A Dynamic Approach to the Dividend Discount Model

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We derive a dynamic model of the firm with endogenous investment and leverage ratio within the framework of the dividend discount model (DDM). Our valuation model incorporates two relevant components, namely, managerial flexibility and long-run growth. We dispense with any utility specification capturing the preferences of shareholders and obtain closed-form solutions for the firm problem. A standard parameterization suggests that the value of the real options and long-run growth opportunities can easily represent more than 8% and 10% of share price, respectively. We also find that these two components of the stock price are both complements and countercyclical. We finally identify industries where valuation models that do not incorporate these features can lead to considerable underpricing of securities.

Keywords: Dividend Discount Model; Gordon Growth Model; Real Options; Long-Run Growth; Dynamic Programming.

JEL Classification: G31, G32

1. Introduction

The traditional dividend discount model (DDM) has been used as one of the main firm valuation tools for decades. In its simplest formulation, this
methodology consists in projecting the level of economic activity of the firm in the future years and then discounting the resulting cash flows to the present. One of the fundamental features of this model is that it does not require the specification of a utility function that represents shareholder preferences. This important simplification is possible due to the separation principle (see, e.g., Copeland et al., 2005). Another attractive feature of the model is that it can handle long-run growth (e.g., the Gordon Growth Model).

We derive a dynamic model of the firm in the spirit of the DDM. In our model, the firm chooses the level of investment and how to finance those assets in each period. In turn, these optimal decisions determine the future sequence of expected cash flows in closed-form. While preserving the two main features of the DDM, our model explicitly incorporates the value of various real options available to the firm. We use the model to decompose the stock price in its primitive components: The book value of equity plus the future excess profits stemming from current assets, real options, and long-run growth. We then ascertain the importance of each of these components for different industries and explain the firm characteristics with the largest effect on each of them. This exercise can be particularly useful to guide investment strategies employed by money managers. Finally, we show that the value of managerial flexibility and secular growth are both complements and countercyclical.\(^1\)

We interpret managerial flexibility as the possibility of the firm to adapt itself to the future realization of uncertainty. In other words, we introduce the ability of the CEO to adapt the firm during the business cycle in order to capitalize on good fortune or to mitigate loss. We follow the corporate finance literature and model the business cycle as mean-reverting profit shocks. During these short-term oscillations, when profits go up, the manager increases firm assets to take advantage of the incremental value (the expansion option). Because profits are persistent, as long as they continue to be high, the CEO will maintain the firm large (the extension option). After profits go down, the CEO responds by shrinking firm assets to reduce the impact of bad results (the contraction option). While profit shocks continue to be bad, the manager postpones new investments, keeping the firm small until better times come (the deferral option). Thus, the real options emerge

\(^1\)Lazzati and Menichini (2014b) show that this model can, indeed, be successfully used for firm valuation. For instance, it can explain a large proportion of the cross-sectional variation in stock prices (above 70%).
naturally in our model because the CEO has the possibility to adjust the firm over the business cycle in order to maximize share price. Though our model incorporates four of the most important real options available to the firm, there are others that we do not include (e.g., the possibility to exit the market, buy another firm, etc.). We believe our model captures an important proportion of the value of managerial flexibility.

Many firms have experienced secular growth with an overlay of short-term cycles. Typically, they are manufacturers of foods, drinks, drugs, and cigarettes, among others. Despite its relevance, virtually all dynamic programming models of the firm in corporate finance have been developed without the possibility of long-run growth. In these models, relevant variables simply fluctuate over time around a long-run constant. We introduce long-run growth into the dynamic model by assuming that the productivity of capital increases over time because of, for instance, technological progress. As a consequence, firm assets in our model fluctuate over the business cycle around a growing long-run trend. From a theoretical standpoint, introducing secular growth requires a specific normalization of growing variables that we describe along the analysis.

We use the dynamic model to ascertain the proportion of share price that is represented by the value of managerial flexibility and long-run growth. We parameterize the model with standard values used in the corporate finance literature and find that the value of the real options explains around 8% of the market value of equity. If we consider that in reality the firm also has other real options, such as the exit option, the conclusion is that the value of managerial flexibility can account for a substantial proportion of share price. The present value of growth opportunities is also an important component as it represents another 10% of market equity. Furthermore, both proportions turn out to be countercyclical over the business cycle because the value of the real options and secular growth are less sensitive to profit shocks than is share price. This result suggests that the undervaluation problem of a pricing model that does not incorporate the real options is more severe during economic recessions than expansions. We also find that the value of managerial flexibility increases with the possibility to grow in the long-run. That is, these two components are complementary.

We finally do a comparative statics analysis of managerial flexibility and secular growth, as well as a cross-sectional comparison of different industries. We find that the elasticity of the capital input in the production function is the main determinant of the value of the real options. This characteristic has received a lot of attention in the Industrial Organization.
literature. Our results highlight its relevance for managerial flexibility. As it has been previously documented, we also find that the volatility and persistence of profit shocks are important determinants of the value of the real options.\textsuperscript{2} Not surprisingly, we find that the market cost of capital and the growth rate are the two most important determinants of the present value of growth opportunities. Finally, we characterize the industries where managerial flexibility and/or long-run growth play an essential role in firm value. For instance, we find that for firms in the Oil and Gas Extraction industry the real options is a key component of the stock price, while for corporations in the Chemical industry the expected future growth represents a large proportion of share price. For these types of industries the pricing model we propose has the largest benefits.

1.1. Literature review

Discounted cash flow models in general, and the DDM in particular, can be traced back to Williams (1938). Copeland \textit{et al.} (2005) and Damodaran (2011), among many others, present a broad and updated review of the large body of literature about these traditional valuation models. We contribute to this literature by deriving a dynamic specification of that canonical model that makes explicit the value of the real options available to the firm. Since we obtain the full sequence of future cash-flows in closed-form from optimal firm behavior, we provide some microeconomic foundations for the DDM.

Several papers in corporate finance use dynamic programming models of the firm to explain firm behavior. For instance, Moyen (2004, 2007), Hennessy and Whited (2005, 2007), Gamba and Triantis (2008), Tserlukevich (2008), Gomes and Schmid (2010), and DeAngelo \textit{et al.} (2011), among others.\textsuperscript{3} Most of these papers assume shareholders are risk-neutral. This assumption allows them to use linear utility functions to capture shareholders’ preferences. We develop the dynamic model of the firm within the framework of the DDM, which assumes the separation principle holds and, therefore, does not require any utility specification. According to this principle, in the context of a perfect capital market, the manager maximizes the lifetime expected utility of all current shareholders by maximizing the price of current shares (i.e., shareholders’ wealth) independently of their individual subjective preferences. In this context, the CEO needs to know only the appropriate cost of capital of the firm to use as the discount rate. Thus,

\textsuperscript{2}See, for instance, Dixit and Pindyck (1994).

\textsuperscript{3}See Strebulaev and Whited (2012) for a comprehensive review of this literature.
by eliminating the assumption of risk-neutrality and using a risk-adjusted discount rate, our model becomes a useful asset pricing tool. The possibility of using dynamic programming models of the firm as valuation models is suggested by Dixit and Pindyck (1994).

In addition, our work improves the extant literature on dynamic programming models of the firm by allowing for long-run growth. As we show in our study, incorporating this feature into the model is important for firms in which future investment opportunities account for a large part of the stock price, such as companies in the Chemical industry. On the other hand, the previous papers contain other realistic features (i.e., diverse kinds of frictions, such as costly adjustment of capital, costly issuance of debt and/or equity, etc.) that we omit in this work. These features could be, nevertheless, introduced into the present model for asset pricing purposes. We keep the model simple for three reasons. First, it allows us to describe the primitive forces driving the results in a transparent way. Second, in a related paper, Lazzati and Menichini (2014a) show that this simple model is able to rationalize several key empirical regularities reported by the literature in corporate finance (e.g., the negative relation between profitability and leverage, the existence of zero-debt firms, and the inverse association between dividends and investment-cash flow sensitivities). Third, Lazzati and Menichini (2014b) show that this model produces very successful results for firm valuation.

Real options have been studied extensively in the economics and finance literature. For example, McDonald and Siegel (1986) investigate the optimal timing of irreversible investments. Pindyck (1988) studies how randomness and irreversibility of investments impact the value of the real options held by the firm, as well as firm’s capacity and market value. Dixit (1989) develops a model of investment decisions with “hysteresis” in a stochastic setting. Kulatilaka and Marcus (1992) show the systematic undervaluation of investments produced by the traditional discounted cash flow methodology in the presence of real options. Trigeorgis (1993a) shows how to value the operating options of the firm, such as the option to defer, expand, and abandon investments. In spite of the significant theoretical contribution of this literature, most papers focus on the valuation of individual real options in isolation. Our paper contributes to this strand of literature by providing a

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4 The list of important papers in this literature is vast, including, but not limited to, Baldwin and Ruback (1986), Paddock et al. (1988), Ingersoll and Ross (1992), Quigg (1993), Moel and Tufano (2002), and Lee et al. (2009).

dynamic programming model of the firm based on the canonical DDM that allows us to calculate the total value of several real options simultaneously (i.e., expansion, extension, contraction, and deferral). In addition, we show that the value of the real options is countercyclical and complementary to the value of secular growth.

The paper is organized as follows. In Sec. 2, we propose a dynamic version of the DDM and derive its analytic solution. In Sec. 3, we separate the different components of the market value of equity and study the economic importance of the real options as well as long-run growth. The comparative statics analysis of the different components of share price and the cross-sectional study are in Sec. 4. Section 5 concludes. Appendix A contains the proofs, while Appendix B describes the calibration of model parameters.

2. A Dynamic Model of the Firm

In this section, we propose a dynamic valuation model that incorporates two key aspects of the firm, namely, managerial flexibility and secular growth. We view our model as a dynamic version of the standard DDM that could be used as a powerful valuation tool. We obtain closed-form solutions for the firm problem, which help us to overcome the black box criticism often received by structural models of the firm, as it becomes clear the role played by each of the different model parts in explaining firm behavior. Furthermore, the analytic results are fundamental for the decomposition of share price in its primitive components that we perform in Sec. 3.

