall our assumption that $x \in R_k = [(-k-1-\alpha)h, (-k+\alpha)h]$.) We therefore split the integral into two parts at that point. Noting the expansion

$$\log\bigl(\sin(t)\bigr) = \log(t) - \frac{1}{6}t^2 - \frac{1}{180}t^4 + O(t^6)_{t\to 0^+}, \tag{10}$$

we have

$$-\int_{(k-\alpha)h}^{-x} \log \left| \sin\left(\frac{x+t}{2}\right) \right| \, dt = \bigl(x + (k-\alpha)h\bigr) \left[\log\left(-\frac{x+(k-\alpha)h}{2} \right) - 1 \right] + O(h^3)$$

and

$$\begin{aligned} -\int_{-x}^{(k+1+\alpha)h} &\log \left| \sin\left(\frac{x+t}{2}\right) \right| \, dt \\ &= \bigl(x + (k+1+\alpha)h\bigr) \left[1 - \log\left(\frac{x+(k+1+\alpha)h}{2} \right) \right] + O(h^3). \end{aligned}$$

Adding these expressions together and applying Lemma 12, we obtain

$$-\int_{(k-\alpha)h}^{(k+1+\alpha)h} \log \left| \sin\left(\frac{x+t}{2}\right) \right| \, dt \le (1+2\alpha)h - (1+2\alpha)h \log\left(\frac{(\alpha+1/2)h}{2} \right) + O(h^3).$$

For the third integral, we use another trapezoidal rule bound and combine the result with the loose terms $(1/2)h(a_1 + b_1)$ to yield

$$\begin{aligned} -\int_{-(1-\alpha)h}^{(1-\alpha)h} \log \left| \sin\left(\frac{x+t}{2}\right) \right| \, dt + \frac{1}{2}h(a_1+b_1) &\le \left(\alpha - \frac{1}{2}\right) h(a_1+b_1) \\ &\le \left(\alpha - \frac{1}{2}\right) h \log \left| \sin\left(\frac{(k+1-2\alpha)h}{2} \right) \sin\left(\frac{(k-1)h}{2} \right) \right|, \end{aligned} \tag{11}$$

where the second inequality follows from Lemma 9 and the fact that $\alpha < 1/2$. By Lemma 8 and (10), we now have

$$\frac{1}{2}h(b_k + c_{k+1}) \le h \log\left(\frac{(\alpha+1/2)h}{2} \right) + O(h^3). \tag{12}$$

Putting all of these results together, we conclude that

$$hS_k(x) \leq -\pi \log(4) + \left(\alpha - \frac{1}{2}\right) h \log \left| \sin\left(\frac{(k+1-2\alpha)h}{2}\right) \sin\left(\frac{(k-1)h}{2}\right) \right| \\ - 2\alpha h \log\left(\frac{(\alpha+1/2)h}{2}\right) + (1+2\alpha)h + O(h^3). \quad (13)$$

Dividing through by h, exponentiating, and suitably relaxing the constants that emerge, we obtain the claimed bound in this case.

Case 2: $k = 1$. This case is similar to the previous one. In particular, all the same integral bounds apply except that the second inequality in (11) is meaningless because the argument to the logarithm function vanishes. We replace (11) and (12) with

$$-\int_{-(1-\alpha)h}^{(1-\alpha)h} \log\left|\sin\left(\frac{x+t}{2}\right)\right| dt + \frac{1}{2}h(a_1 + 2b_1 + c_2) \leq \left(\alpha - \frac{1}{2}\right) ha_1 + \frac{1}{2}hc_2 + \alpha h b_1 \\ \leq \left(\alpha - \frac{1}{2}\right) h \log\big((1-\alpha)h\big) + \left(\alpha + \frac{1}{2}\right) h \log\left(\frac{(1+2\alpha)h}{2}\right) + O(h^3),$$

where the second inequality follows from Lemma 11 and (10). Combining this with the other results just established, we obtain

$$hS_1(x) \leq -\pi \log(4) + \left(\alpha - \frac{1}{2}\right) h \log\big((1-\alpha)h\big) + \left(\alpha + \frac{1}{2}\right) h \log\left(\frac{(1+2\alpha)h}{2}\right) \\ - (1+2\alpha)h \log\left(\frac{(\alpha+1/2)h}{2}\right) + (1+2\alpha)h + O(h^3),$$

and this implies the claimed bound for this case.