In our model, time is discrete and the firm makes decisions at the end of every time period (e.g., quarter, year, etc.). The life horizon of the firm is infinite, which implies that shareholders believe it will run forever. We use dynamic programming as the solution concept to derive the optimal policies as well as the market value of equity. In order to handle long-run growth, we write a tilde on \( X \) (i.e., \( \tilde{X} \)) to indicate that the variable is growing in the long-run. The risk-free rate of interest in the economy is indicated by \( r_f \).

The CEO makes investment and financing decisions such that the market value of equity is maximized. Variable \( \tilde{K}_t \) represents the book value of assets in period \( t \). The assets of the firm \( \tilde{K}_t \) will vary (i.e., increase or decrease) over time, reflecting the investment decisions. In each period, installed capital depreciates at constant rate \( \delta > 0 \). The debt of the firm in

\[6\text{Specifically, we use discrete-time, infinite-horizon, stochastic dynamic programming.}\]

\[7\text{We use the terms market value of equity, share price, and stock price interchangeably in this paper.}\]
period \( t \), \( \tilde{D}_t \), matures in one period and is rolled over at the end of every period.\(^8\) We assume the coupon rate \( c_B \) equals the market cost of debt \( r_B \), which implies that book value of debt \( \tilde{D}_t \) equals market value of debt \( \tilde{B}_t \).\(^9\) The amount of outstanding debt \( \tilde{B}_t \) will increase and decrease over time according to financing decisions. Similar to DeAngelo et al. (2011), we assume debt remains risk-free over the firm’s life. This assumption helps us to rationalize the phenomenon of debt conservatism documented by Graham (2000), as shown in Lazzati and Menichini (2014a). Therefore, the firm can always repay its debt in full, and the market cost of debt \( r_B \) equals the risk-free interest rate \( r_f \).\(^{10}\)

We introduce randomness into the model through the profit shock \( z_t \). It is common in the corporate finance literature to assume that random shocks follow an AR(1) process in logs

\[
\ln(z_t) = \ln(c) + \rho \ln(z_{t-1}) + \varepsilon_t, \tag{1}
\]

where \( \rho \in (0, 1) \) is the autoregressive parameter that defines the persistence of profit shocks. In other words, a high \( \rho \) makes periods of high profit innovations (e.g., economic expansions) and low profit shocks (e.g., recessions) last more on average, and vice versa. The innovation term \( \varepsilon_t \) is assumed to be an \( \text{iid} \) normal random variable with mean 0 and variance \( \sigma^2 \). Constant \( c > 0 \) is a drift in logs that scales the moments of the distribution of \( z_t \). This parameter has a direct impact on expected profits and, thus, regulates the size of the firm.\(^{11}\) These assumptions result in lognormal conditional and unconditional distributions for profit shocks.

Gross profits in period \( t \) are defined by the following function

\[
\tilde{Y}_t = (1 + g)^{t(1-\alpha)} z_t \tilde{K}_t^\alpha, \tag{2}
\]

where \( z_t \) is the profit shock in period \( t \) and parameter \( \alpha \in (0, 1) \) represents the elasticity of capital input. The level of technology in period \( t \) takes the form of \((1 + g)^{t(1-\alpha)} \), which implies the firm grows at constant rate \( g \geq 0 \) in each period. With this factor, the firm becomes a scaled up replica of itself

\(^8\)Alternatively, \( \tilde{D}_t \) could be interpreted as a perpetuity that the firm increases and decreases as needed at the end of every period.

\(^9\)It is straightforward to generalize this component and assume a coupon rate \( c_B \) different from the market cost of debt \( r_B \). Without any loss of generality and to simplify notation, we assume they are equal.

\(^{10}\)Koziol (2014) describes an interesting way to introduce default risk and bankruptcy costs to the DDM.

\(^{11}\)The usual normalization in the literature is \( c = 1 \), which makes term \( \ln(c) \) disappear from Eq. (1).
over time, and we use this feature in a required normalization of growing variables that we describe below. Equation (2) says that gross profits also depend on a Cobb–Douglas production function with decreasing returns to scale in capital input.

Every period, the firm pays operating costs $f\tilde{K}_t$ (with $f > 0$) and corporate earnings are taxed at rate $\tau \in (0, 1)$. Therefore, the firm’s net profits in period $t$ are

$$\tilde{N}_t = (\tilde{Y}_t - f\tilde{K}_t - \delta\tilde{K}_t - r_B\tilde{B}_t)(1 - \tau).$$

Finally, the restriction $(f + \delta)(1 - \tau) \leq 1$ guarantees the market value of equity is weakly positive. With all the previous information, we can state the accounting cash flow equation or, equivalently, the cash flow that the firm pays to equity-holders in period $t$ as

$$\tilde{L}_t = \tilde{N}_t - [(\tilde{K}_{t+1} - \tilde{K}_t) - (\tilde{B}_{t+1} - \tilde{B}_t)].$$

Equation (3) is usually called the levered cash flow of the firm and implies that the dividend paid to shareholders in period $t$ equals net profits minus the change in equity. We let rate $r_s$ represent the market cost of equity and rate $r_A$ denote the market cost of capital. For existence of the market value of equity, we impose the usual restriction that the secular growth rate must be lower than the market cost of capital (i.e., $g < r_A$). Given the current state of the firm at $t = 0$, $(\tilde{K}_0, \tilde{B}_0, z_0)$, the problem of the CEO is to choose an infinite sequence of functions $\{\tilde{K}_{t+1}, \tilde{B}_{t+1}\}_{t=0}^\infty$, such that the market value of equity is maximized. Specifically, we use the Adjusted Present Value method of firm valuation introduced by Myers (1974) to solve the problem of the firm, which we describe in Appendix A. We let $E_0$ indicate the expectation operator given information at $t = 0$ (i.e., $\tilde{K}_0, \tilde{B}_0, z_0$). The stock price can thereby be expressed as

$$\tilde{S}_0(\tilde{K}_0, \tilde{B}_0, z_0) = \max_{\{\tilde{K}_{t+1}, \tilde{B}_{t+1}\}_{t=0}^\infty} E_0 \sum_{t=0}^\infty \frac{1}{\prod_{j=0}^t (1 + r_s)} \tilde{L}_t$$

subject to the restriction of risk-free debt. Formally, we say debt is risk-free if, in every period, the after-shock book value of equity is weakly positive. In other words, net profits plus the sale of assets, $\tilde{N}_t + \tilde{K}_t$, must be sufficient to cover debt, $\tilde{B}_t$. This condition is equivalent to a weakly positive net-worth covenant. This type of covenant is often used with short-term debt contracts

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12From a practical perspective, if we use the present model letting $(f + \delta)(1 - \tau) > 1$, the probability of a negative share price is almost zero for standard values of the parameters.
(see, e.g., Leland, 1994), and fits nicely with the one-period debt in our model.

The analytic solution of the firm problem is presented in the following proposition.

**Proposition 1.** The optimal decisions of the firm are given by

$$
\tilde{K}_{t+1}^*(z_t) = (1 + g)^t E[z_{t+1} | z_t]^{1/\alpha} W^* \quad \text{and} \quad \tilde{B}_{t+1}^*(z_t) = \ell^* \tilde{K}_{t+1}^*(z_t),
$$

(5)

where $E[z_{t+1} | z_t] = cz_t e^\delta t^2$ and the time-invariant part of optimal capital takes the form

$$
W^* = \left( \frac{r_A}{1 - \tau} + f + \delta \right)^{1/\alpha}.
$$

The optimal book leverage ratio is given by

$$
\ell^* = \frac{1 - (f + \delta)(1 - \tau)}{1 + r_B(1 - \tau)}.
$$

(6)

The market value of equity is

$$
\tilde{S}_t(\tilde{K}_t, \tilde{B}_t, z_t) = [(1 + g)^{t(1 - \alpha)} z_t \tilde{K}_t^\alpha - f \tilde{K}_t - \delta \tilde{K}_t - r_B \tilde{B}_t](1 - \tau) + \tilde{K}_t - \tilde{B}_t + (1 + g)^t M(z_t) P^*,
$$

(7)

where function $M(z_t)$ can be expressed as:

$$
M(z_t) = e^{-\frac{1}{2}\sigma^2} \frac{\alpha}{(1 - \alpha)^2} \left\{ \frac{1 + g}{1 + r_A} E[z_{t+1}^{1/(1-\alpha)} | z_t] + \left( \frac{1 + g}{1 + r_A} \right)^2 E[z_{t+2}^{1/(1-\alpha)} | z_t] + \cdots \right\}
$$

with the general term given by

$$
E[z_{t+n}^{1/(1-\alpha)} | z_t] = (c^{1-k/n} z_t^{\rho/n} e^{\sigma^2 [1-(1-p)^n]} (1-\alpha))^{1/\alpha}, \quad n = 1, 2, \ldots.
$$

Finally, variable $P^*$ takes the form

$$
P^* = (W^{*n} - fW^* - \delta W^*)(1 - \tau) - r_A W^* + \left( \frac{1 + r_A}{1 + r_B} \right) r_B \tau \ell^* W^*.
$$

The proof of Proposition 1 is in Appendix A. We next describe each of its components.

Expression (5) shows optimal next-period capital, $\tilde{K}_{t+1}^*(z_t)$, and debt, $\tilde{B}_{t+1}^*(z_t)$, as explicit functions of the economic fundamentals of the firm. Optimal capital increases with the growth rate $g$, the drift parameter $c$, and
the volatility of innovations $\sigma$ because, as shown in Eq. (2), they increment the expected productivity of capital. On the contrary, optimal assets decrease with the market cost of capital $r_A$, operating costs $f$, and depreciation $\delta$. The effect of $\alpha$ and $\rho$ on capital depends on current profit shock $z_t$, though it is generally positive for standard values of the parameters. These firm characteristics also have the same directional effects on the optimal next-period debt, $\tilde{B}_{t+1}^*(z_t)$. Finally, the income tax rate $\tau$ has a negative effect on optimal assets as they become less profitable. It also has a negative effect on optimal debt for the great majority of parameter values, including those used in the paper.

Optimal debt in Eq. (5) is a constant proportion $\ell^*$ of optimal assets. This ratio represents the target leverage of the firm and shows the level of debt that maximizes share price while it keeps debt risk-free. The fact that the firm in our model has a target leverage is consistent with the empirical evidence reported by Graham and Harvey (2001), who find that most surveyed firms actually have some form of target leverage. Equation (6) shows that $\ell^*$ is decreasing in the non-debt tax shields $(f + \delta)$ and the market cost of debt $(r_B)$, and increasing in the income tax rate $(\tau)$. Additionally, $\ell^*$ is strictly less than 1 and bounded below by zero.

The market value of equity is shown in Eq. (7) and represents an analytic solution of the Gordon Growth Model in the stochastic setting. Therefore, it can be used for firm valuation (see, e.g., Lazzati and Menichini, 2014b). The first three terms in Eq. (7) represent the after-shock book value of equity, while the last term is the going-concern value. The latter depends on variable $M(z_t)$, which captures the effect of the infinite sequence of expected profit shocks, and on variable $P^*$, which denotes the dollar return on capital minus the dollar cost of capital at the optimum (plus the interest tax shields). The going-concern value reflects the future excess profits stemming from current assets as well as from the real options available to the firm.

13 The derivation of $\ell^*$ is in Appendix A.
14 This model satisfies the two conditions that are necessary to yield the same stock valuations produced by the residual income and free cash flow models. See Sweeney (2014).
15 We obtain $M(z_t)$ in the following way. Let $A_0 = 0$ and, for $n = 1, 2, \ldots$,

$$A_n = A_{n-1} + \left(\frac{1 + g}{1 + r_A}\right)^n E[1/(1+\alpha) | z_t].$$

Then, we iterate the previous recursion until convergence (i.e., until $A_n = A_{n-1} = A$). Finally, we compute $M(z_t)$ as:

$$M(z_t) = e^{-\frac{1}{\delta} \int_0^t \frac{z_s}{\alpha}} A.$$
and long-run growth. We explain how to compute these different values in the next section. It is easy to show that $P^*$ is strictly positive and that the market value of equity, $\tilde{S}_t(\tilde{K}_t, \tilde{B}_t, z_t)$, is weakly positive, as expected.

It is important to highlight that Eq. (7) deals with two independent aspects of growth, namely, past and future growth. The assumption that the firm is at an arbitrary current period $t$ implies that it has been running and growing for the past $t$ periods. Thus, past growth refers to the $t$ periods of growth that the firm has accumulated since the beginning of its life (i.e., period 0), which enters share price through $(1 + g)^t$. On the contrary, future growth refers to the investment opportunities that the firm expects to take advantage of in the future and enters the stock price through $M(z_t)$. Specifically, future growth appears as $(1 + g)$ on the numerator of the discount factors in $M(z_t)$. This distinction between past and future growth plays a fundamental role in the decomposition of share price that we perform in Sec. 3.

In the next section, we decompose the stock price in its four primitive components and present the model predictions regarding managerial flexibility and secular growth.

3. The Value of Real Options and Long-Run Growth

In this section, we investigate the economic importance of the real options and long-run growth opportunities. To this end, we first decompose the market value of equity into its primitive determinants, namely, the stock price assuming the firm has no managerial flexibility and no future growth opportunities, the value of the real options, the value of long-run growth, and the value of the interaction effect between the last two components. We then use parameter estimates that are standard in the literature to quantify each of these terms for a representative firm.

The aforementioned decomposition of the market value of equity requires us to derive the stock price under different assumptions. First, we obtain the market value of equity assuming that the firm grows in the long-run but has no managerial flexibility. This is the same as assuming that, at the beginning of the firm’s life, the CEO projects the expected level of activity for all future periods and determines the required level of assets and debt for each of those periods in order to maximize shareholders’ wealth. Then the firm is implicitly assumed to follow that path of action passively, ignoring the fact that managers can react to future changes in the environment and make decisions to take advantage of good fortune or to alleviate loss. This behavior is
dynamically inconsistent because the initial plan is no longer optimal as new information arrives.\textsuperscript{16} We formalize this idea by solving the problem of the firm in Eq. (4) in open-loop form instead of using dynamic programming techniques (i.e., closed-loop or feedback form).\textsuperscript{17} That is, we assume that, at time 0, the firm applies the expectation operator to all future cash flows first (conditioning on the initial state, $\tilde{K}_0, \tilde{B}_0, z_0$) and then maximizes the resulting objective function with respect to all decision variables at once (i.e., the firm makes all decisions once and for all at time 0). Because this optimization concept maximizes a deterministic stream of payoffs (i.e., the expected levered cash flows), it captures the assumption of precommitment to a deterministic plan of action. Saying it differently, this model computes maximums of expectations while the real options approach calculates expectations of maximums. By Jensen’s inequality, the latter is always greater than or equal to the former. With these assumptions, the problem of the firm in Eq. (4) becomes

$$\tilde{S}_0(\tilde{K}_0, \tilde{B}_0, z_0)_{WO} = \max_{\{\tilde{K}_{t+1}(\tilde{K}_0, \tilde{B}_0, z_0), \tilde{B}_{t+1}(\tilde{K}_0, \tilde{B}_0, z_0)\}^\infty_{t=0}} E_0 \sum_{t=0}^{\infty} \frac{1}{\prod_{j=0}^{t}(1 + r_S)} \tilde{L}_t.$$  

The subindex WO highlights the fact that the solution corresponds to a firm without real options (but with growth). The solution to this problem is

$$\tilde{S}_t(\tilde{K}_t, \tilde{B}_t, z_t)_{WO} = [(1 + g)^t(1-\alpha)z_t\tilde{K}_t^\alpha - f\tilde{K}_t - \delta\tilde{K}_t - r_B\tilde{B}_t](1 - \tau) + \tilde{K}_t - \tilde{B}_t + (1 + g)^t \tilde{M}(z_t)P^*, \quad (8)$$

where

$$\tilde{M}(z_t) = \left( \frac{1 + g}{1 + r_A} \right) E[z_{t+1}|z_t]^{1-\alpha} + \left( \frac{1 + g}{1 + r_A} \right)^2 E[z_{t+2}|z_t]^{1-\alpha} + \cdots$$

and the general term is

$$E[z_{t+n}|z_t] = c^{\frac{-\rho^n}{1-\rho}} z_t^n e^{\frac{1-\sigma^2}{2(1-\rho)}} E[z_{t+2}|z_t]^{1-\alpha} + \cdots$$

The market value of equity in Eq. (7) differs from that of Eq. (8) via terms $M(z_t)$ and $\tilde{M}(z_t)$. In particular, $M(z_t)$ captures the value of the real options, which is absent in $\tilde{M}(z_t)$. Because the firm in the last model has no managerial flexibility, the market value of equity in Eq. (7) is always greater than or equal to that of Eq. (8).

\textsuperscript{16} In other words, the original policy does not satisfy the Bellman’s Principle of Optimality.

\textsuperscript{17} While open-loop and closed-loop forms produce the same solutions in deterministic settings, they differ in stochastic environments. That difference captures the real options. Thus, this paper also highlights the importance of the recursive formulation of this type of problems.
Second, we derive the stock price for a firm that has managerial flexibility but does not have the possibility to grow in the future. We obtain this value from Eq. (7) assuming $g = 0$ in the discount factor

$$
\tilde{S}_t(\tilde{K}_t, \tilde{B}_t, z_t)_{WG} = [(1 + g)^{t(l(1-\alpha)}z_t\tilde{K}_t^\alpha - f\tilde{K}_t - \delta\tilde{K}_t - r_B\tilde{B}_t](1 - \tau) \\
+ \tilde{K}_t - \tilde{B}_t + (1 + g)^t \hat{M}(z_t)P^*,
$$

(9)

where

$$
\hat{M}(z_t) = e^{-\frac{1}{2}\sigma^2\frac{t}{(l-\alpha)^2}} \left\{ \frac{1}{(1 + r_A)} E[z_{t+1}^{1/(1-\alpha)}|z_t] + \frac{1}{(1 + r_A)^2} E[z_{t+2}^{1/(1-\alpha)}|z_t] + \cdots \right\}.
$$

The subindex WG means that the solution corresponds to a firm without future growth opportunities (but with managerial flexibility). Note that while Eq. (9) does not include the present value of future growth opportunities, it does consider the impact of the $t$ periods of past growth.

We finally find the share price of a firm assuming that it has no managerial flexibility and no possibilities of future growth. This value is obtained from Eq. (8) assuming $g = 0$ in the discount factor

$$
\tilde{S}_t(\tilde{K}_t, \tilde{B}_t, z_t)_{AP} = [(1 + g)^{t(l(1-\alpha)}z_t\tilde{K}_t^\alpha - f\tilde{K}_t - \delta\tilde{K}_t - r_B\tilde{B}_t](1 - \tau) \\
+ \tilde{K}_t - \tilde{B}_t + (1 + g)^t \overline{M}(z_t)P^*,
$$

(10)

where

$$
\overline{M}(z_t) = \frac{1}{(1 + r_A)} E[z_{t+1}|z_t]^{\frac{1}{l-\alpha}} + \frac{1}{(1 + r_A)^2} E[z_{t+2}|z_t]^{\frac{1}{l-\alpha}} + \cdots.
$$

Equation (10) represents the part of share price that depends purely on the expected profits of the firm’s current assets. The first three terms denote the after-shock book value of equity, while the last term reflects the present value of the future excess profits deriving from current assets. The subindex AP refers to the fact that this part of the stock price is explained by the value of assets in place. As with Eq. (9), while $\tilde{S}_t(\tilde{K}_t, \tilde{B}_t, z_t)_{AP}$ does not include the present value of future growth opportunities, it does include the $t$ accumulated periods of past growth.