Case 3: $k = N$. Since $C_N(x)$ has no terms, we have, in this case,

$$hS_N(x) \leq \int_{-\pi}^{\pi} - \int_{-\pi}^{-(N-\alpha)h} - \int_{-(1-\alpha)h}^{(1-\alpha)h} - \int_{(N-\alpha)h}^{\pi} \log\left|\sin\left(\frac{x+t}{2}\right)\right| dt \\ + \frac{1}{2}h(a_1 + a_N + b_1 + b_N).$$

We can bound the third integral and the loose terms $(1/2)h(a_1 + b_1)$ using (11); however, we cannot use (9) to bound the second and fourth integrals. Instead, noting that there is a singularity at $-x$ (or a periodic image thereof) within the domain of integration, we use (10) to find that

$$-\int_{(N-\alpha)h}^{-x} \log\left|\sin\left(\frac{x+t}{2}\right)\right| dt = \big(x + (N-\alpha)h\big)\left[\log\left(-\frac{x+(N-\alpha)h}{2}\right) - 1\right] + O(h^3)$$

and

$$-\int_{-x}^{2\pi-(N-\alpha)h} \log\left|\sin\left(\frac{x+t}{2}\right)\right| dt \\ = \big(2\pi + x - (N-\alpha)h\big)\left[1 - \log\left(\frac{2\pi + x - (N-\alpha)h}{2}\right)\right] + O(h^3).$$

Noting that $2\pi + x - (N-\alpha)h = x + (N+1+\alpha)h$, we can add these together and use periodicity and Lemma 12 to obtain

$$-\int_{-\pi}^{-(N-\alpha)h} - \int_{(N-\alpha)h}^{\pi} \log\left|\sin\left(\frac{x+t}{2}\right)\right| dt$$
$$\leq (1+2\alpha)h - (1+2\alpha)h\log\left(\frac{(\alpha+1/2)h}{2}\right) + O(h^3).$$

By the same identity, Lemma 8, and (10), we have

$$\frac{1}{2}h(a_N + b_N) \leq h\log\left(\frac{(\alpha+1/2)h}{2}\right) + O(h^3).$$

Putting everything together, we arrive once again at (13), which finishes the argument in this case.

Case 4: $k = 0$. As $B_0(x)$ has no terms, we have

$$hS_0(x) \leq \int_{-\pi}^{\pi} - \int_{-\pi}^{-(N-\alpha)h} - \int_{-(1-\alpha)h}^{(1+\alpha)h} - \int_{(N+\alpha)h}^{\pi} \log\left|\sin\left(\frac{x+t}{2}\right)\right| dt$$
$$+ \frac{1}{2}h(a_1 + a_N + c_1 + c_N).$$

We can take care of the second and fourth integrals and the loose terms $(1/2)h(a_N + c_N)$ using (9). For the third integral, noting that $-x$ lies in the interval of integration, we use (10) one more time to conclude that

$$-\int_{-(1-\alpha)h}^{-x} \log\left|\sin\left(\frac{x+t}{2}\right)\right| dt = \big(x-(1-\alpha)h\big)\left[\log\left(-\frac{x-(1-\alpha)h}{2}\right) - 1\right] + O(h^3)$$

and

$$-\int_{-x}^{(1+\alpha)h} \log\left|\sin\left(\frac{x+t}{2}\right)\right| dt = \big(x+(1+\alpha)h\big)\left[1 - \log\left(\frac{x+(1+\alpha)h}{2}\right)\right] + O(h^3).$$

Adding these together and using Lemma 13, we have

$$-\int_{-(1-\alpha)h}^{(1+\alpha)h} \log\left|\sin\left(\frac{x+t}{2}\right)\right| dt \leq -2h\log\left(\frac{h}{2}\right) + 2h + O(h^3).$$

By Lemma 10 and (10), we have

$$\frac{1}{2}h(a_1 + c_1) \leq h\log\left(\frac{h}{2}\right) + O(h^3).$$

Assembling all these facts, we find that

$$hS_0(x) \leq -\pi\log(4) - h\log\left(\frac{h}{2}\right) + 2h + O(h^3),$$

and upon dividing through by h, exponentiating, and adjusting the constant factors that arise, we obtain the desired result.