Next, we derive the different components of share price using the previous formulas. The value of the real options is given by

$$
\tilde{O}_t(z_t) = \tilde{S}_t(\tilde{K}_t, \tilde{B}_t, z_t)_{WG} - \tilde{S}_t(\tilde{K}_t, \tilde{B}_t, z_t)_{AP} = (1 + g)^t (\hat{M}(z_t) - \overline{M}(z_t))P^*.
$$
That is, the present value of managerial flexibility equals the market value of equity of a firm that takes advantage of the real options but does not expect to grow in the future minus the stock price of a firm with no managerial flexibility and no future growth. Variable $\hat{O}_t(z_t)$ represents the discounted value of the future excess profits stemming from the real options available to the firm.

The value of long-run growth results from

$$\hat{G}_t(z_t) = \hat{S}_t(\hat{K}_t, \hat{B}_t, z_t)_{\text{WO}} - \hat{S}_t(\hat{K}_t, \hat{B}_t, z_t)_{\text{AP}} = (1 + g)^t(\hat{M}(z_t) - \hat{M}(z_t))P^*.$$  

Equation (11) means that the present value of secular growth equals the market value of equity of a firm with no managerial flexibility but with the possibility to grow in the future minus the stock price of a firm with no real options and no future growth. Variable $\hat{G}_t(z_t)$ reflects the present value of the future excess profits resulting from the possibility to grow in the long-run.

We finally compute the value of the interaction effect between managerial flexibility and secular growth as a residual in the following way

$$\hat{I}_t(z_t) = \hat{S}_t(\hat{K}_t, \hat{B}_t, z_t) - [\hat{S}_t(\hat{K}_t, \hat{B}_t, z_t)_{\text{AP}} + \hat{O}_t(z_t) + \hat{G}_t(z_t)]$$

$$= (1 + g)^t[(\hat{M}(z_t) - \hat{M}(z_t)) - (\hat{M}(z_t) - \hat{M}(z_t))]P^*.$$  

Equation (12) implies that the interaction effect depends on $(\hat{M}(z_t) - \hat{M}(z_t)) - (\hat{M}(z_t) - \hat{M}(z_t))$, which is an infinite summation with general term:

$$\left[1 + \frac{g}{r_A}\right]^n - \frac{1}{(1 + r_A)^n} \left(e^{-\frac{r_A}{(1-n)}} - \frac{1}{n} - 1\right)\right] \left[1 + \frac{g}{r_A}\right] E[z_t^{(1/(1-n))} | z_t] - E[z_t^{(1/(1-n))} | z_t^{(1/n)}], \quad n = 1, 2, \ldots$$

This expression shows why $\hat{I}_t(z_t)$ represents an interaction effect between the two components: The first factor reflects the pure impact of secular growth while the second factor denotes the sole influence of the real options.
We next analyze the relevance of these components of share price for a representative firm. Table 1 contains the values we use for each of the model parameters. These values are standard in the corporate finance literature. The autoregressive parameter \((\rho)\) is set at 0.75 while the standard deviation of the innovation term \((\sigma)\) equals 0.20. These parameter values are close to the estimates of DeAngelo et al. (2011). We choose the curvature of the production function \((\alpha)\) to be 0.65, which is close to the parameter estimate of Hennessy and Whited (2007). Because we are working with a representative firm, we let parameter \(c\) be 1. We set operating costs \((f)\) equal to 0.20 and the depreciation rate of capital \((\delta)\) equal to 0.10. Finally, we fix the corporate income tax rate \((\tau)\) at 0.35, the market cost of debt \((r_B)\) at 0.02 (which equals the risk-free interest rate \(r_f\)), the market cost of capital \((r_A)\) at 0.08, and the long-run growth rate \((g)\) at 0.01. This parameterization is consistent with a yearly period.

We assume that the firm is at the beginning of its life (i.e., \(t = 0\)) and the current state \((\tilde{K}_0, \tilde{B}_0, z_0)\) is at the mean of the stationary unconditional distribution of profit shocks

\[
  z_0 = E[z_t], \quad \tilde{K}_0 = E[z_t]^{1/\nu} W^* \quad \text{and} \quad \tilde{B}_0 = \ell^* \tilde{K}_0, \quad (13)
\]
where

\[ E[z_0] = c^{\frac{1}{\tau'}} e^{\frac{1}{2} \tau'^2(1-\mu'^2)} . \]

Table 2 exhibits the values of the current state \((\tilde{K}_0, \tilde{B}_0, z_0)\) according to expression (13). These variables are constructed with the parameter values described in Table 1 and used in the parameterization of the different dynamic models we described above. Table 2 also shows that optimal book leverage ratio \((\ell^*)\) is 0.79. As we explained in Sec. 2, this ratio is the target leverage of the firm and is a constant that depends only on primitive parameters.

Next, we present the main quantitative results of this section. Table 3 shows that the market value of equity, \(\tilde{S}_0(\tilde{K}_0, \tilde{B}_0, z_0)\), is 11.45. The value of the real options, \(\tilde{O}_0(z_0)\), is 0.93 and represents 8.10% of share price. Considering that in reality the firm has more real options than the ones included in \(\tilde{S}_0(\tilde{K}_0, \tilde{B}_0, z_0)\), such as the exit option, the conclusion is that the value of managerial flexibility explains an important fraction of the stock price. Firm value, \(\tilde{B}_0 + \tilde{S}_0(\tilde{K}_0, \tilde{B}_0, z_0)\), turns out to be 14.54 and the real options represent 6.38% of that value. This proportion is consistent with the 6% reported by Quigg (1993) as the value of the land explained by the option to wait to develop. The value of long-run growth, \(\tilde{G}_0(z_0)\), is 1.16 and explains

<table>
<thead>
<tr>
<th>Table 2. Parameterization of the current state of the dynamic DDM.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variable</td>
</tr>
<tr>
<td>----------</td>
</tr>
<tr>
<td>(E[z_0])</td>
</tr>
<tr>
<td>(W^*)</td>
</tr>
<tr>
<td>(\ell^*)</td>
</tr>
<tr>
<td>(\tilde{K}_0)</td>
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<tr>
<td>(\tilde{B}_0)</td>
</tr>
<tr>
<td>(z_0)</td>
</tr>
</tbody>
</table>

*Note: The table shows the base case parameterization of the current state of the dynamic DDM. The variables are the mean of the invariant unconditional distribution of profit shocks \((E[z_0])\), the part of optimal assets that is constant with respect to the passage of time \((W^*)\), optimal book leverage ratio \((\ell^*)\), and the current state of capital \((\tilde{K}_0)\), debt \((\tilde{B}_0)\), and the profit shock \((z_0)\).
10.09% of the market value of equity as well as 7.95% of firm value. The interaction term between managerial flexibility and growth has a value of 0.17 and represents 1.44% and 1.14% of share price and firm value, respectively.

**Fact 1:** Managerial flexibility plus secular growth represent around 20% of share price and 15% of the value of the firm.

One of the main methodological contributions of the present paper is to introduce the possibility to engage in long-term growth. Figure 1 displays the stochastic evolution over time of the market value of equity, \( \tilde{S}_t(\tilde{K}_t, \tilde{B}_t, z_t) \), for a firm that grows at \( g = 0.01 \). The model is simulated over 100 periods starting at moment \( t = 0 \). The parameterization of the model uses the values in Table 1 and the initial state \( (\tilde{K}_0, \tilde{B}_0, z_0) \) is as described in expression (13), with the values shown in Table 2. The solid dotted line shows the stochastic path of market equity for the growing firm. This dotted line oscillates around an exponential solid line that represents the long-run growth trend. As benchmark, we added a dashed line that represents the

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>% of ( \tilde{S}_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{S}_{0\text{AP}} )</td>
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<td>80.36</td>
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<tr>
<td>( \tilde{O}_0 )</td>
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</tr>
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<td>( \tilde{G}_0 )</td>
<td>1.16</td>
<td>10.09</td>
</tr>
<tr>
<td>( \tilde{I}_0 )</td>
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<td>1.44</td>
</tr>
<tr>
<td>( \tilde{S}_0 )</td>
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<td>100.00</td>
</tr>
<tr>
<td>( \tilde{K}_1^* )</td>
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</tr>
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<td>( \tilde{B}_1^* )</td>
<td>3.17</td>
<td>27.64</td>
</tr>
<tr>
<td>( \tilde{K}_0 )</td>
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<td>26.97</td>
</tr>
<tr>
<td>( \tilde{B}_0 + \tilde{S}_0 )</td>
<td>14.54</td>
<td>126.97</td>
</tr>
</tbody>
</table>

Note: The table exhibits the base case results for the dynamic DDM. The variables are the market value of equity with no real options and no future growth, \( \tilde{S}_0(\tilde{K}_0, \tilde{B}_0, z_0)_{\text{AP}} \); the value of the real options, \( \tilde{O}_0(z_0) \); the value of long-run growth, \( \tilde{G}_0(z_0) \); the interaction term between managerial flexibility and secular growth, \( \tilde{I}_0(z_0) \); the market value of equity with managerial flexibility and long-run growth, \( \tilde{S}_0(\tilde{K}_0, \tilde{B}_0, z_0) \); optimal next-period assets, \( \tilde{K}_1^*(z_0) \); optimal next-period debt, \( \tilde{B}_1^*(z_0) \); current debt, \( \tilde{B}_0 \); and the market value of the firm, \( \tilde{B}_0 + \tilde{S}_0(\tilde{K}_0, \tilde{B}_0, z_0) \).
share price of the same firm when it does not grow (i.e., \( g = 0 \)). The horizontal solid line around which this dashed line fluctuates is the mean of the stationary distribution of share price. Because both simulations are created for the same resolution of uncertainty, the dotted and dashed lines exhibit the same peaks and troughs over time. The only difference between the two firms is the assumption about growth. All four lines are normalized by the first value in the simulation of market equity for the no-growth firm.

Figure 2 shows the evolution over time of the stock price (solid line with dots), the value of managerial flexibility (solid line), the present value of growth opportunities (dashed line), and the value of the interaction effect between the real options and growth (dotted line). The simulation in this figure uses the same sequence of profit shocks used to construct market value of equity in Fig. 1 and the four lines are normalized by the first value of the corresponding time series. We next highlight the main feature of Fig. 2.

**Fact 2:** The stock price is considerably more sensitive to profit shocks than are the real options, secular growth, and the interaction term.

The reason for this different behavior is that the stock price, \( \tilde{S}_t(\tilde{K}_t, \tilde{B}_t, \tilde{z}_t) \), receives the full impact of \( \tilde{z}_t \) through \( M(\tilde{z}_t) \). On the contrary, \( \tilde{z}_t \) enters \( \tilde{O}_t(\tilde{z}_t) \), \( \tilde{G}_t(\tilde{z}_t) \), and \( \tilde{I}_t(\tilde{z}_t) \) through \( \tilde{M}(\tilde{z}_t) - \tilde{M}(\tilde{z}_t) \), \( \tilde{M}(\tilde{z}_t) - \tilde{M}(\tilde{z}_t) \),
and \( \frac{M(z_t) - M(z_t)}{C_0} \), respectively, and these subtractions reduce the influence of profit shocks.

In Fig. 3, we can observe how the proportion of share price explained by managerial flexibility (solid line), long-run growth (dashed line), and the interaction effect (dotted line) evolve over time. This simulation is also based on the same realization of uncertainty of Fig. 1. We next summarize the main result in Fig. 3.

**Fact 3:** The proportions of share price explained by managerial flexibility and long-run growth are countercyclical. That is, when profit shocks are high, the fraction of market equity explained by these two terms falls and vice versa.

This pattern is due to the different sensitivities to profit shocks described in Fig. 2 and suggests that both managerial flexibility and growth opportunities are more important during economic recessions than expansions. In other words, the undervaluation problem of the traditional DDM is exacerbated during periods of low profits shocks. Figure 3 shows that the value of those components can easily exceed 25% of share price during bad times, which is higher than the 20% of the stock price in normal periods described in Fact 1. Figure 4 exhibits the same three proportions but with respect to firm value, and the conclusions are the same.
Fig. 3. Simulation of real options and long-run growth as a fraction of market equity. The model is simulated over 100 periods with the parameterization described in Sec. 3. The figure exhibits the evolution over time of the proportion of market value of equity explained by managerial flexibility (solid line), secular growth (dashed line), and the interaction term (dotted line).

Fig. 4. Simulation of real options and long-run growth as a fraction of firm value. The model is simulated over 100 periods with the parameterization described in Sec. 3. The figure exhibits the evolution over time of the proportion of firm value explained by managerial flexibility (solid line), secular growth (dashed line), and the interaction term (dotted line).
In Sec. 4, we elaborate on the key determinants of the primitive components of the stock price and perform a cross-sectional comparison of different industries.

4. Comparative Statics and Cross-Sectional Analysis

In the previous section, we showed that the value of the real options is sensitive to the fluctuations of the business cycle. In Sec. 4.1, we elaborate on the specific firm characteristics that have a fundamental impact on the value of the real options and long-run growth. In Sec. 4.2, we then perform a cross-sectional analysis to compare the relevance of these determinants across different SIC industries.

4.1. Comparative statics analysis

In this subsection, we study how the market value of equity, \( \bar{S}_0(K_0, B_0, z_0) \), and the proportion of share price explained by the real options, \( \bar{O}_0(z_0)/\bar{S}_0(K_0, B_0, z_0) \), and by long-run growth, \( \bar{G}_0(z_0)/\bar{S}_0(K_0, B_0, z_0) \), vary when we change the characteristics of the firm. In order to implement our idea, we change the base case parameter values in Table 1 by up to ± 20%. The analytic solutions make it easy to understand the relative impact of the different model parameters on these terms. Briefly, we find that while the stock price and the value of the real options are very sensitive to the curvature of the production function (\( \alpha \)), the present value of growth opportunities is very sensitive to both the market cost of capital \( (r_A) \) and (naturally) the growth rate \( (g) \). The persistence of profit shocks \( (\rho) \) is also an important determinant of the value of managerial flexibility. We next elaborate on these results.

We assume again that the firm is at period \( t = 0 \). Table 4 displays how market equity value changes with different parameterizations of the model. We find that one of the most important parameters is the curvature of the production function \( (\alpha) \). The higher this parameter, the higher the marginal productivity of capital, which allows the firm to increase optimal assets to take more advantage of profit shocks. A 20% increment in the value of \( \alpha \) (i.e., from 0.65 to 0.78) increases market value of equity by a factor of more than 3, from 11.45 to 36.79. Another important parameter is the drift in logs \( (c) \), which scales the moments of the distribution of profit shocks. The importance of this parameter stems from the fact that it has a direct influence on mean profits and, thus, defines firm size. When \( c \) increases from 1 to 1.2, share price goes up from 11.45 to 63.31, a more than five times increment. The other parameters affect market equity to a lesser extent, with the
Table 4. Comparative statics analysis of market equity value.

<table>
<thead>
<tr>
<th></th>
<th>BC-20%</th>
<th>BC-16%</th>
<th>BC-12%</th>
<th>BC-8%</th>
<th>BC-4%</th>
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<th>BC+8%</th>
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<td>11.57</td>
<td>11.63</td>
<td>11.70</td>
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</tbody>
</table>

Note: The table shows share price, $\tilde{S}(\tilde{K}_0, \tilde{B}_0, z_0)$, for different values of model parameters. The column labeled BC contains the base case parameter values described in Table 1 while the other columns contain proportional changes of those initial values. The parameters are the drift in logs ($c$), the persistence of profit shocks ($\rho$), the standard deviation of the innovation term ($\sigma$), the concavity of the production function ($\alpha$), the operating costs ($f$), the capital depreciation rate ($\delta$), the corporate income tax rate ($\tau$), the market cost of debt ($r_1$), the market cost of capital ($r_0$), and the growth rate ($g$).
persistence of profit shocks ($\rho$) and the market cost of capital ($r_A$) at the top of this second group.

Table 5 exhibits the comparative statics analysis of the fraction of market equity explained by managerial flexibility. It is clear that the elasticity of capital input ($\alpha$) has a significant influence on the value of the real options, specially on the expansion and contraction options. The possibility of the CEO to take advantage of high profits and to protect shareholders from bad earnings by adapting firm size is more valuable when the curvature of the production function is lower. For the base case parameter value of $\alpha = 0.65$, managerial flexibility represents 8.10% of market equity, percentage that increases to 24.41% when $\alpha$ goes up to 0.78. For the same reasons, the volatility of profit shocks ($\sigma$) also has a considerable influence on the value of the real options. The higher the variability of profit shocks, the more the firm can benefit from having the possibility to adapt itself to changes in the environment. When $\sigma$ goes up from the base case value of 0.20 to 0.24, the fraction of share price represented by real options increases from 8.10% to 11.59%.

The persistence of profit shocks ($\rho$) also plays a major role in the valuation of real options. Economic expansions and recessions are longer on average when profit shocks are more persistent, and the value of managerial flexibility increases, specially the value of the extension and deferral options. The reason for this result is that the CEO is able to maintain the firm large during periods of high earnings and to keep it small during periods of low profit shocks. A 20% increase of $\rho$ from 0.75 to 0.90 augments the proportion of real options from 8.10% to 21.69%. The other model parameters have a smaller effect on this proportion.

These results are in line with empirical work showing that the value of managerial flexibility is highly sensitive to the primitive features of the firm. For instance, Quigg (1993) finds that the mean value of the option to wait to invest is 6% of the value of the land, ranging from 1% to 30% according to land characteristics.