All cases have now been handled. The proof is complete. $\square$

Next, we bound Q_k. The result we need is the following:

LEMMA 15. *For sufficiently large K,*

$$Q_k \geq (1-2\alpha)2^{-K}K^{1-2\alpha}\left|\sin\left(\frac{(k+1/2+\alpha)h}{2}\right)\right|^{-2\alpha}.$$

Proof. The proof is similar in structure to that of Lemma 14. Let $S_k = \log(Q_k)$, so that

$$S_k = \sum_{j=1}^{N}\log\left|\sin\left(\frac{(2\alpha-j)h}{2}\right)\right| + \sum_{j=1}^{k}\log\left|\sin\left(\frac{jh}{2}\right)\right| + \sum_{j=k+1}^{N}\log\left|\sin\left(\frac{(2\alpha+j)h}{2}\right)\right|. \tag{14}$$

We will bound S_k using integrals of $\log|\sin(t/2)|$, just as before; but this time, since we seek a lower bound, we use the midpoint rule instead of the trapezoidal rule [2, p. 54]. Assuming $0 \leq \alpha \leq 1/4$, we have

$$hS_k \geq \int_{-\pi}^{\pi} - \int_{(2\alpha-1/2)h}^{h/2} - \int_{(k+1/2)h}^{(k+2\alpha+1/2)h}\log\left|\sin\left(\frac{t}{2}\right)\right|\,dt. \tag{15}$$

We evaluated the first integral in (8), above. We bound the third integral using the midpoint rule:

$$-\int_{(k+1/2)h}^{(k+2\alpha+1/2)h}\log\left|\sin\left(\frac{t}{2}\right)\right|\,dt \geq -2\alpha h\log\left|\sin\left(\frac{(k+1/2+\alpha)h}{2}\right)\right|.$$

For the second integral, we split the interval of integration at the singularity at 0 and use (10) to compute

$$\begin{aligned}-\int_{(2\alpha-1/2)h}^{h/2}\log\left|\sin\left(\frac{t}{2}\right)\right|\,dt = \left(2\alpha-\frac{1}{2}\right)h\log\left(\frac{(1/2-2\alpha)h}{2}\right)\\ -\frac{h}{2}\log\left(\frac{h}{4}\right)+(1-2\alpha)h+O(h^3).\end{aligned}$$

From these results, it follows that

$$\begin{aligned}hS_k \geq{}& -\pi\log(4) - 2\alpha h\log\left|\sin\left(\frac{(k+1/2+\alpha)h}{2}\right)\right|\\ &+\left(2\alpha-\frac{1}{2}\right)h\log\left(\frac{(1/2-2\alpha)h}{2}\right)-\frac{h}{2}\log\left(\frac{h}{4}\right)+(1-2\alpha)h+O(h^3).\end{aligned}$$

Dividing through by h, exponentiating, and suitably adjusting the constant factors that arise, we obtain the claimed result.

If $1/4 < \alpha < 1/2$, the argument is similar except that we have to track the $j = 1$ term in the first sum in the definition of S_k independently. We write

$$hS_k \geq \int_{-\pi}^{\pi} - \int_{(2\alpha-3/2)h}^{h/2} - \int_{(k+1/2)h}^{(k+2\alpha+1/2)h}\log\left|\sin\left(\frac{t}{2}\right)\right|\,dt + h\log\left|\sin\left(\frac{(2\alpha-1)h}{2}\right)\right|.$$

Using (10) one last time, we compute

$$h\log\left|\sin\left(\frac{(2\alpha-1)h}{2}\right)\right| = h\log\left(\frac{(1-2\alpha)h}{2}\right)+O(h^3).$$

and

$$-\int_{(2\alpha-3/2)h}^{h/2} \log\left|\sin\left(\frac{t}{2}\right)\right| dt = \left(2\alpha - \frac{3}{2}\right)\log\left(\frac{(3/2-2\alpha)h}{2}\right) - \frac{h}{2}\log\left(\frac{h}{4}\right) + 2(1-\alpha)h + O(h^3).$$

Therefore,

$$hS_k \geq -\pi\log(4) - 2\alpha h\log\left|\sin\left(\frac{(k+1/2+\alpha)h}{2}\right)\right| + h\log\left(\frac{(1-2\alpha)h}{2}\right) + \left(2\alpha-\frac{3}{2}\right)\log\left(\frac{(3/2-2\alpha)h}{2}\right) - \frac{h}{2}\log\left(\frac{h}{4}\right) + 2(1-\alpha)h + O(h^3).$$

and this implies the claimed bound in the usual way. ▯

Note that the proof of this lemma shows that the constant $1-2\alpha$ can be dropped from the bound when $0 \leq \alpha \leq 1/4$. We have chosen for simplicity to include it in this case anyway because omitting it will at best improve our final results by a small constant factor.

At this point, we have all that we need to bound $|\widetilde{\ell}_0(x)|$ uniformly for $x \in [-\pi,\pi]$ and independent of the points $\widetilde{x}_j$: evaluate the bounds on M_k for $x \in [-\pi,0]\cap R_k$ given by Lemmas 14 and 15, and take the maximum over k. Lemma 7 shows that the result bounds $|\widetilde{\ell}_0(x)|$ for $x \in [-\pi,0]$. By symmetry, the same bound must hold for $x \in [0,\pi]$ as well. Even further, by considering circular rotations of the points $\widetilde{x}_j$, the bound can be seen to apply to $|\widetilde{\ell}_k(x)|$ for $k \neq 0$. Therefore, by (3), we could bound $\widetilde{\Lambda}_N$ by multiplying the bound on $|\widetilde{\ell}_0(x)|$ by K.

We can do better than this, however, because Lemmas 7 and 14 retain some information about how $|\widetilde{\ell}_0(x)|$ varies with x through the hypothesis that $x \in R_k$. We can use this information to get a better bound on $|\widetilde{\ell}_k(x)|$ for $k \neq 0$ than the one just described. The result we need is given by the following lemma, which we could have proved earlier but have delayed until now.

LEMMA 16. *If $x \in R_p^*$, $0 \leq p \leq N$, then for $-N \leq k \leq N$,*

$$|\widetilde{\ell}_k(x)| \leq \begin{cases} \max(M_{-(p+k)}, M_{-(p+k+1)}, M_{-(p+k+2)}) & -N \leq p+k \leq -2 \\ \max(M_0, M_1) & p+k = -1, 0 \\ \max(M_{p+k-1}, M_{p+k}, M_{p+k+1}) & 1 \leq p+k \leq N-1 \\ \max(M_{N-1}, M_N) & p+k = N \\ \max(M_{K-(p+k)}, M_{K-(p+k+1)}, M_{K-(p+k+2)}) & N+1 \leq p+k \leq 2N-1 \\ \max(M_0, M_1) & p+k = 2N. \end{cases} \tag{16}$$

Proof. For $k=0$, the result follows from Lemma 7, which actually gives a stronger bound. The proof for $k \neq 0$ is ultimately just a matter of reducing it to the $k=0$ case by exploiting circular and reflectional symmetry; however, there are some subtleties, so we will spell out the details to make things clear. Note that $x \in R_p^*$ implies $x \in R_p$ by Lemma 4.