Finally, the impact of different parameter values on the proportion of market equity explained by long-run growth appears in Table 6. As expected, the market cost of capital ($r_A$) and the growth rate ($g$) are among the most important parameters. A 20% reduction in the value of $r_A$ (i.e., from 0.08 to 0.064) increases the proportion of secular growth from 10.09% to 12.98% while a 20% increase in the value of $g$ (i.e., from 0.01 to 0.012) augments the fraction of long-run growth from 10.09% to 12.14%. Similar in importance is the drift in logs ($c$), which strongly affects the expected profits of the firm. When $c$ increases from 1 to 1.2, the proportion of secular growth goes up.
Table 5. Comparative statics analysis of real options value.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BC-20%</th>
<th>BC-16%</th>
<th>BC-12%</th>
<th>BC-8%</th>
<th>BC-4%</th>
<th>Base Case (BC)</th>
<th>BC+4%</th>
<th>BC+8%</th>
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</thead>
<tbody>
<tr>
<td>$c$</td>
<td>0.800</td>
<td>0.840</td>
<td>0.880</td>
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<td>1.000</td>
<td>1.040</td>
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</tr>
<tr>
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<td>8.10%</td>
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</tr>
<tr>
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<td>0.660</td>
<td>0.690</td>
<td>0.720</td>
<td>0.750</td>
<td>0.780</td>
<td>0.810</td>
<td>0.840</td>
<td>0.870</td>
<td>0.900</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.160</td>
<td>0.168</td>
<td>0.176</td>
<td>0.184</td>
<td>0.192</td>
<td>0.200</td>
<td>0.208</td>
<td>0.216</td>
<td>0.224</td>
<td>0.232</td>
<td>0.240</td>
</tr>
<tr>
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<td>0.572</td>
<td>0.598</td>
<td>0.624</td>
<td>0.650</td>
<td>0.676</td>
<td>0.702</td>
<td>0.728</td>
<td>0.754</td>
<td>0.780</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.160</td>
<td>0.168</td>
<td>0.176</td>
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</tr>
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<td>8.23%</td>
<td>8.18%</td>
<td>8.14%</td>
<td>8.10%</td>
<td>8.06%</td>
<td>8.02%</td>
<td>7.97%</td>
<td>7.93%</td>
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<td>0.108</td>
<td>0.112</td>
<td>0.116</td>
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<tr>
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<td>0.336</td>
<td>0.350</td>
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<td>0.378</td>
<td>0.392</td>
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</tr>
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<td>0.018</td>
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<td>0.020</td>
<td>0.021</td>
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<td>0.024</td>
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<td>8.10%</td>
<td>8.10%</td>
<td>8.10%</td>
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<td>$\rho$</td>
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<td>0.080</td>
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<td>0.086</td>
<td>0.090</td>
<td>0.093</td>
<td>0.096</td>
</tr>
<tr>
<td>$\rho$</td>
<td>8.42%</td>
<td>8.37%</td>
<td>8.31%</td>
<td>8.25%</td>
<td>8.18%</td>
<td>8.10%</td>
<td>8.02%</td>
<td>7.93%</td>
<td>7.85%</td>
<td>7.75%</td>
<td>7.66%</td>
</tr>
<tr>
<td>$g$</td>
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<td>0.0088</td>
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<td>0.0096</td>
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<td>0.0104</td>
<td>0.0108</td>
<td>0.0112</td>
<td>0.0116</td>
<td>0.0120</td>
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<tr>
<td>$\bar{O}_0/S_0$</td>
<td>8.31%</td>
<td>8.27%</td>
<td>8.23%</td>
<td>8.19%</td>
<td>8.14%</td>
<td>8.10%</td>
<td>8.06%</td>
<td>8.02%</td>
<td>7.97%</td>
<td>7.93%</td>
<td>7.89%</td>
</tr>
</tbody>
</table>

Note: The table shows the proportion of share price, $\bar{S}_0(K_0, B_0, z_0)$, that is explained by the value of the real options, $\bar{O}_0(z_0)$, for different values of model parameters. The column labeled BC contains the base case parameter values described in Table 1 while the other columns contain proportional changes of those initial values. The parameters are the drift in logs ($c$), the persistence of profit shocks ($\rho$), the standard deviation of the innovation term ($\sigma$), the concavity of the production function ($\alpha$), the operating costs ($f$), the capital depreciation rate ($\delta$), the corporate income tax rate ($r$), the market cost of debt ($r_D$), the market cost of capital ($r_A$), and the growth rate ($g$).
Table 6. Comparative statics analysis of secular growth value.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>BC-20%</th>
<th>BC-16%</th>
<th>BC-12%</th>
<th>BC-8%</th>
<th>BC-4%</th>
<th>Base Case (BC)</th>
<th>BC+4%</th>
<th>BC+8%</th>
<th>BC+12%</th>
<th>BC+16%</th>
<th>BC+20%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>0.800</td>
<td>0.840</td>
<td>0.880</td>
<td>0.920</td>
<td>0.960</td>
<td>1.000</td>
<td>1.040</td>
<td>1.080</td>
<td>1.120</td>
<td>1.160</td>
<td>1.200</td>
</tr>
<tr>
<td>$\tilde{G}_0/\tilde{S}_0$</td>
<td>3.46%</td>
<td>4.75%</td>
<td>6.16%</td>
<td>7.58%</td>
<td>8.91%</td>
<td>10.09%</td>
<td>11.11%</td>
<td>11.96%</td>
<td>12.66%</td>
<td>13.24%</td>
<td>13.73%</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.600</td>
<td>0.630</td>
<td>0.660</td>
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<td>0.720</td>
<td>0.750</td>
<td>0.780</td>
<td>0.810</td>
<td>0.840</td>
<td>0.870</td>
<td>0.900</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.160</td>
<td>0.168</td>
<td>0.176</td>
<td>0.184</td>
<td>0.192</td>
<td>0.200</td>
<td>0.208</td>
<td>0.216</td>
<td>0.224</td>
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<td>0.240</td>
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<tr>
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<td>10.34%</td>
<td>10.23%</td>
<td>10.09%</td>
<td>9.93%</td>
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<td>8.63%</td>
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<tr>
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<td>0.546</td>
<td>0.572</td>
<td>0.598</td>
<td>0.624</td>
<td>0.650</td>
<td>0.676</td>
<td>0.702</td>
<td>0.728</td>
<td>0.754</td>
<td>0.780</td>
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<tr>
<td>$\tilde{G}_0/\tilde{S}_0$</td>
<td>10.49%</td>
<td>10.41%</td>
<td>10.33%</td>
<td>10.26%</td>
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<td>9.83%</td>
<td>9.60%</td>
<td>9.25%</td>
<td>8.70%</td>
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<tr>
<td>$f$</td>
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<td>0.184</td>
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<td>0.216</td>
<td>0.224</td>
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<td>0.240</td>
</tr>
<tr>
<td>$\tilde{G}_0/\tilde{S}_0$</td>
<td>10.35%</td>
<td>10.30%</td>
<td>10.25%</td>
<td>10.20%</td>
<td>10.15%</td>
<td>10.09%</td>
<td>10.04%</td>
<td>9.99%</td>
<td>9.94%</td>
<td>9.88%</td>
<td>9.83%</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.080</td>
<td>0.084</td>
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<td>0.100</td>
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<td>0.112</td>
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<tr>
<td>$\tilde{G}_0/\tilde{S}_0$</td>
<td>10.22%</td>
<td>10.20%</td>
<td>10.17%</td>
<td>10.15%</td>
<td>10.12%</td>
<td>10.09%</td>
<td>10.07%</td>
<td>10.04%</td>
<td>10.01%</td>
<td>9.99%</td>
<td>9.96%</td>
</tr>
<tr>
<td>$r$</td>
<td>0.280</td>
<td>0.294</td>
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<td>0.364</td>
<td>0.378</td>
<td>0.392</td>
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<td>0.420</td>
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<tr>
<td>$\tilde{G}_0/\tilde{S}_0$</td>
<td>10.16%</td>
<td>10.14%</td>
<td>10.13%</td>
<td>10.12%</td>
<td>10.11%</td>
<td>10.09%</td>
<td>10.08%</td>
<td>10.06%</td>
<td>10.05%</td>
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</tr>
<tr>
<td>$r_B$</td>
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<td>0.018</td>
<td>0.018</td>
<td>0.019</td>
<td>0.020</td>
<td>0.021</td>
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<td>0.024</td>
</tr>
<tr>
<td>$\tilde{G}_0/\tilde{S}_0$</td>
<td>10.08%</td>
<td>10.09%</td>
<td>10.09%</td>
<td>10.09%</td>
<td>10.09%</td>
<td>10.09%</td>
<td>10.10%</td>
<td>10.10%</td>
<td>10.10%</td>
<td>10.10%</td>
<td>10.10%</td>
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<tr>
<td>$r_A$</td>
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<td>0.067</td>
<td>0.070</td>
<td>0.074</td>
<td>0.077</td>
<td>0.080</td>
<td>0.083</td>
<td>0.086</td>
<td>0.090</td>
<td>0.093</td>
<td>0.096</td>
</tr>
<tr>
<td>$\tilde{G}_0/\tilde{S}_0$</td>
<td>12.98%</td>
<td>12.29%</td>
<td>11.67%</td>
<td>11.10%</td>
<td>10.58%</td>
<td>10.09%</td>
<td>9.64%</td>
<td>9.23%</td>
<td>8.48%</td>
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<td>8.13%</td>
</tr>
<tr>
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<td>0.0088</td>
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<td>0.0096</td>
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<td>0.0112</td>
<td>0.0116</td>
<td>0.0120</td>
</tr>
<tr>
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<td>8.46%</td>
<td>8.87%</td>
<td>9.28%</td>
<td>9.69%</td>
<td>10.09%</td>
<td>10.50%</td>
<td>10.91%</td>
<td>11.32%</td>
<td>11.73%</td>
<td>12.14%</td>
</tr>
</tbody>
</table>

**Note:** The table shows the proportion of share price, $\tilde{S}_0(K_0, B_0, z_0)$, that is explained by the value of long-run growth, $\tilde{G}_0(z_0)$, for different values of model parameters. The column labeled BC contains the base case parameter values described in Table 1 while the other columns contain proportional changes of those initial values. The parameters are the drift in logs ($c$), the persistence of profit shocks ($\rho$), the standard deviation of the innovation term ($\sigma$), the concavity of the production function ($\alpha$), the operating costs ($f$), the capital depreciation rate ($\delta$), the corporate income tax rate ($r$), the market cost of debt ($r_B$), the market cost of capital ($r_A$), and the growth rate ($g$).
from 10.09% to 13.73%. The other model parameters play a smaller role regarding this proportion.

In the next subsection, we compare the importance of real options and long-run growth across industries that sharply differ regarding key parameter values, such as the elasticity of capital input.

4.2. Cross-sectional analysis

In this subsection, we extend our previous results by comparing the relevance of the real options and growth opportunities across different SIC industries. In particular, we focus on three industries that display considerably different capital elasticity. We choose Oil and Gas Extraction (OGE) as an industry with high capital elasticity, Printing and Publishing (PP) as an industry with low capital elasticity, and Chemicals (C) as an industry between those two extremes. We compute the model parameters for each industry using Compustat data and show their values in Table 7. Appendix B describes the data items used to calibrate the parameters for each industry.