First, suppose that $1 \le k \le N$. Then, $1 \le p+k \le 2N$, so only the last four cases in (16) are relevant. Let

$$\hat{x}_j = \begin{cases} \widetilde{x}_{j+k} - kh & -N \le j \le N-k \\ \widetilde{x}_{j+k-K} + 2\pi - kh & N-k+1 \le j \le N. \end{cases}$$

These points are just a circular shift in $[-\pi, \pi]$ of the points $\widetilde{x}_j$ by kh. It follows that $\widetilde{\ell}_k(x) = \hat{\ell}_0(x - kh)$, where $\hat{\ell}_0$ is the (trigonometric) Lagrange basis function for the points $\hat{x}_j$ that takes on the value 1 at $\hat{x}_0$. One can easily check that

$$\hat{x}_j = \begin{cases} x_j + t_{j+k}h & -N \le j \le N-k \\ x_j + t_{j+k-K}h & N-k+1 \le j \le N, \end{cases}$$

where the x_j are the equispaced points (1), and the t_j are defined by (2). Thus, the points $\hat{x}_j$ constitute a set of perturbed equispaced points of the sort that we have been considering. In particular, we can use Lemma 7 to bound $\hat{\ell}_0(x - kh)$ and hence $\widetilde{\ell}_k(x)$. We consider several cases.

Case 1: $1 \le p + k \le N - 1$. Since $x \in R_p$, it follows that $x - kh \in R_{p+k}$, which means that $x - kh$ must belong to one of R^*_{p+k-1}, R^*_{p+k}, and R^*_{p+k+1}, again by Lemma 4. By Lemma 7, $|\hat{\ell}_0(x - kh)| \le \max(M_{p+k-1}, M_{p+k}, M_{p+k+1})$.

Case 2: $p + k = N$ and $(-p - 1/2)h \le x \le (-p + \alpha)h$. We have $x - kh \in R_N$. Moreover, $x - kh \ge (-p - k - 1/2)h = (-N - 1/2)h = -\pi$, so $x - kh \in [-\pi, 0] \cap R_N$. Thus, $x - kh$ belongs to either R^*_N or R^*_{N-1} by Lemma 4, and so Lemma 7 gives $|\hat{\ell}_0(x - kh)| \le \max(M_{N-1}, M_N)$.

Case 3: $p + k = N$ and $(-p - 1 - \alpha)h \le x < (-p - 1/2)h$. Again, we have $x - kh \in R_N$, but this time, $x - kh < \pi$. Nevertheless, $\hat{\ell}_0(x - kh) = \hat{\ell}_0(x - kh + 2\pi)$, and $x - kh + 2\pi \in [0, \pi] \cap -R_N$. By reflecting the problem about 0 (i.e., replacing $\hat{x}_j$ with $-\hat{x}_j$ for each j and $x - kh + 2\pi$ by $-(x - kh + 2\pi) \in [-\pi, 0] \cap R_N$), and applying Lemma 7, we obtain $|\hat{\ell}_0(x - kh)| \le \max(M_{N-1}, M_N)$ as in the previous case.

Case 4: $N + 1 \le p + k \le 2N - 1$. Just as in the previous case, we will look not at $\hat{\ell}_0(x - kh)$ but at $\hat{\ell}_0(x - kh + 2\pi)$. Noting that $2\pi = Kh$, we see that $x - kh + 2\pi \in -R_{K-(p+k+1)}$. Since $x \ge -\pi$ and $k \le N$, we have $x - kh + 2\pi \ge -\pi + (K - N)h = h/2 > 0$. Thus, $x - kh + 2\pi \in [0, \pi] \cap -R_{K-(p+k+1)}$. Reflecting about 0 as was done in the previous case and noting that $-(x - kh + 2\pi)$ must belong to one of $R^*_{K-(p+k)}$, $R^*_{K-(p+k+1)}$, and $R^*_{K-(p+k+2)}$ by Lemma 4, we may apply Lemma 7 to conclude that $|\hat{\ell}_0(x - kh)| \le \max(M_{K-(p+k)}, M_{K-(p+k+1)}, M_{K-(p+k+2)})$.

Case 5: $p + k = 2N$. This is handled exactly the same as the previous case except that since $x - kh + 2\pi \in [0, \pi] \cap -R_0$, we have that $-(x - kh + 2\pi)$ can belong only to one of R^*_0 and R^*_1. Therefore, $|\hat{\ell}_0(x - kh)| \le \max(M_0, M_1)$.

For $-N \le k \le -1$, the argument is similar. In this case, the circularly shifted points $\hat{x}_j$ are

$$\hat{x}_j = \begin{cases} \widetilde{x}_{j+k} - kh & -N-k \le j \le N \\ \widetilde{x}_{j+k+K} - 2\pi - kh & -N \le j \le -N-k-1, \end{cases}$$

so that

$$\hat{x}_j = \begin{cases} x_j + t_{j+k}h & -N-k \le j \le N \\ x_j + t_{j+k+K}h & -N \le j \le -N-k-1. \end{cases}$$

Just as before, we have $\widetilde{\ell}_k(x) = \hat{\ell}_0(x - kh)$. Noting that $-N \le p + k \le N - 1$, the proof again breaks into cases as follows.

Case 1: $1 \le p+k \le N-1$. Just as in the previous Case 1, we have $x-kh \in R_{p+k}$, and the result follows in exactly the same way.

Case 2: $p + k = 0$ *and* $(-p - 1 - \alpha)h \le x \le -ph$. Here, $x - kh \in R_0$, and the restriction on x forces $x - kh \le 0$, so in fact, $x - kh \in [-\pi, 0] \cap R_0$. Therefore, $x - kh$ belongs to one of R_0^* and R_1^* by Lemma 4, and so by Lemma 7 we have $|\hat{\ell}_0(x - kh)| \le \max(M_0, M_1)$.

Case 3: $p + k = 0$ *and* $-ph < x \le (-p + \alpha)h$. Now $x - kh \in R_0$, but $0 < x - kh \le \alpha h$. To bound $\hat{\ell}_0(x - kh)$ in this case, we reflect the problem about 0 as we did in some of the cases for positive k above. Since $[-\alpha h, \alpha h] \subset R_0$, we have $-(x - kh) \in [-\pi, 0] \cap R_0$, and so Lemma 7 tells us that $|\hat{\ell}_0(x - kh)| \le \max(M_0, M_1)$ once again.

Case 4: $p + k = -1$ *and* $(-p - 1 - \alpha)h \le x \le (-p - 1)h$. In this case, $x - kh \in [-\alpha h, 0]$ and hence belongs to $[-\pi, 0] \cap R_0$. Applying Lemma 7, we have $|\hat{\ell}_0(x-kh)| \le \max(M_0, M_1)$ just as in the previous two cases.

Case 5: $p+k = -1$ *and* $(-p-1)h < x \le (-p+\alpha)h$. Now, $x-kh \in [0, \pi] \cap -R_0$. Reflecting in 0 and using Lemma 7 yet again, we have $|\hat{\ell}_0(x - kh)| \le \max(M_0, M_1)$.

Case 6: $-N \le p+k \le -2$. We have $x-kh \in -R_{-(p+k+1)}$. Since $-R_{-(p+k+1)} \subset [0, \pi]$, we reflect in 0 and observe that, by Lemma 4, $-(x - kh)$ belongs to one of $R^*_{-(p+k)}$, $R^*_{-(p+k+1)}$, and $R^*_{-(p+k+2)}$. Applying Lemma 7 one last time, we obtain $|\hat{\ell}_0(x - kh)| \le \max(M_{-(p+k)}, M_{-(p+k+1)}, M_{-(p+k+2)})$.

All cases have been handled. The proof is finished. ∎

The point of Lemma 16 is that it allows us to bound $\widetilde{\Lambda}_N$ by summing the bounds of Lemma 7 over k instead of maximizing them over k and multiplying by K as described previously.