Table 8 exhibits our main findings. OGE firms have the greatest proportion of share price explained by managerial flexibility, 9.42%, while C

<table>
<thead>
<tr>
<th>Parameter</th>
<th>OGE Firms</th>
<th>C Firms</th>
<th>PP Firms</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
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<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
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<td>0.5603</td>
</tr>
<tr>
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<td>0.1787</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.6905</td>
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<td>0.5823</td>
</tr>
<tr>
<td>$f$</td>
<td>0.2079</td>
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<td>0.3746</td>
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<tr>
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<td>0.0925</td>
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<tr>
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<td>$\tau_B$</td>
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<td>$\tau_A$</td>
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<td>0.0043</td>
<td>0.0347</td>
<td>0.0251</td>
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</tbody>
</table>

Note: The table presents the values used to parameterize the dynamic DDM for three different SIC industries: Oil and Gas Extraction (OGE), Chemicals (C), and Printing and Publishing (PP). The parameters are the drift in logs ($c$), the persistence of profit shocks ($\rho$), the standard deviation of the innovation term ($\sigma$), the concavity of the production function ($\alpha$), the operating costs ($f$), the capital depreciation rate ($\delta$), the corporate income tax rate ($\tau$), the market cost of debt ($\tau_B$), the market cost of capital ($\tau_A$), and the growth rate ($g$).
Table 8. Cross-sectional value of market equity, real options, and long-run growth.

<table>
<thead>
<tr>
<th>Variable</th>
<th>OGE Firms</th>
<th>C Firms</th>
<th>PP Firms</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Value</td>
<td>% of $\tilde{S}_0$</td>
<td>% of $\tilde{B}_0 + \tilde{S}_0$</td>
</tr>
<tr>
<td>$\tilde{S}_{0,\text{AP}}$</td>
<td>18.91</td>
<td>85.11</td>
<td>66.79</td>
</tr>
<tr>
<td>$\tilde{O}_0$</td>
<td>2.09</td>
<td>9.42</td>
<td>7.39</td>
</tr>
<tr>
<td>$\tilde{G}_0$</td>
<td>1.06</td>
<td>4.77</td>
<td>3.74</td>
</tr>
<tr>
<td>$\tilde{I}_0$</td>
<td>0.16</td>
<td>0.70</td>
<td>0.55</td>
</tr>
<tr>
<td>$\tilde{S}_0$</td>
<td>22.22</td>
<td>100.00</td>
<td>78.48</td>
</tr>
<tr>
<td>$\tilde{K}^*_1$</td>
<td>8.54</td>
<td>38.45</td>
<td>30.18</td>
</tr>
<tr>
<td>$\tilde{B}^*_1$</td>
<td>6.53</td>
<td>29.37</td>
<td>23.05</td>
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<tr>
<td>$\tilde{B}_0$</td>
<td>6.09</td>
<td>27.42</td>
<td>21.52</td>
</tr>
<tr>
<td>$\tilde{B}_0 + \tilde{S}_0$</td>
<td>28.31</td>
<td>127.42</td>
<td>100.00</td>
</tr>
</tbody>
</table>

Note: The table exhibits the results of the dynamic DDM for three different SIC industries: Oil and Gas Extraction (OGE), Chemicals (C), and Printing and Publishing (PP). The variables are the market value of equity with no real options and no future growth, $\tilde{S}_0(K_0, B_0, z_0)_{\text{AP}}$; the value of the real options, $\tilde{O}_0(z_0)$; the value of long-run growth, $\tilde{G}_0(z_0)$; the interaction term between managerial flexibility and secular growth, $\tilde{I}_0(z_0)$; the market value of equity with managerial flexibility and long-run growth, $\tilde{S}_0(K_0, B_0, z_0)$; optimal next-period assets, $\tilde{K}^*_1(z_0)$; optimal next-period debt, $\tilde{B}^*_1(z_0)$; current debt, $\tilde{B}_0$; and the market value of the firm, $\tilde{B}_0 + \tilde{S}_0(K_0, B_0, z_0)$. 
firms are in second place with 3.22% and PP firms are third with 1.31%. Although these industries exhibit considerable variation in the values of all other parameters, the curvature of the production function ($\alpha$) remains as the main driver of the value of the real options, as it defines how much the CEO can take advantage of the future business cycles. For instance, PP firms have higher persistence of profit shocks ($\rho$) than do OGE firms, but the proportion of share price explained by managerial flexibility is roughly seven times less, mainly as a consequence of their lower elasticity of capital. These findings are consistent with the results in the previous subsection.

Finally, it is noticeable the impact of the growth rate ($g$) on the present value of growth opportunities. OGE firms have a modest growth rate of 0.43% per year, which explains 4.77% of the stock price. On the contrary, C firms, with a growth rate of 3.47% per year, have 35.49% of their stock price explained by future growth opportunities. Between these two extremes are PP firms with a growth rate of 2.51% per year, which explains 25.24% of share price.

In summary, for industries such as OGE, which display low curvature of the production function, pricing methods that omit managerial flexibility substantially undervalue share price. Similarly, for industries such as C, which experience considerable growth over time, models of stock valuation that do not incorporate the possibility of secular growth (e.g., most of the existing dynamic programming models of the firm) produce an important downward bias in the estimates. **We conclude that for these cases the valuation model we derive in this article displays the largest benefits.**

5. Conclusion

We derive a dynamic model of the firm within the framework of the DDM. The model we propose includes two important features: real options and secular growth. The introduction of secular growth enhances the existing dynamic programming models of the firm in corporate finance, which so far have been developed assuming the firm does not have long-run growth opportunities. Furthermore, being based on the separation principle, our model does not require any assumption about shareholders’ preferences, as long as we discount future cash flows with an appropriately risk-adjusted discount rate. We believe all these features make our dynamic model a useful asset pricing tool.

The introduction of real options and secular growth into the valuation model is essential to value the firm correctly. By decomposing the stock price
in its primitive components, we are able to quantify for different industries the magnitude of the underpricing problem created by neglecting those features. For instance, we find that for firms in the Oil and Gas Extraction industry the proportion of share price explained by managerial flexibility can easily exceed 9%. Thus, valuation models that do not consider managerial flexibility would lead to a large underpricing of this type of firms. Analogously, we find that for firms in the Chemical industry more than 30% of their stock price can be explained by future growth opportunities. Then, employing valuation models that do not handle long-run growth would produce severely undervalued estimates. We find that the proportions of share price represented by both managerial flexibility and secular growth are countercyclical, which suggests that the aforementioned undervaluation problems become aggravated during economic recessions. Finally, our sensitivity analysis highlights the importance of the elasticity of capital as a key determinant of the value of the real options.

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Appendix A. Proofs
The proof of Proposition 1 requires an intermediate result that we present next.

Lemma A.1. Restricting debt to be risk-free, the maximum level of book leverage in each period is given by

\[ \ell^* = \frac{1 - (f + \delta)(1 - \tau)}{1 + r_B(1 - \tau)}. \]  

(A.1)

Proof. We say debt is risk-free if, in every period, the following inequality is true for all \( z' \)

\[ (z'K'' - fK' - \delta K' - r_B\ell K')(1 - \tau) + K' - \ell K' \geq 0. \]

That is, risk-free debt implies that next-period, after-shock book value of equity must be weakly positive for all \( z' \).\(^{19}\) In other words, net profits,

\(^{19}\)The same result is true if we define risk-free debt using the market value of equity as opposed to the book value of equity. That is, in both cases we arrive at Eq. (A.1) as the maximum level of book leverage consistent with risk-free debt. In order to simplify notation, we use the book value of equity.
\((z'K'' - fK' - \delta K' - r_B\ell K')(1 - \tau), plus the sale of assets, \(K'\), must be sufficient to cover debt, \(\ell K'\).

Given that the worst-case scenario is \(z' = 0\), the maximum book leverage ratio consistent with risk-free debt, \(\ell^*\), satisfies
\((-fK' - \delta K' - r_B\ell^* K')(1 - \tau) + K' - \ell^* K' = 0.

Working on the previous expression, we can derive the maximum level of book leverage as
\[
\ell^* = \frac{1 - (f + \delta)(1 - \tau)}{1 + r_B(1 - \tau)}
\]
which completes the proof.  

**Proof of Proposition 1.** The maximization in Eq. (4) requires a normalization of growing variables that keeps the expectation of the payoff function in the future periods bounded. This normalization is equivalent to the one used to find the solution of the canonical Gordon Growth Model. Let vector \(\bar{X}_t = \{\bar{K}_t, \bar{B}_t, \bar{Y}_t, \bar{N}_t, \bar{L}_t, \bar{S}_t\}\) contain the growing variables of the model. We then transform vector \(\bar{X}_t\) in the following way: \(X_t = \frac{\bar{X}_t}{(1 + g)^t}\).

Using the normalized variables and modifying the payoff function accordingly, the market value of equity can be expressed as
\[
S_0(K_0, B_0, z_0) = \max_{\{K_{t+1}, B_{t+1}\} \geq 0} E_0 \sum_{t=0}^{\infty} \frac{(1 + g)^t}{\prod_{j=0}^{t}(1 + r_{S_j})} L_{ut}
\]
subject to keeping debt risk-free. Because we use the Adjusted Present Value method of firm valuation, we solve the problem of the firm in Eq. (A.2) in three steps. First, we determine the value of the unlevered firm, \(S_{u_0}(K_0, z_0)\). Second, we solve for optimal debt and compute the present value of the financing side effects. Finally, we obtain the value of the levered firm in Eq. (A.2).