LEMMA 17. *We have*

$$\widetilde{\Lambda}_N \le 9 \sum_{k=0}^{N} M_k. \tag{17}$$

Proof. Suppose that $x \in [-\pi, 0] \cap R_p^*$, $0 \le p \le N$. We can use Lemma 16 to bound the sum in (3) for this value of x by summing the right-hand side of (16) over $-N \le k \le N$. This is equivalent to summing it over the values of $p + k$ such that $-N + p \le p + k \le N + p$, and this is certainly bounded above by the sum over the larger range $-N \le p + k \le 2N$. Writing j in place of $p + k$, it follows that

$$\begin{aligned}\sum_{k=-N}^{N} |\widetilde{\ell}_k(x)| \le \sum_{j=-N}^{-2} \max(M_{-j}, M_{-(j+1)}, M_{-(j+2)}) + \sum_{j=1}^{N-1} \max(M_{j-1}, M_j, M_{j+1}) \\ + \sum_{j=N+1}^{2N-1} \max(M_{K-j}, M_{K-(j+1)}, M_{K-(j+2)}) \\ + 3\max(M_0, M_1) + \max(M_{N-1}, M_N).\end{aligned}$$

Since $\max(a,b) \le a+b$ when $a, b \ge 0$, we can convert the maxima into sums to obtain

$$\sum_{k=-N}^{N} |\widetilde{\ell}_k(x)| \le \sum_{j=2}^{N} M_j + \sum_{j=1}^{N-1} M_j + \sum_{j=0}^{N-2} M_j + \sum_{j=0}^{N-2} M_j + \sum_{j=1}^{N-1} M_j + \sum_{j=2}^{N} M_j$$
$$+ \sum_{j=2}^{N} M_j + \sum_{j=1}^{N-1} M_j + \sum_{j=0}^{N-2} M_j + 3M_0 + 3M_1 + M_{N-1} + M_N$$

after simplifying the indices of summation. We immediately obtain

$$\sum_{k=-N}^{N} |\widetilde{\ell}_k(x)| \le 9 \sum_{j=0}^{N} M_j.$$

Since the right-hand side of this inequality is independent of p, this bound actually holds for all $x \in [-\pi, 0]$. Even further, since the M_j are independent of both x and the points (2), by symmetry, it holds for all $x \in [-\pi, \pi]$. The result now follows from (3). □

We can now prove Theorem 2.1 from the main article.

Proof of Theorem 2.1. We use Lemmas 14 and 15 to bound the right-hand side of (17). For K sufficiently large and $k = 0, 1$, we have

$$M_k \le \frac{5}{1-2\alpha} K^{2\alpha} \left| \sin\left(\frac{(k+1/2+\alpha)\pi}{K} \right) \right|^{2\alpha}$$
$$\le \frac{5}{1-2\alpha} K^{2\alpha} \left| \frac{(k+1/2+\alpha)\pi}{K} \right|^{2\alpha} \le \frac{10\pi}{1-2\alpha}, \quad (18)$$

while for $2 \le k \le N$,

$$M_k \le \frac{3}{1-2\alpha} K^{4\alpha-1} \frac{\left| \sin\left(\frac{(k+1/2+\alpha)\pi}{K} \right) \right|^{2\alpha}}{\left| \sin\left(\frac{(k+1-2\alpha)\pi}{K} \right) \sin\left(\frac{(k-1)\pi}{K} \right) \right|^{1/2-\alpha}}$$
$$\le \frac{3}{1-2\alpha} K^{4\alpha-1} \frac{\left| \sin\left(\frac{(k+1/2+\alpha)\pi}{K} \right) \right|^{2\alpha}}{\left| \sin\left(\frac{(k-1)\pi}{K} \right) \right|^{1-2\alpha}}.$$

In deriving the last expression, we have used the inequality

$$\left| \sin\left(\frac{(k+1-2\alpha)\pi}{K} \right) \right| \ge \left| \sin\left(\frac{(k-1)\pi}{K} \right) \right|,$$

which clearly holds for $2 \le k \le N-1$ and for $k = N$ with $1/4 \le \alpha < 1/2$, since in those cases, $(k+1-2\alpha)\pi/K \in [0, \pi/2]$, and $k+1-2\alpha \ge k > k-1$. To see that it holds for $k = N$ with $0 < \alpha < 1/4$ as well, note that in this case

$$\sin\left(\frac{(N+1-2\alpha)\pi}{K} \right) = \sin\left(\frac{(N+2\alpha)\pi}{K} \right)$$

by the symmetry of sine about $\pi/2$. Since $N+2\alpha \in [0,\pi/2]$ and $N+2\alpha > N-1$, the inequality follows.

Using the inequalities $|\sin(x)| \le |x|$ for $x \in \mathbb{R}$ and $|\sin(x)| \ge (2/\pi)|x|$ for $|x| \le \pi/2$, we can simplify the bound on M_k for $2 \le k \le N$ even further to

$$M_k \le \frac{3}{1-2\alpha} K^{4\alpha-1} \frac{\left|\frac{(k+1/2+\alpha)\pi}{K}\right|^{2\alpha}}{\left|\frac{2(k-1)}{K}\right|^{1-2\alpha}} \le \frac{3\pi}{1-2\alpha}\frac{(k+1)^{2\alpha}}{(k-1)^{1-2\alpha}}. \tag{19}$$

The result now follows from summing the bounds on the M_k established in (18) and (19) and bounding the sum by interpreting it as a midpoint rule approximation[2] to the integral of a function that is concave-up (note that $N+1/2 = K/2$):

$$\begin{aligned}\sum_{k=2}^{N} \frac{(k+1)^{2\alpha}}{(k-1)^{1-2\alpha}} &\le \int_{3/2}^{N+1/2} \frac{(x+1)^{2\alpha}}{(x-1)^{1-2\alpha}}\,dx \\ &\le (K/2+1)^{2\alpha}\int_{3/2}^{K/2}\frac{dx}{(x-1)^{1-2\alpha}} = \frac{(K^2/4-1)^{2\alpha}-(K/4+1/2)^{2\alpha}}{2\alpha} \\ &\le \frac{(K^2/4)^{2\alpha}-(K/4)^{2\alpha}}{2\alpha} \le \frac{K^{4\alpha}-1}{2\alpha}. \end{aligned}\tag{20}$$

The bound (20) is given in terms of $K = 2N+1$. To obtain the result in terms of N stated in the main article, we seek a universal constant C' such that $(2N+1)^{4\alpha}-1 \le C'(N^{4\alpha}-1)$ for $N \ge 2$ and $0 < \alpha < 1/2$. A straightforward computation using calculus shows that $C' = 8$ will suffice. $\square$

We close with a word about why our argument falls short of establishing the stronger bound on $\widetilde{\Lambda}_N$ that we conjecture involving $N^{2\alpha}$ instead of $N^{4\alpha}$. As summarized in the opening paragraphs of this appendix, our argument proceeds by choosing the perturbed points $\widetilde{x}_j$ to maximize $|\widetilde{\ell}_k|$ for a fixed value of k, bounding the maximum, and then summing the bounds. This is a different (and easier) problem than choosing the points to maximize the sum $\sum_{k=-N}^{N} |\widetilde{\ell}_k|$ and bounding that maximum instead.

In symbols, our argument bounds $\widetilde{\Lambda}_N$ by bounding the rightmost expression in the following chain of inequalities:

$$\widetilde{\Lambda}_N \le \max_{\widetilde{x}_{-N},\ldots,\widetilde{x}_N} \max_{x\in[-\pi,\pi]} \sum_{k=-N}^{N} |\widetilde{\ell}_k(x)| \le \max_{x\in[-\pi,\pi]} \sum_{k=-N}^{N} \max_{\widetilde{x}_{-N},\ldots,\widetilde{x}_N} |\widetilde{\ell}_k(x)|.$$

The loss enters in the passage to the rightmost expression from the one in the middle. To prove the stronger bound, one needs to consider the $|\widetilde{\ell}_k|$ all together at once in the sum instead of individually as we have done here.

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