The market value of equity for the unlevered firm can be expressed as:
\[
S_{u_0}(K_0, z_0) = \max_{\{K_{t+1}\} \geq 0} E_0 \sum_{t=0}^{\infty} \left( \frac{1 + g}{1 + r_A} \right)^t L_{ut},
\]

where \(L_{ut} = N_{ut} - (K_{t+1} - K_t)\) and \(N_{ut} = (Y_t - fK_t - \delta K_t)(1 - \tau)\). We let normalized variables with primes indicate values in the next period and normalized variables with no primes denote current values. Then, the

---

\(20\)The restriction \((f + \delta)(1 - \tau) \leq 1\) described in Sec. 2 also guarantees that \(\ell^* \geq 0\).
Bellman equation for the problem of the firm in Eq. (A.3) is given by

\[
S_u(K, z) = \max_{K'} \left\{ (zK^\alpha - fK - \delta K)(1 - \tau) - (1 + g)K' + K \right. \\
+ \left. \frac{(1 + g)}{(1 + r_A)} E[S_u(K', z') | z] \right\}.
\] (A.4)

We use the guess and verify method as the proof strategy. Thus, we start by guessing that the solution is given by

\[
S_u(K, z) = (zK^\alpha - fK - \delta K)(1 - \tau) + K + M(z)P_u^*,
\] (A.5)

where

\[
M(z) = e^{\frac{1}{2} \sigma^2 \frac{z - z_0}{(r_A)^2}} \sum_{n=1}^{\infty} \left\{ \left( \frac{1 + g}{1 + r_A} \right)^n \left( e^{\frac{1}{1-\tau}} z^{\rho n} e^{\frac{1}{1-\tau} (1 - \rho) (1 - \alpha)} \right)^{\frac{1}{1-\tau}} \right\},
\] (A.6)

\[
P_u^* = (W^* - fW^* - \delta W^*)(1 - \tau) - r_A W^*,
\]

\[
W^* = \left( \frac{\alpha}{r_A + f + \delta} \right)^{\frac{1}{1-\tau}}.
\] (A.7)

We obtained this initial guess as the solution of Eq. (A.4) by the backward induction method.

We now verify our guess. To this end, let us write

\[
S_u(K, z) = \max_{K'} \{ F(K', K, z) \}
\]

with \( F \) defined as the objective function in Eq. (A.4).

The FOC for this problem is

\[
\partial F(K', K, z)/\partial K' = -(1 + g) + \frac{(1 + g)}{(1 + r_A)} \left[ (E[z'| z]K^{\ast_\alpha - 1} - f - \delta)(1 - \tau) + 1 \right]
\]

\[
= 0
\]

and optimal capital turns out to be

\[
K^* = E[z'| z]^{\frac{1}{1-\tau}} W^*
\]

where \( W^* \) is as in Eq. (A.7).

Finally, the market value of equity for the unlevered firm becomes

\[
S_u(K, z) = (zK^\alpha - fK - \delta K)(1 - \tau) - (1 + g)K^* + K \\
+ \frac{(1 + g)}{(1 + r_A)} \left[ (E[z'| z]K^{\ast_\alpha - 1} - fK^* - \delta K^*)(1 - \tau) + K^* \right. \\
+ \left. E[M(z') | z] P_u^* \right]
\]
\[
(\frac{z}{\text{K}} - fK - \delta K)(1 - \tau) + K - (1 + g)E[z' | z] \frac{1}{1 + \tau} W^* \\
+ \frac{(1 + g)}{(1 + r_A)} \{E[z' | z] \frac{1}{1 + \tau} \} [W^{**} - fW^* - \delta W^*(1 - \tau) + W^*] \\
+ E[M(z') | z] P^*_u \}
\]

which is equivalent to our initial guess in Eq. (A.5).

Next, we obtain optimal debt. In each period, the firm solves the following problem

\[
B^* = \max_{B'} \left\{ B' - \frac{1}{(1 + r_B)} B'[1 + r_B(1 - \tau)] \right\}
\]

subject to the restriction of risk-free debt. Because \(\tau > 0\), the firm increases debt as much as possible (as long as it remains risk-free) in order to maximize the tax benefits of debt. Then, optimal debt is

\[
B^* = \ell^* K^*
\]

where

\[
\ell^* = \frac{1 - (f + \delta)(1 - \tau)}{1 + r_B(1 - \tau)},
\]

as shown in Lemma A.1. The present value of the financing side effects turns out to be

\[
Q(z) = \left( \frac{1 + g}{1 + r_A} \right) \left\{ \left( \frac{1 + r_A}{1 + r_B} \right) r_B \tau B^* + E[Q(z') | z] \right\}
\]

\[
= M(z) \left( \frac{1 + r_A}{1 + r_B} \right) r_B \tau \ell^* W^*,
\]

where \(M(z)\) is as in Eq. (A.6). Under this financial policy, the amount of debt and interest payments will vary with the future asset cash flows (i.e., they depend on future firm performance). Then, because future interest tax shields will have a level of risk in line with that of the firm cash flows, we use the cost of capital, \(r_A\), as the discount rate.

The third step consists in obtaining the market value of equity for the levered firm. If we assume the firm used debt \(B\) in the previous period, and
now has to pay interest \( r_B B(1 - \tau) \), then the stock price for the levered firm is

\[
S(K, B, z) = S_u(K, z) + M(z) \left( \frac{1 + r_A}{1 + r_B} \right) r_B \ell^* W^* - B - r_B B(1 - \tau)
\]

\[
= (zK^\alpha - fK - \delta K - r_B B)(1 - \tau) + K - B + M(z) P^*,
\]

where variable \( P^* \) takes the form

\[
P^* = (W^{*n} - fW^* - \delta W^*)(1 - \tau) - r_A W^* + \left( \frac{1 + r_A}{1 + r_B} \right) r_B \ell^* W^*.
\]

The last part of the proof consists in transforming normalized variables back into growing variables. For this step, we return to the initial notation with growing variables, where next-period assets are \( \tilde{K}_{t+1} \) and current-period assets are \( \tilde{K}_t \). Then, the required transformation is:

\[
\tilde{X}_t = (1 + g)^{t+1} E[z_{t+1} | z_t] \frac{1}{1-\alpha} W^* \quad \text{and} \quad \tilde{B}_t = \ell^* \tilde{K}_{t+1}(z_t)
\]

and the growing market value of equity is

\[
\tilde{S}_t(\tilde{K}_t, \tilde{B}_t, z_t) = [(1 + g)^{t(1-\alpha)} z_t \tilde{K}_t^\alpha - f \tilde{K}_t - \delta \tilde{K}_t - r_B \tilde{B}_t](1 - \tau)
\]

\[
+ \tilde{K}_t - \tilde{B}_t + (1 + g)^{t} M(z_t) P^*
\]

as shown in Proposition 1.

**Corollary A.2.** The interaction effect between managerial flexibility and growth is weakly positive. That is

\[
\tilde{I}_t(z_t) \geq 0. \quad (A.8)
\]

**Proof.** Because variable \( P^* \) is strictly positive, inequality (A.8) is equivalent to saying that \( (M(z_t) - \tilde{M}(z_t)) - (\tilde{M}(z_t) - \overline{M}(z_t)) \geq 0 \) is true. Then, the proof consists in showing that all terms in that infinite summation are non-negative. That is, we need to show that, for \( n = 1, 2, \ldots \), the following inequality is true

\[
\left[ \left( \frac{1 + g}{1 + r_A} \right)^n - \frac{1}{(1 + r_A)^n} \right] (e^{-\frac{1}{2} \sigma^2} \sigma \sqrt{n} z_{t+n}^{1/(1-\alpha)} E[z_{t+n+1}^{1/(1-\alpha)} | z_t] - E[z_{t+n} | z_t]^{1/\alpha}) \geq 0.
\]

The first factor can be rearranged as \( \frac{(1 + g)^n - 1}{(1 + r_A)^n} \) and, because \( 0 \leq g < r_A \), it is weakly positive. The second factor is also non-negative and, in order to prove this, we need to show that, for all \( n > 0 \), the following inequality is true.
which can be expanded as:
\[
\exp\left(\frac{-1}{2\sigma^2} \frac{1}{(1-\alpha)^2} \left( \frac{1}{z_{t+n}} \right) \left( \frac{\epsilon_{t+n}^2(1-\rho^2\sigma^2)}{1-\rho^2} \right) \right)^{1/(1-\alpha)} \geq \left( \left( \frac{1}{z_{t+n}} \right) \left( \frac{\epsilon_{t+n}^2(1-\rho^2\sigma^2)}{1-\rho^2} \right) \right)^{1/(1-\alpha)}.
\] (A.9)

After working inequality (A.9) algebraically, it reduces to
\[
\rho^2 \geq \rho^{2n}.
\]

Given that \( \rho \in (0,1) \), the last inequality is true for \( n = 1, 2, \ldots \), which completes the proof. \( \square \)

**Appendix B. Calibration of Model Parameters**

We need to find parameter values for \( c, \rho, \sigma, \alpha, f, \delta, \tau, r_B, r_A, \) and \( g \) for each of the three industries. We calibrate the model using Compustat annual data for all firms in each of the three SIC codes (i.e., Oil and Gas Extraction (OGE) is SIC 13, Printing and Publishing (PP) is SIC 27, and Chemicals (C) is SIC 28). The sample covers the period 1990–2013 and includes 9476 firm-years for the OGE industry, 1859 firm-years for the PP industry, and 11,162 firm-years for the C industry.

In order to obtain parameter \( f \), we average the ratio Selling, General, and Administrative Expense (XSGA)/Assets – Total (AT) for all firm-years in each industry. We follow the same procedure to get \( \delta \) as the ratio of Depreciation and Amortization (DP) over Assets – Total (AT), and \( \tau \) as the fraction Income Taxes – Total (TXT)/Pretax Income (PI). We trim these ratios at the lower and upper one-percentiles to reduce the effect of outliers and errors in the data. Following Moyen (2004), we obtain parameters \( \rho, \sigma, \alpha \), and \( \delta \) for each industry using the firm’s autoregressive profit shock process of Eq. (1) and the gross profits Eq. (2). The data we use with these equations are Gross Profit (GP) and Assets – Total (AT). Given that we are working with representative firms, we set \( c = 1 \) for the three industries. We keep the assumption that the risk-free interest rate \((r_f = r_B)\) is 0.02. We derive \( r_A \) using CAPM with the corresponding (unlevered) industry betas estimated by Fama and French (1997) and assuming an expected market return \((r_M)\) of 0.08. Finally, we obtain \( g \) for each industry from Jorgenson and Stiroh (2000).
References


